

● **Indefinite Integral (Primitive or Antiderivative)**

If for a function g defined on some interval $I \subset \mathbb{R}$, $\frac{d}{dx}(g(x)) = f(x)$, $\forall x \in I$, then $g(x)$ is called a primitive or antiderivative or indefinite integral of f with respect to x .

It is denoted by $\int f(x) dx$. The function f is the integrand and x is the variable of integration.

- If f and g are both differentiable on (a, b) and if $f'(x) = g'(x)$, $\forall x \in (a, b)$, then $f(x) = g(x) + c$, where c is a constant.

- If $\frac{d}{dx}(F(x)) = f(x)$, then $\int f(x) dx = F(x) + c$, where c is an arbitrary constant.

- If f and g are integrable on (a, b) , then

(i) $f + g$ is also integrable and $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$.

(ii) $f - g$ is also integrable and $\int (f(x) - g(x)) dx = \int f(x) dx - \int g(x) dx$.

(iii) For any $k \in \mathbb{R}$, $k \cdot f$ is also integrable and $\int k \cdot f(x) dx = k \int f(x) dx$.

● **Standard integrals**

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \in \mathbb{R} - \{-1\}, x \in \mathbb{R}^+$

(2) $\int \frac{1}{x} dx = \log |x| + c, x \in \mathbb{R} - \{0\}$

(3) $\int \cos x dx = \sin x + c, \forall x \in \mathbb{R}$

(4) $\int \sin x dx = -\cos x + c, \forall x \in \mathbb{R}$

(5) $\int \sec^2 x dx = \tan x + c, x \neq (2k-1)\frac{\pi}{2}, k \in \mathbb{Z}$

(6) $\int \operatorname{cosec}^2 x dx = -\cot x + c, x \neq k\pi, k \in \mathbb{Z}$

(7) $\int \sec x \tan x dx = \sec x + c, x \neq (2k-1)\frac{\pi}{2}, k \in \mathbb{Z}$

(8) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c, x \neq k\pi, k \in \mathbb{Z}$

(9) $\int a^x dx = \frac{a^x}{\log_e a} + c, a \in \mathbb{R}^+ - \{1\}, x \in \mathbb{R}$

(10) $\int e^x dx = e^x + c, \forall x \in \mathbb{R}$

(11) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c = -\frac{1}{a} \cot^{-1} \frac{x}{a} + c', a \in \mathbb{R} - \{0\}, x \in \mathbb{R}$

$$\int \frac{1}{a + bx^2} dx = \frac{1}{\sqrt{ab}} \tan^{-1}\left(\sqrt{\frac{b}{a}} x\right) + c, (a, b > 0)$$

(12) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c, a \in \mathbb{R} - \{0\}$

$$(13) \quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{x+a}{x-a} \right| + c, \quad a \in \mathbb{R} - \{0\}$$

$$(14) \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c = -\cos^{-1} \frac{x}{a} + c', \quad x \in (-a, a), \quad a > 0$$

$$(15) \quad \int \frac{dx}{\sqrt{x^2 + a^2}} = \log |x + \sqrt{x^2 + a^2}| + c, \quad \forall x \in \mathbb{R}$$

$$(16) \quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \log |x + \sqrt{x^2 - a^2}| + c, \quad |x| > |a| > 0$$

$$(17) \quad \int \frac{dx}{|x| \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c = -\frac{1}{a} \operatorname{cosec}^{-1} \frac{x}{a} + c', \quad |x| > |a| > 0$$

For (18) to (21), $I = (k\pi, (2k+1)\frac{\pi}{2})$ or $((2k-1)\frac{\pi}{2}, k\pi)$, $k \in \mathbb{Z}$

$$(18) \quad \int \tan x \, dx = \log |\sec x| + c = -\log |\cos x| + c$$

$$(19) \quad \int \cot x \, dx = \log |\sin x| + c = -\log |\operatorname{cosec} x| + c$$

$$(20) \quad \int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + c = \log \left| \tan \frac{x}{2} \right| + c$$

$$(21) \quad \int \sec x \, dx = \log |\sec x + \tan x| + c = \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + c$$

$$(22) \quad \int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + c, \quad (x^2 > a^2)$$

$$(23) \quad \int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + c$$

$$(24) \quad \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c, \quad (a > 0)$$

$$(25) \quad \int e^{ax} \sin (bx + k) \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin (bx + k) - b \cos (bx + k)] + c, \quad (a, b > 0)$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin (bx + k - \alpha) + c,$$

where, $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}, \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}, \alpha \in (0, 2\pi)$

$$(26) \quad \int e^{ax} \cos (bx + k) \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos (bx + k) + b \sin (bx + k)] + c, \quad (a, b > 0)$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos (bx + k - \alpha) + c,$$

where, $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}, \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}, \alpha \in (0, 2\pi)$

(1) Method of Substitution for Integration

Let $g: [\alpha, \beta] \rightarrow \mathbb{R}$ be continuous on $[\alpha, \beta]$ and differentiable on (α, β) . $g'(t)$ is continuous on (α, β) and $g'(t) \neq 0, \forall t \in (\alpha, \beta)$. Also, the range of g is a subset of $[a, b]$ means $R_g \subset [a, b]$ and $f: [a, b] \rightarrow \mathbb{R}$ is continuous. Then $x = g(t)$ gives, $\int f(x) \, dx = \int f(g(t)) \cdot g'(t) \, dt$.

(2) If $\int f(x) dx = F(x)$, then $\int f(ax + b) dx = \frac{1}{a} F(ax + b)$, where $f : I \rightarrow \mathbb{R}$ is continuous on some interval I . ($a \neq 0$)

(3) $\int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$, ($n \neq -1$, $f(x) > 0$). f and f' are continuous and $f'(x) \neq 0$.

$$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$$

(4) $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$, where f is continuous on $[a, b]$ and differentiable on (a, b) and f' is continuous and non-zero, $\forall x \in [a, b]$ and $f(x) \neq 0$, $\forall x \in [a, b]$.

(5) **Rule of Integration by Parts** : If f and g are differentiable on (a, b) and f' and g' are continuous on (a, b) , then

$$\int f(x) \cdot g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx.$$

If we take $f(x) = u$ and $g'(x) = v$, we get

$$\int u \cdot v dx = u \int v dx - \int \left(\frac{d}{dx}(u) \int v dx \right) dx.$$

(6) $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

(7) **Trigonometric Substitutions** :

Integrand	Substitution
(i) $\sqrt{x^2 + a^2}, \frac{1}{\sqrt{x^2 + a^2}}$	$x = a \tan \theta$ or $x = a \cot \theta$
(ii) $\sqrt{x^2 - a^2}, \frac{1}{\sqrt{x^2 - a^2}}$	$x = a \sec \theta$ or $x = a \operatorname{cosec} \theta$
(iii) $\sqrt{a^2 - x^2}, \frac{1}{\sqrt{a^2 - x^2}}$	$x = a \sin \theta$ or $x = a \cos \theta$
(iv) $\sqrt{\frac{a-x}{a+x}}, \sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
(v) $\sqrt{2ax - x^2}$	$x = 2a \sin^2 \theta$
$\sqrt{2ax - x^2} = \sqrt{a^2 - (x-a)^2}$	$x - a = a \sin \theta$ or $x - a = a \cos \theta$
(vi) $\sqrt{2ax + x^2}$	$x = 2a \tan^2 \theta$
(vii) $\sqrt{a-x}, \sqrt{\frac{x}{a-x}}, \sqrt{x(a-x)}$	$x = a \sin^2 \theta, x = a \cos 2\theta$
(viii) $\sqrt{a+x}, \sqrt{\frac{x}{a+x}}, \sqrt{x(a+x)}$	$x = a \tan^2 \theta, x = a \cos 2\theta$
(ix) $\sqrt{\frac{x-\alpha}{\beta-x}}, \sqrt{(x-\alpha)(\beta-x)}$	$x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

- (8) If the integrand is $\frac{1}{a+b\sin x}$, $\frac{1}{a+b\cos x}$ or $\frac{1}{a+b\sin x+c\cos x}$, then substitute $\tan \frac{x}{2} = t$. Then

$$dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2} \text{ and } \cos x = \frac{1-t^2}{1+t^2}$$

The given integrand will transform into integrand in t .

- (9) If the integrand is $\sin^m x \cos^n x$, $m, n \in \mathbb{N}$, then

- (i) If both m and n are odd, put $\sin x = t$ if $m \geq n$ and $\cos x = t$ otherwise.
- (ii) If m is odd and n is even, then $\cos x = t$.
- (iii) If m is even and n is odd, then $\sin x = t$.
- (iv) If m and n are both even, then we use power reducing formulae

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \text{ and } \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

- (10) (a) $\int \frac{dx}{ax^2+bx+c}$ or (b) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

If we express $ax^2 + bx + c$ in the form of a perfect square, the integral (a) can be obtained using standard forms (11), (12) or (13) and (b) can be obtained using standard forms (14) or (15) or (16).

- (11) (a) $\int \frac{Ax+B}{ax^2+bx+c} dx$ or (b) $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}} dx$

We will find out two constants m and n such that,

$$\begin{aligned} Ax + B &= m[\text{derivative of } ax^2 + bx + c] + n \\ &= m(2ax + b) + n \\ &= (2ma)x + (mb + n) \end{aligned}$$

Comparing coefficients of x on both sides we get, $m = \frac{A}{2a}$ and $n = B - mb$.

$$\begin{aligned} (a) \quad \int \frac{Ax+B}{ax^2+bx+c} dx &= \int \frac{m(2ax+b)+n}{ax^2+bx+c} dx \\ &= m \int \frac{2ax+b}{ax^2+bx+c} dx + n \int \frac{1}{ax^2+bx+c} dx \\ &= m \log |ax^2+bx+c| + n I_2 \end{aligned}$$

I_2 can be obtained using method 10(a).

$$\begin{aligned} (b) \quad \int \frac{Ax+B}{\sqrt{ax^2+bx+c}} dx &= \int \frac{m(2ax+b)+n}{\sqrt{ax^2+bx+c}} dx \\ &= m \int (2ax+b)(ax^2+bx+c)^{-\frac{1}{2}} dx + n \int \frac{dx}{\sqrt{ax^2+bx+c}} \\ &= 2m (ax^2+bx+c)^{\frac{1}{2}} + n I_2 \end{aligned}$$

I_2 can be obtained using method 10(b).

$$(12) \int \frac{dx}{a + b \cos^2 x}, \int \frac{dx}{a + b \sin^2 x} \text{ or } \int \frac{dx}{a \cos^2 x + b \sin^2 x + c}$$

Here, we divide each term by $\cos^2 x \neq 0$. In denominator $\sec^2 x$ is replaced by $1 + \tan^2 x$ and then we assume $\tan x = t$ giving $\sec^2 x \, dx = dt$. Then integrate the resulting rational algebraic function of t .

$$(13) \int \frac{a \sin x + b \cos x + c}{p \sin x + q \cos x + r} dx \text{ or } \int \frac{ae^x + be^{-x} + c}{pe^x + qe^{-x} + r} dx$$

Here, first we find constants, l , m and n , such that

$$\text{NUM} = l (\text{DEN}) + m \frac{d}{dx} (\text{DEN}) + n$$

Find l , m and n by comparing the coefficients of $\cos x$, $\sin x$ and constant term (or e^x , e^{-x} and constant term). Then write integral as, $l \int \frac{dx}{\text{DEN}} + m \int \frac{\frac{d}{dx}(\text{DEN})}{\text{DEN}} + n \int \frac{dx}{\text{DEN}}$ and then integrate.

$$(14) \int \frac{a \tan x + b}{p \tan x + q} dx \text{ or } \int \frac{ae^{2x} + b}{pe^{2x} + q} dx$$

First we transform them as $\int \frac{a \sin x + b \cos x}{p \sin x + q \cos x} dx$ or $\int \frac{ae^x + be^{-x}}{pe^x + qe^{-x}} dx$ respectively, then

$$\text{NUM} = l (\text{DEN}) + m \frac{d}{dx} (\text{DEN})$$

Find l , m and integrate, using (13) above.

$$(15) \int \frac{dx}{a \sin x \pm b \cos x}$$

Use $a = r \cos \alpha$ and $b = r \sin \alpha$ to put the integral in the form $\frac{1}{r} \int \frac{dx}{\sin(x \pm \alpha)}$. Now use formula $\int \operatorname{cosec}(x \pm \alpha) dx$.

$$(16) (a) \int \sqrt{ax^2 + bx + c} \, dx \text{ and } (b) \int (Ax + B) \sqrt{ax^2 + bx + c} \, dx$$

(a) If we express $ax^2 + bx + c$ in the form of a perfect square, the integral can be obtained using (22), (23) or (24).

(b) We will find out constant m and n , such that

$$(Ax + B) = m \left(\frac{d}{dx} (ax^2 + bx + c) \right) + n.$$

Comparing coefficient of x on both sides we get $m = \frac{A}{2a}$ and $n = B - mb$.

$$\begin{aligned} \int (Ax + B) \sqrt{ax^2 + bx + c} \, dx &= \int [m(2ax + b) + n] (ax^2 + bx + c)^{\frac{1}{2}} \, dx \\ &= \int m(2ax + b) (ax^2 + bx + c)^{\frac{1}{2}} \, dx + \int n \sqrt{ax^2 + bx + c} \, dx \end{aligned}$$

$$= \frac{m(ax^2 + bx + c)^{\frac{3}{2}}}{\frac{3}{2}} + n I_2$$

I_2 can be obtained using method (a).

$$(17) \quad (i) \quad \int \frac{x^2 + 1}{x^4 + kx^2 + 1} dx, \quad \int \frac{x^2 - 1}{x^4 + kx^2 + 1} dx, \quad k \in \mathbb{Z}$$

To evaluate these types of integrals, divide the numerator and denominator by $x^2 \neq 0$

$$\text{We get } \int \frac{\left(1 + \frac{1}{x^2}\right) dx}{x^2 + k + \frac{1}{x^2}}, \quad \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{x^2 + k + \frac{1}{x^2}}.$$

then put $\left(x - \frac{1}{x}\right) = t$ or $\left(x + \frac{1}{x}\right) = t$ respectively. Then integrand is converted in terms of a function of t and can be obtained by using standard integrals.

$$(ii) \quad \int \frac{x^2 + a^2}{x^4 + kx^2 + a^4} dx, \quad \int \frac{x^2 - a^2}{x^4 + kx^2 + a^4} dx.$$

To evaluate these types of integrals divide the numerator and denominator by $x^2 \neq 0$ and then putting $x - \frac{a^2}{x} = t$ or $x + \frac{a^2}{x} = t$ respectively.

$$(18) \quad \int \frac{dx}{P(x)\sqrt{Q(x)}} \quad (\text{where, } P(x) \text{ and } Q(x), \text{ are linear or quadratic expressions in } x.)$$

Following table gives suitable substitution for the given integrals :

Integral	Substitution
$\int \frac{dx}{(ax+b)\sqrt{cx+d}}$	$cx + d = t^2$
$\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$	$px + q = t^2$
$\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$	$px + q = \frac{1}{t}$
$\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$	$x = \frac{1}{t} \text{ and } c + dt^2 = u^2$

(19) Method of Partial Fractions :

- (i) If in a rational function $\frac{p(x)}{q(x)}$, degree of $p(x) <$ degree of $q(x)$, then $\frac{p(x)}{q(x)}$ is called **Proper rational function**.

- (ii) If the degree of $p(x) \geq$ the degree of $q(x)$, then $\frac{p(x)}{q(x)}$ is called an **Improper rational function**. If $\frac{p(x)}{q(x)}$ is an improper rational function, we divide $p(x)$ by $q(x)$ till we reach the remainder whose degree is less than that of the denominator. We express $\frac{p(x)}{q(x)} = r(x) + \frac{p_1(x)}{q(x)}$, where degree of $p_1(x)$ is less than that of $q(x)$, so that $\frac{p_1(x)}{q(x)}$ is a proper rational function.

Thus, we can uniquely express an improper rational function as a sum of a rational polynomial and a proper rational function. Now we study the method of integrating a proper rational function. We consider different cases depending on types of factors of $q(x)$.

Case 1 : Real, linear and non-repeated factors :

Let $q(x)$ have n real linear non-repeated factors $x - \beta_1, x - \beta_2, \dots, x - \beta_n$ i.e. $q(x) = (x - \beta_1)(x - \beta_2) \dots (x - \beta_n)$.

Then we express $\frac{p(x)}{q(x)} = \frac{A_1}{x - \beta_1} + \frac{A_2}{x - \beta_2} + \dots + \frac{A_n}{x - \beta_n}$, where $A_1, A_2, \dots, A_n$ are unknown constants. We can always determine these constants and integrate the right hand side.

Case 2 : Real, linear repeated and non-repeated factors :

If $q(x) = (x - \beta_1)^2(x - \beta_3) \dots (x - \beta_n)$, then take

$$\frac{p(x)}{q(x)} = \frac{A_1}{x - \beta_1} + \frac{A_2}{(x - \beta_1)^2} + \frac{A_3}{x - \beta_3} + \dots + \frac{A_n}{(x - \beta_n)}, \text{ where } A_1, A_2, A_3, \dots, A_n \text{ are unknown constants}$$

which can be determined.

Case 3 : One real quadratic and other linear non-repeated factors :

If $q(x) = (ax^2 + bx + c)(x - \beta_3)(x - \beta_4) \dots (x - \beta_n)$, then let

$$\frac{p(x)}{q(x)} = \frac{A_1x + A_2}{ax^2 + bx + c} + \frac{A_3}{x - \beta_3} + \dots + \frac{A_n}{x - \beta_n}.$$

where $A_1, A_2, A_3, \dots, A_n$ are unknown constant to be determined.

Case 4 : Real repeated quadratic factor which cannot be factorised further and other linear non-repeated factors :

If $q(x) = (ax^2 + bx + c)^2(x - \beta_5) \dots (x - \beta_n)$, then take

$$\frac{p(x)}{q(x)} = \frac{A_1x + A_2}{ax^2 + bx + c} + \frac{A_3x + A_4}{(ax^2 + bx + c)^2} + \frac{A_5}{x - \beta_5} + \dots + \frac{A_n}{x - \beta_n},$$

where $A_1, A_2, A_3, \dots, A_n$ are unknown constant to be determined.

Following table gives partial fractions corresponding some different types of rational functions :

No.	Proper Rational Function	Partial Fraction
1.	$\frac{p(x)}{(x-\alpha)(x-\beta)}$	$\frac{A}{x-\alpha} + \frac{B}{x-\beta}$
2.	$\frac{p(x)}{(x-\alpha)(x-\beta)(x-\gamma)}$	$\frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}$
3.	$\frac{p(x)}{(x-\alpha)^2}$	$\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2}$
4.	$\frac{p(x)}{(x-\alpha)^2(x-\beta)}$	$\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2} + \frac{C}{x-\beta}$
5.	$\frac{p(x)}{(x-\alpha)(ax^2+bx+c)}$	$\frac{A}{x-\alpha} + \frac{Bx+C}{ax^2+bx+c}$
6.	$\frac{p(x)}{(x-\alpha)^3}$	$\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2} + \frac{C}{(x-\alpha)^3}$
7.	$\frac{p(x)}{(ax^2+bx+c)^2}$	$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}$

: Multiple Choice Questions :

(1) $\int \left(\frac{x}{p} + \frac{q}{x} + x^p + q^x + pqx \right) dx = \dots\dots$ $x \in R^+, (p \neq -1, 0), q > 0$

(A) $\frac{1}{p} - \frac{q}{x^2} + p \cdot x^{p-1} + q^x \cdot \log q + pq + c$

(B) $\frac{x^2}{2p} + \frac{q}{x^2} + p \cdot x^{p-1} + q^x \log q + pq \frac{x^2}{2} + c$

(C) $\frac{x^2}{2p} + q \log |x| + \frac{x^{p+1}}{p+1} + q^x \log q + pq \frac{x^2}{2} + c$

(D) $\frac{x^2}{2p} + q \log |x| + \frac{x^{p+1}}{p+1} + \frac{q^x}{\log q} + pq \frac{x^2}{2} + c$

Sol. : $I = \int \left(\frac{x}{p} + \frac{q}{x} + x^p + q^x + pqx \right) dx$
 $= \frac{1}{p} \int x dx + q \int \frac{1}{x} dx + \int x^p dx + \int q^x dx + \int pqx dx$
 $= \frac{1}{p} \cdot \frac{x^2}{2} + q \log |x| + \frac{x^{p+1}}{p+1} + \frac{q^x}{\log q} + pq \frac{x^2}{2} + c$

Ans. : (D)

(2) $\int e^{2a \log x} + e^{3x \log a} dx = \dots\dots$ $x, a \in R^+$

(A) $\frac{x^{2a}}{2a} + a^{3x} \log 3 + c$

(B) $\frac{x^{2a+1}}{2a+1} + a^{3x} \log a + c$

(C) $\frac{x^{2a+1}}{2a+1} + \frac{a^{3x}}{\log a} + c$

(D) $\frac{x^{2a+1}}{2a+1} + \frac{a^{3x}}{3 \log a} + c$

$$\begin{aligned}
 \text{Sol. : } I &= \int e^{2a \log x} + e^{3x \log a} dx \\
 &= \int e^{\log x^{2a}} + e^{\log a^{3x}} dx \\
 &= \int (x^{2a} + a^{3x}) dx \\
 &= \frac{x^{2a+1}}{2a+1} + \frac{a^{3x}}{3 \log a} + c
 \end{aligned}$$

Ans. : (D)

$$(3) \quad \int e^{x \log a} \cdot e^x dx = \dots\dots$$

$$(A) \quad \frac{a^x}{\log ae} + c \quad (B) \quad \frac{e^x}{1 + \log a} + c \quad (C) \quad \frac{a^x e^x}{\log a} + c \quad (D) \quad \frac{(ae)^x}{\log a + 1} + c$$

$$\text{Sol. : } I = \int e^{x \log a} \cdot e^x dx$$

$$a \in R^+, a \neq e^{-1}$$

$$= \int e^{\log a^x} \cdot e^x dx$$

$$= \int a^x e^x dx$$

$$= \int (ae)^x dx$$

$$= \frac{(ae)^x}{\log ae} + c$$

$$= \frac{(ae)^x}{\log a + \log e} + c$$

$$= \frac{(ae)^x}{\log a + 1} + c$$

Ans. : (D)

$$(4) \quad \int \frac{3^x + 4^x}{5^x} dx = \dots\dots + c$$

$$(A) \quad \frac{3^x \log 3}{5^x \log 5} + \frac{4^x \log 4}{5^x \log 5}$$

$$(B) \quad \left(\frac{3}{5}\right)^x (\log 3 - \log 5) + \left(\frac{4}{5}\right)^x (\log 4 - \log 5)$$

$$(C) \quad \frac{\left(\frac{3}{5}\right)^x}{\log 3 + \log 5} + \frac{\left(\frac{4}{5}\right)^x}{\log 4 + \log 5}$$

$$(D) \quad \frac{\left(\frac{3}{5}\right)^x}{\log \left(\frac{3}{5}\right)} + \frac{\left(\frac{4}{5}\right)^x}{\log \left(\frac{4}{5}\right)}$$

$$\text{Sol. : } I = \int \frac{3^x + 4^x}{5^x} dx$$

$$= \int \left(\frac{3^x}{5^x} + \frac{4^x}{5^x} \right) dx$$

$$= \int \left(\left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x \right) dx$$

$$= \frac{\left(\frac{3}{5}\right)^x}{\log \left(\frac{3}{5}\right)} + \frac{\left(\frac{4}{5}\right)^x}{\log \left(\frac{4}{5}\right)} + c$$

Ans. : (D)

(5) If $g'(x) = \sec^2 x + \operatorname{cosec}^2 x - 2$ and $g\left(\frac{\pi}{4}\right) = 1$, then $g(x) = \dots$

- (A) $\tan x + \cot x - 2x + \frac{\pi}{2}$ (B) $\tan x - \cot x - 2x + \frac{\pi}{2} + 1$
 (C) $\tan x + \cot x + \frac{\pi}{2} + 1$ (D) $\tan x - \cot x + 2x - \frac{\pi}{2} + 1$

Sol. : Here, $g'(x) = \sec^2 x + \operatorname{cosec}^2 x - 2$

$$\therefore g(x) = \int (\sec^2 x + \operatorname{cosec}^2 x - 2) dx$$

$$g(x) = \tan x - \cot x - 2x + c$$

$$\text{Now, } g\left(\frac{\pi}{4}\right) = 1$$

$$\therefore \tan \frac{\pi}{4} - \cot \frac{\pi}{4} - 2\left(\frac{\pi}{4}\right) + c = 1$$

$$1 - 1 - \frac{\pi}{2} + c = 1$$

$$\therefore c = \frac{\pi}{2} + 1$$

$$\therefore g(x) = \tan x - \cot x - 2x + \frac{\pi}{2} + 1$$

Ans. : (B)

(6) If $f'(x) = 1 - \frac{3}{x^2}$ and $f(1) = 6$, then $f(2) = \dots$

- (A) $\frac{11}{2}$ (B) $\frac{5}{2}$ (C) $\frac{3}{2}$ (D) $\frac{1}{2}$

Sol. : $f'(x) = 1 - \frac{3}{x^2} = 1 - 3x^{-2}$

$$\therefore f(x) = \int (1 - 3x^{-2}) dx$$

$$\therefore f(x) = x - \frac{3x^{-1}}{-1} + c$$

$$\therefore f(x) = x + \frac{3}{x} + c$$

$$\text{Now, } f(1) = 6$$

$$1 + 3 + c = 6$$

$$\therefore c = 2$$

$$\therefore f(x) = x + \frac{3}{x} + 2$$

$$\therefore f(2) = 2 + \frac{3}{2} + 2 = \frac{11}{2}$$

Ans. : (A)

(7) $\int \frac{\sec x}{\sec x - \tan x} dx = \dots$

- (A) $x - \log | \operatorname{cosec} x - \cot x + x | + c$ (B) $x + \log \left| \tan \frac{x}{2} \right| + c$
 (C) $\sec x + \tan x + c$ (D) $\sec x \tan x + \sec^2 x + c$

$$\text{Sol. : } I = \int \frac{\sec x}{\sec x - \tan x} dx$$

$$\begin{aligned}
&= \int \frac{\sec x (\sec x + \tan x)}{(\sec x - \tan x)(\sec x + \tan x)} dx \\
&= \int (\sec^2 x + \sec x \tan x) dx \\
&= \tan x + \sec x + c
\end{aligned}$$

Ans. : (C)

$$(8) \quad \int \frac{\sin^2 x dx}{1 - \cos x} = \dots\dots$$

$$(A) \quad x - \sin x + c$$

$$(B) \quad x + \sin x + c$$

$$(C) \quad \log |1 - \cos x| + c$$

$$(D) \quad 2 \log |\sin x| + c$$

$$\begin{aligned}
\text{Sol. : I} &= \int \frac{\sin^2 x dx}{1 - \cos x} \\
&= \int \frac{1 - \cos^2 x}{1 - \cos x} dx \\
&= \int \frac{(1 + \cos x)(1 - \cos x)}{(1 - \cos x)} dx \\
&= \int (1 + \cos x) dx \\
&= x + \sin x + c
\end{aligned}$$

Ans. : (B)

$$(9) \quad \int \frac{x^4 + 2}{x^2 + 1} dx = \dots\dots$$

$$(A) \quad \frac{x^3}{3} + x + \tan^{-1} x + c$$

$$(B) \quad \frac{x^3}{3} - x + 3\tan^{-1} x + c$$

$$(C) \quad \frac{x^3}{3} - x + \frac{1}{3}\tan^{-1} x + c$$

$$(D) \quad \frac{x^3}{3} + x + \frac{1}{3}\tan^{-1} x + c$$

$$\begin{aligned}
\text{Sol. : I} &= \int \frac{x^4 + 2}{x^2 + 1} dx \\
&= \int \frac{(x^4 - 1) + 3}{(x^2 + 1)} dx \\
&= \int \left(x^2 - 1 + \frac{3}{x^2 + 1} \right) dx \\
&= \frac{x^3}{3} - x + 3\tan^{-1} x + c
\end{aligned}$$

Ans. : (B)

$$(10) \quad \int x\sqrt{x+4} dx = \dots\dots + c$$

$$(A) \quad 2\sqrt{x+4} - 8(x+4)^{-\frac{1}{2}}$$

$$(B) \quad \frac{1}{3}(x+4)^{\frac{3}{2}} - \frac{4}{3}(x+4)^{\frac{1}{2}}$$

$$(C) \quad \frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}}$$

$$(D) \quad \frac{2}{5}(x+4)^{\frac{5}{2}} + \frac{4}{3}(x+4)^{\frac{3}{2}}$$

$$\begin{aligned}
\text{Sol. : I} &= \int x\sqrt{x+4} \, dx \\
&= \int [(x+4) - 4] \sqrt{x+4} \, dx \\
&= \int (x+4) \sqrt{x+4} \, dx - 4 \int \sqrt{x+4} \, dx \\
&= \frac{(x+4)^{\frac{5}{2}}}{\frac{5}{2}} - 4 \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
&= \frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}} + c
\end{aligned}$$

Ans. : (C)

$$(11) \int \frac{x-1}{\sqrt{x+4}} \, dx = \dots\dots$$

$$(A) \quad \frac{2}{3}(x+4)^{\frac{3}{2}} - 10\sqrt{x+4} + c$$

$$(B) \quad \frac{-2}{\sqrt{x+4}} + 10\sqrt{x+4} + c$$

$$(C) \quad \frac{2}{3}(x+4)^{\frac{3}{2}} - \frac{5}{2}\sqrt{x+4} + c$$

$$(D) \quad \frac{2}{3}(x+4)^{\frac{3}{2}} + \frac{5}{2}\sqrt{x+4} + c$$

$$\begin{aligned}
\text{Sol. : I} &= \int \frac{x-1}{\sqrt{x+4}} \, dx \\
&= \int \frac{(x+4)-5}{\sqrt{x+4}} \, dx \\
&= \int \left(\sqrt{x+4} - \frac{5}{\sqrt{x+4}} \right) \, dx \\
&= \int (x+4)^{\frac{1}{2}} \, dx - 5 \int (x+4)^{-\frac{1}{2}} \, dx \\
&= \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} - 5 \frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}} + c \\
&= \frac{2}{3}(x+4)^{\frac{3}{2}} - 10\sqrt{x+4} + c
\end{aligned}$$

Ans. : (A)

$$(12) \int \frac{\cos 2x - 1}{\cos 2x + 1} \, dx = \dots\dots$$

$$(A) \quad \tan x - x + c \quad (B) \quad -\cot x - x + c \quad (C) \quad x + \tan x + c \quad (D) \quad x - \tan x + c$$

$$\begin{aligned}
\text{Sol. : I} &= \int \frac{\cos 2x - 1}{\cos 2x + 1} \, dx \\
&= -\int \tan^2 x \, dx \\
&= -\int (\sec^2 x - 1) \, dx \\
&= x - \tan x + c
\end{aligned}$$

Ans. : (D)

(13) If $\int \frac{\cos 4x + 1}{\cot x - \tan x} dx = P \cos 4x + c$, then $P = \dots$

(A) $-\frac{1}{2}$

(B) $-\frac{1}{4}$

(C) $-\frac{1}{8}$

(D) $\frac{1}{8}$

Sol. :
$$\begin{aligned} & \int \frac{\cos 4x + 1}{\cot x - \tan x} dx \\ &= \int \frac{2\cos^2 2x}{\left(\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}\right)} dx \\ &= \int \frac{2\cos^2 2x \cdot \sin x \cdot \cos x}{\cos^2 x - \sin^2 x} dx \\ &= \int \frac{\cos^2 2x \cdot \sin 2x}{\cos 2x} dx \\ &= \frac{1}{2} \int 2 \sin 2x \cos 2x dx \\ &= \frac{1}{2} \int \sin 4x dx \\ &= \frac{1}{2} \left(-\frac{\cos 4x}{4} \right) + c \\ &= -\frac{1}{8} \cos 4x + c \end{aligned}$$

So, $P = -\frac{1}{8}$

Ans. : (C)

(14) $\int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} dx = \dots$

(A) $-\cot \frac{x}{2} + c$

(B) $-\frac{1}{2} \cot \frac{x}{2} + c$

(C) $-\tan \frac{x}{2} + c$

(D) $\cot \frac{x}{2} + c$

Sol. :
$$\begin{aligned} & \int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} dx \\ &= \int \frac{\left(\frac{1}{\sin x}\right)}{\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)} dx \\ &= \int \frac{1}{1 - \cos x} dx \\ &= \int \frac{1}{2 \sin^2 \frac{x}{2}} dx \\ &= \frac{1}{2} \int \operatorname{cosec}^2 \frac{x}{2} dx \\ &= \frac{1}{2} \left(-\frac{\cot \frac{x}{2}}{\frac{1}{2}} \right) + c \\ &= -\cot \frac{x}{2} + c \end{aligned}$$

Second method :
$$\int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x) dx}{1}$$

$$= \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x) dx$$

$$= -\cot x - \operatorname{cosec} x + c$$

$$= -\frac{1 + \cos x}{\sin x} + c$$

$$= -\frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= -\cot \frac{x}{2} + c$$

Ans. : (A)

$$(15) \int \frac{\cos 2x - \cos 2\beta}{\cos x - \cos \beta} dx = \dots\dots$$

$$(A) \quad 2(\sin x + x \cos \beta) + c$$

$$(B) \quad 2(\sin x + \sin \beta) + c$$

$$(C) \quad 2(-\sin x + x \cos \beta) + c$$

$$(D) \quad -2(\sin x + \sin \beta) + c$$

$$\begin{aligned} \text{Sol. : } & \int \frac{\cos 2x - \cos 2\beta}{\cos x - \cos \beta} dx \\ &= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \beta - 1)}{(\cos x - \cos \beta)} dx \\ &= \int \frac{2(\cos^2 x - \cos^2 \beta)}{(\cos x - \cos \beta)} dx \\ &= 2 \int (\cos x + \cos \beta) dx \\ &= 2(\sin x + x \cos \beta) + c \end{aligned}$$

Ans. : (A)

$$(16) \int \frac{1 + \cos x}{\sin x \cos x} dx = \dots\dots$$

$$(A) \quad \log |\sin x| + \log |\cos x| + c$$

$$(B) \quad \log \left| \tan x \cdot \tan \frac{x}{2} \right| + c$$

$$(C) \quad \log \left| 1 + \tan \frac{x}{2} \right| + c$$

$$(D) \quad \log \left| \sec \frac{x}{2} + \tan \frac{x}{2} \right| + c$$

$$\begin{aligned} \text{Sol. : } & \int \frac{1 + \cos x}{\sin x \cos x} dx \\ &= \int \frac{1}{\sin x \cos x} dx + \int \frac{1}{\sin x} dx \\ &= 2 \int \frac{1}{\sin 2x} dx + \int \operatorname{cosec} x dx \\ &= 2 \int \operatorname{cosec} 2x dx + \int \operatorname{cosec} x dx \\ &= 2 \frac{\log |\tan x|}{2} + \log \left| \tan \frac{x}{2} \right| + c \\ &= \log \left| \tan x \cdot \tan \frac{x}{2} \right| + c \end{aligned}$$

Ans. : (B)

$$(17) \int \sec^2 x \cdot \operatorname{cosec}^2 x dx = \dots\dots$$

$$(A) \quad \cot x + \tan x + c$$

$$(B) \quad \cot x - \tan x + c$$

$$(C) \quad -\cot x - \tan x + c$$

$$(D) \quad -\cot x + \tan x + c$$

$$\begin{aligned} \text{Sol. : } & \int \sec^2 x \cdot \operatorname{cosec}^2 x dx \\ &= \int \frac{1}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx \end{aligned}$$

$$\begin{aligned}
&= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx \\
&= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx \\
&= \tan x - \cot x + c
\end{aligned}$$

Ans. : (D)

$$(18) \int \frac{1 + \cos x}{\cos x (1 - \cos x)} dx = \dots$$

$$(A) \quad 2 \log |\cos x| - \cot \frac{x}{2} + c$$

$$(B) \quad \log |\sec x + \tan x| - 2 \cot \frac{x}{2} + c$$

$$(C) \quad \log |\tan \frac{x}{2}| + 2 \cot \frac{x}{2} + c$$

$$(D) \quad \log |\sec x + \tan x| - \cot \frac{x}{2} + c$$

$$\begin{aligned}
\text{Sol. : } &\int \frac{1 + \cos x}{\cos x (1 - \cos x)} dx \\
&= \int \frac{(1 - \cos x) + 2 \cos x}{\cos x (1 - \cos x)} dx \\
&= \int \sec x dx + \int \frac{2}{1 - \cos x} dx \\
&= \int \sec x dx + 2 \int \frac{1}{2 \sin^2 \frac{x}{2}} dx \\
&= \int \sec x dx + \int \operatorname{cosec}^2 \frac{x}{2} dx \\
&= \log |\sec x + \tan x| - 2 \cot \frac{x}{2} + c
\end{aligned}$$

Ans. : (B)

$$(19) \text{ If } \int \frac{2x^2 + 3}{(x^2 - 1)(x^2 + 4)} dx = a \log \left| \frac{x+1}{x-1} \right| + b \tan^{-1} \frac{x}{2} + c, \text{ then } a = \dots \text{ and } b = \dots$$

$$(A) \quad \frac{1}{2}, \frac{1}{2}$$

$$(B) \quad -\frac{1}{2}, \frac{1}{2}$$

$$(C) \quad \frac{1}{2}, -\frac{1}{2}$$

$$(D) \quad -\frac{1}{2}, -\frac{1}{2}$$

$$\begin{aligned}
\text{Sol. : } &\int \frac{2x^2 + 3}{(x^2 - 1)(x^2 + 4)} dx \\
&= \int \frac{(x^2 + 4) + (x^2 - 1)}{(x^2 - 1)(x^2 + 4)} dx \\
&= \int \frac{1}{x^2 - 1} dx + \int \frac{1}{x^2 + 4} dx \\
&= \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c \\
&= -\frac{1}{2} \log \left| \frac{x+1}{x-1} \right| + \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c
\end{aligned}$$

$$a = -\frac{1}{2}, b = \frac{1}{2}$$

Ans. : (B)

$$(20) \int \frac{\sin^3 3x}{\cos^5 3x} dx = \dots\dots$$

- (A) $\tan^4 x + c$ (B) $\tan 6x + c$ (C) $\frac{1}{12} \tan^4 3x + c$ (D) $\frac{1}{3} \tan^4 3x + c$

$$\text{Sol. : } \int \frac{\sin^3 3x}{\cos^5 3x} dx$$

$$= \int \tan^3 3x \cdot \sec^2 3x dx$$

$$\text{Let } \tan 3x = t, \quad \text{So, } 3\sec^2 3x dx = dt$$

$$\therefore \sec^2 3x dx = \frac{1}{3} dt$$

$$I = \frac{1}{3} \int t^3 \cdot dt$$

$$= \frac{1}{3} \times \frac{t^4}{4} + c = \frac{1}{12} \tan^4 3x + c$$

Ans. : (C)

$$(21) \int \frac{\cot x}{\sqrt{\sin x}} dx = \dots\dots$$

- (A) $\frac{-2}{\sqrt{\sin x}} + c$ (B) $\frac{-1}{2\sqrt{\sin x}} + c$ (C) $\frac{1}{2\sqrt{\sin x}} + c$ (D) $2\sqrt{\sin x} + c$

$$\text{Sol. : } \int \frac{\cot x}{\sqrt{\sin x}} dx$$

$$= \int \frac{\cos x}{\sin x \cdot (\sin x)^{\frac{1}{2}}} dx$$

$$= \int (\sin x)^{\frac{-3}{2}} \cos x dx$$

$$= \frac{(\sin x)^{\frac{-3}{2}+1}}{\frac{-3}{2}+1} + c = \frac{(\sin x)^{\frac{-1}{2}}}{\frac{-1}{2}} + c = \frac{-2}{\sqrt{\sin x}} + c$$

Ans. : (A)

$$(22) \text{ If } f'(x) = \frac{1}{\cos^2 x \sqrt{1 + \tan x}} \text{ and } f(0) = 4, \text{ then } f(x) = \dots\dots$$

- (A) $\sqrt{1 + \tan x} + 4$ (B) $\frac{2}{3} (1 + \tan x)^{\frac{3}{2}}$
 (C) $2(\sqrt{1 + \tan x} + 1)$ (D) $2(\sqrt{1 + \tan x} - 1)^{\frac{2}{3}}$

$$\text{Sol. : } f'(x) = \frac{1}{\cos^2 x \sqrt{1 + \tan x}}$$

$$\therefore f(x) = \int \frac{\sec^2 x dx}{\sqrt{1 + \tan x}}$$

Let $\tan x = t$.

So, $\sec^2 x \, dx = dt$

$$\therefore f(x) = \int \frac{dt}{\sqrt{1+t}}$$

$$\therefore = 2\sqrt{1+\tan x} + c$$

Now, $f(0) = 4$

$$\therefore 2\sqrt{1+\tan 0} + c = 4$$

$$\therefore c = 2$$

$$\therefore f(x) = 2\sqrt{1+\tan x} + 2$$

$$= 2(\sqrt{\tan x + 1} + 1)$$

Ans. : (C)

(23) If $f(x) = \frac{1}{3-2\cos^2 x + 6\sin^2 x}$ and has antiderivative $F(x) = \frac{1}{3} \tan^{-1}(g(x)) + c$, then $g(x) = \dots\dots$

(A) $3\tan x$ (B) $\sqrt{3}\tan x$ (C) $2\tan x$ (D) $\tan x$

Sol. : $f(x) = \frac{1}{3-2\cos^2 x + 6\sin^2 x}$

$$\therefore F(x) = \int \frac{dx}{3-2\cos^2 x + 6\sin^2 x}$$

$$= \int \frac{\sec^2 x \, dx}{3\sec^2 x - 2 + 6\tan^2 x}$$

$$= \int \frac{\sec^2 x \, dx}{3(1+\tan^2 x) - 2 + 6\tan^2 x}$$

$$= \int \frac{\sec^2 x \, dx}{9\tan^2 x + 1}$$

$$= \int \frac{dt}{9t^2 + 1}$$

(Let $\tan x = t$. So, $\sec^2 x \, dx = dt$)

$$= \frac{1}{3} \tan^{-1}(3t) + c$$

$$= \frac{1}{3} \tan^{-1}(3\tan x) + c$$

$$\therefore g(x) = 3\tan x$$

Ans. : (A)

(24) $\int \frac{e^{\sqrt{x}} \sin(e^{\sqrt{x}})}{\sqrt{x}} \, dx = \dots\dots$

(A) $e^{\sqrt{x}} [\sin(e^{\sqrt{x}}) + \cos(e^{\sqrt{x}})] + c$

(B) $e^{\sqrt{x}} (\cos\sqrt{x} - \sin\sqrt{x}) + c$

(C) $-2\cos(e^{\sqrt{x}}) + c$

(D) $-2e^{\sqrt{x}} \cos(e^{\sqrt{x}}) + c$

$$\text{Sol. : } I = \int \frac{e^{\sqrt{x}} \sin(e^{\sqrt{x}})}{\sqrt{x}} dx$$

$$\text{Let } e^{\sqrt{x}} = t$$

$$\therefore e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}} dx = dt$$

$$\therefore \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2dt$$

$$\therefore I = 2 \int \sin t dt = -2\cos t + c = -2\cos(e^{\sqrt{x}}) + c \quad \text{Ans. : (C)}$$

$$(25) \text{ If } \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \log |f(x)| + c, \text{ then } f(x) = \dots$$

$$(A) (\sin x + \cos x)^2 \quad (B) (\sin x - \cos x) \quad (C) (\cos x + \sin x) \quad (D) \cos 2x$$

$$\text{Sol. : } I = \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

$$\text{Let } \cos x + \sin x = t.$$

$$\text{So, } (-\sin x + \cos x)dx = dt$$

$$\therefore I = \int \frac{dt}{t} = \log |t| + c = \log |\cos x + \sin x| + c$$

$$\therefore f(x) = \cos x + \sin x \quad \text{Ans. : (C)}$$

$$(26) \int \frac{\cos^3 x}{\sin^2 x + \sin x} dx = \dots$$

$$(A) \tan^{-1} \left(\frac{\cos x}{1 + \sin x} \right) + c$$

$$(B) \log |\sin x + 1| + c$$

$$(C) \log |1 + \cos x| + \cos x + c$$

$$(D) \log |\sin x| - \sin x + c$$

$$\text{Sol. : } I = \int \frac{\cos^3 x}{\sin^2 x + \sin x} dx$$

$$= \int \frac{(1 - \sin^2 x) \cos x}{\sin x (\sin x + 1)} dx$$

$$= \int \frac{(1 - \sin x) \cos x}{\sin x} dx$$

$$\text{Let } \sin x = t.$$

$$\text{So, } \cos x dx = dt$$

$$\therefore I = \int \frac{1-t}{t} dt = \int \frac{1}{t} dt - \int dt = \log |t| - t + c = \log |\sin x| - \sin x + c \quad \text{Ans. : (D)}$$

$$(27) \int \frac{\sin x}{\sin x - \cos x} dx = \dots\dots$$

$$(A) \quad \frac{x}{2} + \log |\cos x - \sin x| + c$$

$$(B) \quad \frac{x}{2} + \frac{1}{2} \log |\sin x - \cos x| + c$$

$$(C) \quad \frac{x}{2} + \frac{1}{2} \log |\cos x + \sin x| + c$$

$$(D) \quad \frac{x}{2} - \frac{1}{2} \log |\cos x + \sin x| + c$$

(IIT : 1978)

$$\begin{aligned} \text{Sol. : } & \int \frac{\sin x}{\sin x - \cos x} dx \\ &= \frac{1}{2} \int \frac{\sin x - \cos x + \cos x + \sin x}{(\sin x - \cos x)} dx \\ &= \frac{1}{2} \int \left(1 + \frac{(\cos x + \sin x)}{\sin x - \cos x} \right) dx \\ &= \frac{1}{2} (x + \log |\sin x - \cos x|) + c \\ &= \frac{x}{2} + \frac{1}{2} \log |\sin x - \cos x| + c \end{aligned}$$

Ans. : (B)

$$(28) \text{ If } \int \frac{x^2}{(a+bx)^2} dx = P(a+bx) + \frac{Q}{(a+bx)} + R \log |a+bx| + c, \text{ then } P + Q + R = \dots\dots .$$

$$(A) \quad \frac{1+2a-a^2}{b^3}$$

$$(B) \quad \frac{(1+a)^2}{b^3}$$

$$(C) \quad \frac{1-2a-a^2}{b^3}$$

$$(D) \quad \frac{(1-a)^2}{3}$$

(IIT : 1979)

$$\text{Sol. : } I = \int \frac{x^2}{(a+bx)^2} dx.$$

$$\text{Let } a + bx = t$$

$$\text{So, } b dx = dt \text{ and } dx = \frac{1}{b} dt$$

$$I = \frac{1}{b^3} \int \frac{(t-a)^2}{t^2} dt$$

$$= \frac{1}{b^3} \int \left(\frac{t^2 - 2at + a^2}{t^2} \right) dt$$

$$= \frac{1}{b^3} \int \left(1 - \frac{2a}{t} + \frac{a^2}{t^2} \right) dt$$

$$= \frac{1}{b^3} \left[t - 2a \log |t| + a^2 \left(\frac{-1}{t} \right) \right] + c$$

$$= \frac{(a+bx)}{b^3} - \frac{2a}{b^3} \log |a+bx| - \frac{a^2}{b^3(a+bx)} + c$$

$$\therefore P = \frac{1}{b^3}, Q = \frac{-a^2}{b^3}, R = -\frac{2a}{b^3}$$

$$\therefore P + Q + R = \frac{1-2a-a^2}{b^3}$$

Ans. : (C)

$$(29) \int \frac{dx}{\sqrt{\sin^3 x \cdot \cos x}} = \dots\dots$$

- (A) $\frac{2}{\sqrt{\tan x}} + c$ (B) $\frac{-2}{\sqrt{\tan x}} + c$ (C) $2\sqrt{\tan x} + c$ (D) $-2\sqrt{\tan x} + c$

$$\text{Sol. : } \int \frac{dx}{\sqrt{\sin^3 x \cdot \cos x}}$$

$$= \int \frac{dx}{\sqrt{\frac{\sin^3 x}{\cos^3 x} \cdot \cos^4 x}}$$

$$= \int \frac{\sec^2 x \, dx}{\sqrt{\tan^3 x}}$$

$$= \int \frac{dt}{\sqrt{t^3}}$$

(Let $\tan x = t$. So, $\sec^2 x \, dx = dt$)

$$= \int t^{-\frac{3}{2}} \, dt$$

$$= \frac{-2}{\sqrt{t}} + c$$

$$= \frac{-2}{\sqrt{\tan x}} + c$$

Ans. : (B)

$$(30) \int \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} \, dx = \dots\dots$$

- (A) $\frac{1}{a^2 + b^2} \log |a^2 \cos^2 x + b^2 \sin^2 x| + c$ (B) $\frac{1}{b^2 - a^2} \log |a^2 \cos^2 x + b^2 \sin^2 x| + c$
 (C) $\frac{a^2}{2b^2} \log |a^2 \cos^2 x + b^2 \sin^2 x| + c$ (D) $\frac{b^2}{2a^2} \log |a^2 \cos^2 x + b^2 \sin^2 x| + c$

$$\text{Sol. : } I = \int \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} \, dx$$

$$\text{Let } a^2 \cos^2 x + b^2 \sin^2 x = t$$

$$\therefore (-a^2 \sin 2x + b^2 \sin 2x) \, dx = dt$$

$$\therefore (b^2 - a^2) \sin 2x \, dx = dt$$

$$\therefore \sin 2x \, dx = \frac{1}{b^2 - a^2} \, dt$$

$$\therefore I = \frac{1}{b^2 - a^2} \int \frac{dt}{t}$$

$$= \frac{1}{b^2 - a^2} \log |t| + c$$

$$= \frac{1}{b^2 - a^2} \log |a^2 \cos^2 x + b^2 \sin^2 x| + c$$

Ans. : (B)

$$(31) \int \frac{(7x^6 + 7^x \log 7)}{7^x + x^7} dx = \dots\dots$$

$$(A) \quad 7^x \log |1 + x^7| + c$$

$$(B) \quad (x + 1) \log |7^x + x^7| + c$$

$$(C) \quad \log |7^x + x^7| + c$$

$$(D) \quad \frac{1}{\log x} \log |7^x + x^7| + c$$

$$\text{Sol. : } I = \int \frac{(7x^6 + 7^x \log 7)}{7^x + x^7} dx$$

$$\text{Let } 7^x + x^7 = t.$$

$$\text{So, } (7^x \log 7 + 7x^6) dx = dt$$

$$I = \int \frac{dt}{t} = \log |t| + c = \log |7^x + x^7| + c$$

Ans. : (C)

$$(32) \int \log (\tan x) \cdot \sec x \operatorname{cosec} x dx = \dots\dots$$

$$(A) \quad \log |\sec x \cdot \operatorname{cosec} x| + c$$

$$(B) \quad \log |\log (\tan x)| + c$$

$$(C) \quad \frac{1}{2} [\log (\tan x)]^2 + c$$

$$(D) \quad \frac{1}{2} \log (\tan x) + c$$

$$\text{Sol. : } I = \int \log (\tan x) \cdot \sec x \operatorname{cosec} x dx$$

$$\text{Let } \log (\tan x) = t$$

$$\therefore \left(\frac{1}{\tan x} \cdot \sec^2 x \right) dx = dt$$

$$\therefore \left(\frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x} \right) dx = dt$$

$$\therefore \sec x \operatorname{cosec} x dx = dt$$

$$I = \int t dt = \frac{t^2}{2} + c = \frac{1}{2} [\log (\tan x)]^2 + c$$

Ans. : (C)

$$(33) \int \sqrt{\frac{\cos x - \cos^3 x}{1 - \cos^3 x}} dx = \dots\dots$$

$$(0 < x < \frac{\pi}{2})$$

$$(A) \quad \frac{1}{3} \cos^{-1} (\cos^{\frac{3}{2}} x) + c$$

$$(B) \quad \frac{1}{3} \sin^{-1} (\sin^{\frac{3}{2}} x) + c$$

$$(C) \quad \frac{2}{3} \sin^{-1} (\cos^{\frac{3}{2}} x) + c$$

$$(D) \quad -\frac{2}{3} \sin^{-1} (\cos^{\frac{3}{2}} x) + c$$

$$\text{Sol. : } I = \int \sqrt{\frac{\cos x - \cos^3 x}{1 - \cos^3 x}} dx$$

$$= \int \frac{\sqrt{\cos x \cdot \sin x}}{\sqrt{1 - \cos^3 x}} dx$$

$$\text{Let } \cos^{\frac{3}{2}} x = t$$

$$\therefore \frac{-3}{2} \cdot \cos^{\frac{1}{2}} x \sin x dx = dt$$

$$\therefore \sqrt{\cos x} \cdot \sin x \, dx = -\frac{2}{3} dt$$

$$\therefore I = \frac{-2}{3} \int \frac{dt}{\sqrt{1-t^2}} = \frac{-2}{3} \sin^{-1}(t) + c = \frac{-2}{3} \sin^{-1}(\cos^{\frac{3}{2}} x) + c \quad \text{Ans. : (D)}$$

$$(34) \int \frac{\log(x+1) - \log x}{x(x+1)} dx = \dots$$

$$(A) -\log \log \left(\frac{x+1}{x} \right) + c$$

$$(B) \log(x+1)^2 - \log x^2 + c$$

$$(C) -\log \left(\frac{x+1}{x} \right) + c$$

$$(D) \frac{-1}{2} \left[\log \left(\frac{x+1}{x} \right) \right]^2 + c$$

$$\text{Sol. : } I = \int \frac{\log(x+1) - \log x}{x(x+1)} dx$$

$$\text{Let } \log(x+1) - \log x = t$$

$$\therefore \left(\frac{1}{x+1} - \frac{1}{x} \right) dx = dt$$

$$\therefore \frac{-1}{x(x+1)} dx = dt$$

$$\therefore I = - \int t \, dt$$

$$= -\frac{t^2}{2} + c = \frac{-1}{2} [\log(x+1) - \log x]^2 + c = \frac{-1}{2} \left[\log \left(\frac{x+1}{x} \right) \right]^2 + c \quad \text{Ans. : (D)}$$

$$(35) \int \frac{e^{\log \left(1 + \frac{1}{x^2} \right)}}{x^2 + \frac{1}{x^2}} dx = \dots$$

$$(A) \frac{1}{2} \tan^{-1} \left(\frac{x^2-1}{\sqrt{2}x} \right) + c$$

$$(B) \tan^{-1} \left(\frac{x^2-1}{\sqrt{2}x} \right) + c$$

$$(C) \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2-1}{\sqrt{2}x} \right) + c$$

$$(D) \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2-1}{2x} \right) + c$$

$$\text{Sol. : } I = \int \frac{e^{\log \left(1 + \frac{1}{x^2} \right)}}{x^2 + \frac{1}{x^2}} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx$$

$$= \int \frac{\left(1 + \frac{1}{x^2} \right)}{\left(x - \frac{1}{x} \right)^2 + 2} dx$$

$$= \int \frac{dt}{t^2 + 2} \quad \left(\text{Let } x - \frac{1}{x} = t. \text{ So, } \left(1 + \frac{1}{x^2}\right) dx = dt \right)$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) + c = \frac{1}{\sqrt{2}} \tan^{-1} \frac{\left(x - \frac{1}{x}\right)}{\sqrt{2}} + c = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + c \quad \text{Ans. : (C)}$$

$$(36) \int \frac{dx}{\sqrt{3} \sin x + \cos x} = \dots$$

$$(A) \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

$$(B) \frac{1}{2} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{2} \right) \right| + c$$

$$(C) \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{12} \right) \right| + c$$

$$(D) \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

$$\text{Sol. : } I = \int \frac{dx}{\sqrt{3} \sin x + \cos x}$$

$$= \frac{1}{2} \int \frac{dx}{\frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x}$$

$$= \frac{1}{2} \int \frac{dx}{\sin \left(x + \frac{\pi}{6} \right)} = \frac{1}{2} \int \operatorname{cosec} \left(x + \frac{\pi}{6} \right) dx = \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c \quad \text{Ans. : (D)}$$

$$(37) \int \frac{dx}{\cos x - \sin x} = \dots$$

$$(A) \frac{1}{2\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{8} \right) \right| + c$$

$$(B) \frac{1}{\sqrt{2}} \log \left| \cot \left(\frac{x}{2} \right) \right| + c$$

$$(C) \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{3\pi}{8} \right) \right| + c$$

$$(D) \frac{1}{2\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{3\pi}{8} \right) \right| + c$$

$$\text{Sol. : } \int \frac{dx}{\cos x - \sin x}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\cos \left(x + \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sqrt{2}} \int \sec \left(x + \frac{\pi}{4} \right) dx = \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{8} + \frac{\pi}{4} \right) \right| + c = \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{3\pi}{8} \right) \right| + c$$

Ans. : (C)

$$(38) \text{ If } \int \frac{1}{1 + \sin x} dx = \tan(p(x)) + c, \text{ then } p(x) = \dots$$

$$(A) x + \frac{\pi}{2}$$

$$(B) \frac{x}{2} + \frac{\pi}{4}$$

$$(C) \frac{x}{2} - \frac{\pi}{4}$$

$$(D) \frac{x}{2} - \frac{\pi}{8}$$

$$\text{Sol. : } \int \frac{1}{1 + \sin x} dx$$

$$= \int \frac{1}{1 + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int \frac{dx}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}$$

$$= \frac{1}{2} \int \sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) dx$$

$$= \frac{1}{2} \frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)}{\left(\frac{-1}{2}\right)} + c = -\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + c = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + c$$

Ans. : (C)

$$(39) \int \sqrt{1 + \sin \frac{x}{2}} dx = \dots\dots \quad 0 < x < 2\pi$$

$$(A) \quad 2\left(\cos \frac{x}{4} - \sin \frac{x}{4}\right) + c$$

$$(B) \quad 4\left(\cos \frac{x}{4} - \sin \frac{x}{4}\right) + c$$

$$(C) \quad 4\left(\sin \frac{x}{4} - \cos \frac{x}{4}\right) + c$$

$$(D) \quad 2\left(\sin \frac{x}{4} - \cos \frac{x}{4}\right) + c$$

$$\text{Sol. : } \int \sqrt{1 + \sin \frac{x}{2}} dx$$

$$= \int \sqrt{\sin^2 \frac{x}{4} + \cos^2 \frac{x}{4} + 2\sin \frac{x}{4} \cos \frac{x}{4}} dx$$

$$= \int \left(\sin \frac{x}{4} + \cos \frac{x}{4}\right) dx \quad \left(0 < \frac{x}{4} < \frac{\pi}{2}\right)$$

$$= 4 \left(\sin \frac{x}{4} - \cos \frac{x}{4}\right) + c$$

Ans. : (C)

$$(40) \int \frac{\sin x dx}{\sqrt{1 + \sin x}} = \dots\dots \quad 0 < x < \pi$$

$$(A) \quad 2\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right) + \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{4}\right) \right| + c$$

$$(B) \quad 2\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) + \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2}\right) \right| + c$$

$$(C) \quad 2\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) - \sqrt{2} \log \left| \tan \left(\frac{\pi}{8} + \frac{x}{4}\right) \right| + c$$

$$(D) \quad 2\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right) - \frac{1}{2} \log \left| \tan \left(\frac{x}{4} - \frac{\pi}{8}\right) \right| + c$$

$$\text{Sol. : } \int \frac{\sin x dx}{\sqrt{1 + \sin x}} \quad (0 < x < \pi)$$

$$\begin{aligned}
&= \int \frac{(1 + \sin x) - (1)}{\sqrt{1 + \sin x}} dx \\
&= \int \sqrt{1 + \sin x} dx - \int \frac{1}{\sqrt{1 + \sin x}} dx \\
&= \int \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) dx - \int \frac{dx}{\sin \frac{x}{2} + \cos \frac{x}{2}} \quad \left(1 + \sin x = \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2} \right) \\
&= \int \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) dx - \frac{1}{\sqrt{2}} \int \frac{1}{\frac{1}{\sqrt{2}} \sin \frac{x}{2} + \frac{1}{\sqrt{2}} \cos \frac{x}{2}} dx \\
&= \int \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) dx - \frac{1}{\sqrt{2}} \int \frac{dx}{\sin \left(\frac{x}{2} + \frac{\pi}{4} \right)} \\
&= \int \sin \frac{x}{2} dx + \int \cos \frac{x}{2} dx - \frac{1}{\sqrt{2}} \int \operatorname{cosec} \left(\frac{x}{2} + \frac{\pi}{4} \right) dx \\
&= -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} - \frac{1}{\sqrt{2}} \times \left(\frac{1}{2} \right) \log \left| \tan \left(\frac{x}{4} + \frac{\pi}{8} \right) \right| + c \\
&= 2 \left(\sin \frac{x}{2} - \cos \frac{x}{2} \right) - \sqrt{2} \log \left| \tan \left(\frac{\pi}{8} + \frac{x}{4} \right) \right| + c \quad \text{Ans. : (C)}
\end{aligned}$$

(41) If $\int \frac{(3x+2)^8}{(4x+3)^{10}} dx = A \left(\frac{3x+2}{4x+3} \right)^B + c$, then $A \cdot B = \dots$

- (A) 10 (B) 1 (C) 8 (D) 9

Sol. : $I = \int \frac{(3x+2)^8}{(4x+3)^{10}} dx$

$$= \int \left(\frac{3x+2}{4x+3} \right)^8 \times \frac{1}{(4x+3)^2} dx$$

Let $\frac{3x+2}{4x+3} = t$.

So, $\frac{(4x+3)3 - (3x+2)4}{(4x+3)^2} dx = dt$

$$\therefore \frac{1}{(4x+3)^2} dx = dt$$

$$\therefore I = \int t^8 dt = \frac{t^9}{9} + c$$

$$= \frac{1}{9} \left(\frac{3x+2}{4x+3} \right)^9 + c$$

$$\therefore A = \frac{1}{9}, B = 9$$

$$\therefore AB = 1$$

Ans. : (B)

$$(42) \quad \int \frac{dx}{\sqrt{1 + \sec^2 x}} = \dots\dots \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

$$(A) \quad \sin^{-1} \left(\frac{\sin x}{\sqrt{2}} \right) + c$$

$$(B) \quad \sin^{-1} \left(\frac{\cos x}{\sqrt{2}} \right) + c$$

$$(C) \quad \cos^{-1} \left(\frac{\sin x}{\sqrt{2}} \right) + c$$

$$(D) \quad \cos^{-1} \left(\frac{\cos x}{\sqrt{2}} \right) + c$$

$$\text{Sol. : } \int \frac{dx}{\sqrt{1 + \sec^2 x}} \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

$$I = \int \frac{\cos x}{\sqrt{\cos^2 x + 1}} dx \quad (|\cos x| = \cos x)$$

$$= \int \frac{\cos x}{\sqrt{2 - \sin^2 x}} dx$$

$$\text{Let } \sin x = t.$$

$$\text{So, } \cos x \, dx = dt$$

$$\therefore I = \int \frac{dt}{\sqrt{2 - t^2}}$$

$$= \sin^{-1} \left(\frac{t}{\sqrt{2}} \right) + c$$

$$= \sin^{-1} \left(\frac{\sin x}{\sqrt{2}} \right) + c$$

Ans. : (A)

$$(43) \quad \int \sqrt{1 + \operatorname{cosec} x} \, dx = \dots\dots \quad \left(0 < x < \frac{\pi}{2}\right)$$

$$(A) \quad 2\sin^{-1} \sqrt{\sin x} + c$$

$$(B) \quad 2\sin^{-1} \sqrt{\cos x} + c$$

$$(C) \quad 2\cos^{-1} \sqrt{\cos x} + c$$

$$(D) \quad 2\sin^{-1} \sqrt{\sin x} + c$$

$$\text{Sol. : } I = \int \sqrt{1 + \operatorname{cosec} x} \, dx$$

$$= \int \frac{\sqrt{1 + \sin x}}{\sqrt{\sin x}} dx$$

$$\text{Let } \sin x = t^2. \quad (t > 0)$$

$$\text{So, } \cos x \, dx = 2t \, dt$$

$$I = \int \frac{\sqrt{1 + t^2}}{t} \cdot \frac{2t \, dt}{\sqrt{1 - t^4}} = 2 \int \frac{1}{\sqrt{1 - t^2}} dt$$

$$= 2 \sin^{-1} (t) + c$$

$$= 2\sin^{-1} \sqrt{\sin x} + c$$

Ans. : (D)

$$(44) \quad \int \sqrt{\frac{a-x}{a+x}} dx = \dots\dots \quad (a > 0)$$

$$(A) \quad \sin^{-1}x + \sqrt{a^2 - x^2} + c$$

$$(B) \quad a \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2} + c$$

$$(C) \quad \frac{a}{2} \sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2} + c$$

$$(D) \quad \frac{1}{a} \sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2} + c$$

$$\text{Sol. : I} = \int \sqrt{\frac{a-x}{a+x}} dx$$

$$= \int \frac{a-x}{\sqrt{a^2 - x^2}} dx \quad (\text{Multiply numerator and denominator by } \sqrt{a-x})$$

$$= a \int \frac{1}{\sqrt{a^2 - x^2}} dx - \int \frac{x dx}{\sqrt{a^2 - x^2}}$$

$$= a \int \frac{1}{\sqrt{a^2 - x^2}} dx + \frac{1}{2} \int \frac{-2x dx}{\sqrt{a^2 - x^2}}$$

$$= a \int \frac{1}{\sqrt{a^2 - x^2}} dx + \frac{1}{2} \int (a^2 - x^2)^{-\frac{1}{2}} (-2x) dx$$

$$= a \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2} \times \frac{(a^2 - x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= a \sin^{-1}\left(\frac{x}{a}\right) + (a^2 - x^2)^{\frac{1}{2}} + c$$

$$= a \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2} + c$$

Ans. : (B)

$$\text{Second method :} = \int \sqrt{\frac{a-x}{a+x}} dx = \int \tan \frac{\theta}{2} (-a \sin \theta) d\theta \quad (\text{Let } x = a \cos \theta, 0 < \theta < \pi)$$

$$= a \int \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} (-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}) d\theta$$

$$= -a \int (1 - \cos \theta) d\theta$$

$$= -a (\theta - \sin \theta) + c' = -a \cos^{-1} \frac{x}{a} + \sqrt{a^2 - x^2} + c'$$

$$= a \sin^{-1} \frac{x}{a} + \sqrt{a^2 - x^2} + c$$

Ans. : (B)

(45) If $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx = Px \cos^{-1}x + Q\sqrt{1-x^2} + c$, then $P + Q = \dots$

- (A) $\frac{1}{2}$ (B) 0 (C) $-\frac{1}{2}$ (D) -1

Sol. : $I = \int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$

Let $x = \cos 2\theta$.

So, $dx = -2\sin 2\theta d\theta$ ($0 < \theta < \frac{\pi}{2}$)

$$\begin{aligned} I &= \int \tan^{-1} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} (-2\sin 2\theta) d\theta \\ &= -2 \int \tan^{-1} \sqrt{\tan^2 \theta} \cdot \sin 2\theta d\theta \\ &= -2 \int \theta \cdot \sin 2\theta d\theta \\ &= -2 \left[\theta \cdot \left(\frac{-\cos 2\theta}{2} \right) + \int \left(1 \cdot \frac{\cos 2\theta}{2} \right) d\theta \right] \quad (\text{Integration by parts}) \\ &= -2 \left[-\frac{\theta}{2} \cos 2\theta + \frac{1}{2} \frac{\sin 2\theta}{2} \right] + c \\ &= \theta \cdot \cos 2\theta - \frac{1}{2} \sin 2\theta + c \\ &= \frac{1}{2} \cos^{-1}x \cdot x - \frac{1}{2} \sqrt{1-x^2} + c \quad (0 < 2\theta < \pi) \\ &= \frac{x}{2} \cos^{-1}x - \frac{1}{2} \sqrt{1-x^2} + c \end{aligned}$$

$\therefore P = \frac{1}{2}, Q = -\frac{1}{2}$

So, $P + Q = 0$

Ans. : (B)

(46) If $\int \tan^4 x dx = \alpha x + \beta \tan x + \gamma \tan^3 x + c$, then $\alpha + \beta + \gamma = \dots$

- (A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) -1

Sol. : $I = \int \tan^4 x dx$

$$\begin{aligned} &= \int \tan^2 x (\sec^2 x - 1) dx \\ &= \int (\tan^2 x \sec^2 x - \tan^2 x) dx \\ &= \int (\tan x)^2 \sec^2 x dx - \int (\sec^2 x - 1) dx \\ &= \frac{\tan^3 x}{3} - \tan x + x + c \end{aligned}$$

$\therefore \alpha = 1, \beta = -1, \gamma = \frac{1}{3}$

$\therefore \alpha + \beta + \gamma = \frac{1}{3}$

Ans. : (C)

$$(47) \int \frac{1}{\cos x + \sin x + 1} dx = \dots\dots$$

$$(A) \log \left| 1 + \tan \frac{x}{2} \right| + c$$

$$(B) \frac{1}{2} \log \left| 1 + \tan \frac{x}{2} \right| + c$$

$$(C) 2 \log \left| 1 + \tan \frac{x}{2} \right| + c$$

$$(D) \frac{1}{2} \log \left| 1 - \tan \frac{x}{2} \right| + c$$

$$\text{Sol. : } I = \int \frac{1}{\cos x + \sin x + 1} dx$$

$$\text{Let } \tan \frac{x}{2} = t.$$

$$\text{So, } \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2} \text{ and } dx = \frac{2dt}{1+t^2}$$

$$I = \int \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} + 1} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{1-t^2+2t+1+t^2}$$

$$= 2 \int \frac{dt}{2t+2} = \int \frac{dt}{t+1} = \log |t+1| + c = \log \left| 1 + \tan \frac{x}{2} \right| + c$$

Ans. : (A)

$$(48) \int \frac{dx}{\sin x - \cos x + \sqrt{2}} = \dots\dots$$

$$(A) \frac{1}{\sqrt{2}} \tan \left(\frac{\pi}{8} + \frac{x}{2} \right) + c$$

$$(B) -\frac{1}{\sqrt{2}} \tan \left(\frac{\pi}{8} + \frac{x}{2} \right) + c$$

$$(C) \frac{1}{\sqrt{2}} \cot \left(\frac{\pi}{8} + \frac{x}{2} \right) + c$$

$$(D) -\frac{1}{\sqrt{2}} \cot \left(\frac{\pi}{8} + \frac{x}{2} \right) + c$$

$$\text{Sol. : } I = \int \frac{dx}{\sin x - \cos x + \sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{\sqrt{2}} \sin x - \frac{1}{\sqrt{2}} \cos x \right) + 1}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{1 - \left(\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{1 - \cos \left(x + \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{2 \sin^2 \left(\frac{x}{2} + \frac{\pi}{8} \right)}$$

$$= \frac{1}{2\sqrt{2}} \int \operatorname{cosec}^2 \left(\frac{\pi}{8} + \frac{x}{2} \right) dx$$

$$= \frac{1}{2\sqrt{2}} \left(\frac{1}{2} \right) \left[-\cot \left(\frac{\pi}{8} + \frac{x}{2} \right) \right] + c$$

$$= -\frac{1}{\sqrt{2}} \cot \left(\frac{\pi}{8} + \frac{x}{2} \right) + c$$

Ans. : (D)

$$(49) \int \frac{\operatorname{cosec} x}{\cos^2 \left(1 + \log \tan \frac{x}{2}\right)} dx = \dots$$

$$(A) \sec^2 \left[1 + \log \left(\tan \frac{x}{2}\right)\right] + c$$

$$(B) -\tan \left(1 + \log \tan \frac{x}{2}\right) + c$$

$$(C) \sin^2 \left[1 + \log \left(\tan \frac{x}{2}\right)\right] + c$$

$$(D) \tan \left(1 + \log \tan \frac{x}{2}\right) + c$$

$$\text{Sol. : } I = \int \frac{\operatorname{cosec} x}{\cos^2 \left(1 + \log \tan \frac{x}{2}\right)} dx$$

$$\text{Let, } 1 + \log \tan \frac{x}{2} = u. \quad \text{So, } \frac{1}{\tan \frac{x}{2}} \cdot \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = du$$

$$\therefore \operatorname{cosec} x dx = du$$

$$I = \int \frac{1}{\cos^2 u} du$$

$$= \int \sec^2 u du = \tan u + c = \tan \left(1 + \log \tan \frac{x}{2}\right) + c$$

Ans. : (D)

$$(50) \int \sec^{\frac{2}{3}} x \cdot \operatorname{cosec}^{\frac{4}{3}} x dx = \dots$$

$$(A) -3(\tan x)^{\frac{-1}{3}} + c \quad (B) 3(\tan x)^{\frac{-1}{3}} + c \quad (C) -3(\tan x)^{\frac{1}{3}} + c \quad (D) (\tan x)^{\frac{-1}{3}} + c$$

$$\text{Sol. : } I = \int \sec^{\frac{2}{3}} x \cdot \operatorname{cosec}^{\frac{4}{3}} x dx$$

$$= \int \frac{1}{\cos^{\frac{2}{3}} x \cdot \sin^{\frac{4}{3}} x} dx$$

$$= \int \frac{\left(\frac{1}{\cos^2 x}\right)}{\frac{\cos^{\frac{2}{3}} x}{\cos^2 x} \cdot \sin^{\frac{4}{3}} x} dx$$

$$= \int \frac{\sec^2 x}{\tan^{\frac{4}{3}} x} dx$$

$$= \int t^{-\frac{4}{3}} \cdot dt.$$

$$(\text{Let, } \tan x = t. \text{ So, } \sec^2 x dx = dt)$$

$$= \frac{t^{-\frac{1}{3}}}{-\frac{1}{3}} + c$$

$$= -3(\tan x)^{-\frac{1}{3}} + c$$

Ans. : (A)

$$(51) \int \cos^{-\frac{3}{7}} x \cdot \sin^{-\frac{11}{7}} x dx = \dots$$

$$(A) \log |\sin^{\frac{4}{7}} x| + c \quad (B) \frac{-7}{4} \tan^{\frac{4}{7}} x + c \quad (C) \frac{-7}{4} \cot^{\frac{4}{7}} x + c \quad (D) \log |\cos^{\frac{3}{7}} x| + c$$

$$\begin{aligned}
\text{Sol. : } I &= \int \cos^{\frac{-3}{7}} x \cdot \sin^{\frac{-11}{7}} x \, dx \\
&= \int \frac{1}{\sin^{\frac{11}{7}} x \cdot \cos^{\frac{3}{7}} x} \, dx \\
&= \int \frac{dx}{\tan^{\frac{11}{7}} x \cdot \cos^{\frac{3}{7}} x \cdot \cos^{\frac{11}{7}} x} \quad \left(\text{Dividing numerator and denominator by } \cos^{\frac{11}{7}} x \right) \\
&= \int \frac{\sec^2 x}{(\tan x)^{\frac{11}{7}}} \, dx \\
&= \frac{(\tan x)^{\frac{-11}{7}+1}}{\frac{-11}{7}+1} + c = \frac{-7}{4} (\tan x)^{\frac{-4}{7}} + c = \frac{-7}{4} (\cot x)^{\frac{4}{7}} + c \quad \text{Ans. : (C)}
\end{aligned}$$

(52) If $\int \frac{(1+x)e^x}{\cos^2(xe^x)} \, dx = f(x) + c$, then $f(x) = \dots$

- (A) $\cos(x \cdot e^x)$ (B) $2\tan^{-1}x$ (C) $\tan(x \cdot e^x)$ (D) $\sin(x \cdot e^x)$

$$\text{Sol. : } I = \int \frac{(1+x)e^x}{\cos^2(xe^x)} \, dx$$

Let, $x \cdot e^x = t$

$\therefore (x \cdot e^x + e^x) \, dx = dt$

So, $(x+1)e^x \, dx = dt$

$$I = \int \frac{dt}{\cos^2 t} = \int \sec^2 t \, dt = \tan t + c = \tan(x \cdot e^x) + c$$

$\therefore f(x) = \tan(x \cdot e^x)$

Ans. : (C)

(53) $\int x \cdot (x^x)^x \cdot (2 \log x + 1) \, dx = \dots$

- (A) $(x^x)^x + c$ (B) $\log(x)^x + c$ (C) $x^x + c$ (D) $x(x)^x + c$

$$\text{Sol. : } I = \int x \cdot (x^x)^x \cdot (2 \log x + 1) \, dx$$

Substituting $(x^x)^x = t$ we get $x^{x^2} = t$

$\therefore x^2 \cdot \log x = \log t$

$\therefore (x^2 \cdot \frac{1}{x} + \log x \cdot 2x) \, dx = \frac{1}{t} \, dt$

$x(1 + 2 \log x) \cdot (x^x)^x \, dx = dt$

$\therefore I = \int dt = t + c = (x^x)^x + c$

Ans. : (A)

(54) $\int \frac{\sin x}{\sin(x - \frac{\pi}{4})} \, dx = \dots$

(A) $\frac{x}{\sqrt{2}} - \frac{1}{\sqrt{2}} \log \left| \sin \left(x - \frac{\pi}{4} \right) \right| + c$ (B) $\frac{x}{\sqrt{2}} + \frac{1}{\sqrt{2}} \log \left| \sin \left(x - \frac{\pi}{4} \right) \right| + c$

(C) $\frac{x}{\sqrt{2}} - \frac{1}{\sqrt{2}} \log \left| \cos \left(x - \frac{\pi}{4} \right) \right| + c$ (D) $\frac{x}{2} + \frac{1}{\sqrt{2}} \log \left| \cos \left(x - \frac{\pi}{4} \right) \right| + c$

$$\text{Sol. : } \int \frac{\sin x}{\sin \left(x - \frac{\pi}{4}\right)} dx$$

$$= \int \frac{\sin \left(\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}\right)}{\sin \left(x - \frac{\pi}{4}\right)} dx$$

$$= \int \frac{\sin \left(x - \frac{\pi}{4}\right) \cos \frac{\pi}{4} + \cos \left(x - \frac{\pi}{4}\right) \sin \frac{\pi}{4}}{\sin \left(x - \frac{\pi}{4}\right)} dx$$

$$= \int \frac{1}{\sqrt{2}} dx + \frac{1}{\sqrt{2}} \int \cot \left(x - \frac{\pi}{4}\right) dx$$

$$= \frac{x}{\sqrt{2}} + \frac{1}{\sqrt{2}} \log \left| \sin \left(x - \frac{\pi}{4}\right) \right| + c$$

Ans. : (B)

$$(55) \int \frac{\sin 2x}{\sin 5x \sin 3x} dx = \dots\dots$$

$$(A) \quad 3 \log |\sin 3x| - 5 \log |\sin 5x| + c$$

$$(B) \quad \frac{1}{3} \log |\sin 3x| - \frac{1}{5} \log |\sin 5x| + c$$

$$(C) \quad \frac{1}{3} \log |\sin 3x| + \frac{1}{5} \log |\sin 5x| + c$$

$$(D) \quad \log |\sin 3x| - \log |\sin 5x| + c$$

$$\text{Sol. : } \int \frac{\sin 2x}{\sin 5x \sin 3x} dx$$

$$= \int \frac{\sin (5x - 3x)}{\sin 5x \sin 3x} dx$$

$$= \int \frac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x} dx$$

$$= \int (\cot 3x - \cot 5x) dx$$

$$= \frac{1}{3} \log |\sin 3x| - \frac{1}{5} \log |\sin 5x| + c$$

Ans. : (B)

$$(56) \int (x+1)(x+2)^7(x+3) dx = \dots\dots$$

$$(A) \quad \frac{(x+2)^{10}}{10} - \frac{(x+2)^8}{8} + c$$

$$(B) \quad \frac{(x+1)^2}{2} - \frac{(x+2)^8}{8} - \frac{(x+3)^2}{2} + c$$

$$(C) \quad \frac{(x+2)^{10}}{10} + c$$

$$(D) \quad \frac{(x+1)^2}{2} + \frac{(x+2)^8}{8} + \frac{(x+3)^2}{2} + c$$

$$\text{Sol. : } \int (x+1)(x+2)^7(x+3) dx$$

Substituting, $x+2 = t$. we get $dx = dt$

$$I = \int (t-1)(t^7)(t+1) dt$$

$$\begin{aligned}
&= \int (t^2 - 1) t^7 dt \\
&= \int (t^9 - t^7) dt \\
&= \frac{t^{10}}{10} - \frac{t^8}{8} + c \\
&= \frac{(x+2)^{10}}{10} - \frac{(x+2)^8}{8} + c
\end{aligned}$$

Ans. : (A)

$$(57) \int \frac{dx}{\cos(x-\alpha) \cos(x-\beta)} = \dots$$

- (A) $\frac{1}{\cos(\alpha-\beta)} \log \left| \frac{\cos(x-\alpha)}{\cos(x-\beta)} \right| + c$ (B) $\frac{1}{\cos(\alpha-\beta)} \log \left| \frac{\sin(x-\alpha)}{\sin(x-\beta)} \right| + c$
- (C) $\frac{1}{\sin(\alpha-\beta)} \log \left| \frac{\cos(x-\alpha)}{\cos(x-\beta)} \right| + c$ (D) $\frac{1}{\sin(\alpha-\beta)} \log \left| \frac{\sin(x-\alpha)}{\sin(x-\beta)} \right| + c$

Sol. : $I = \int \frac{dx}{\cos(x-\alpha) \cos(x-\beta)}$

$$\begin{aligned}
&= \frac{1}{\sin(\alpha-\beta)} \int \frac{\sin[(x-\beta)-(x-\alpha)]}{\cos(x-\alpha) \cos(x-\beta)} dx \\
&= \frac{1}{\sin(\alpha-\beta)} \int \frac{\sin(x-\beta) \cos(x-\alpha) - \cos(x-\beta) \sin(x-\alpha)}{\cos(x-\alpha) \cos(x-\beta)} dx \\
&= \frac{1}{\sin(\alpha-\beta)} \int [\tan(x-\beta) - \tan(x-\alpha)] dx \\
&= \frac{1}{\sin(\alpha-\beta)} [\log |\sec(x-\beta)| - \log |\sec(x-\alpha)|] + c \\
&= \frac{1}{\sin(\alpha-\beta)} \log \left| \frac{\sec(x-\beta)}{\sec(x-\alpha)} \right| + c \\
&= \frac{1}{\sin(\alpha-\beta)} \log \left| \frac{\cos(x-\alpha)}{\cos(x-\beta)} \right| + c
\end{aligned}$$

Ans. : (C)

$$(58) \int \operatorname{cosec}\left(x - \frac{\pi}{6}\right) \operatorname{cosec}\left(x - \frac{\pi}{3}\right) dx = \dots$$

- (A) $2 \log \left| \frac{\sin\left(x - \frac{\pi}{3}\right)}{\cos\left(x - \frac{\pi}{6}\right)} \right| + c$ (B) $2 \log \left| \frac{\cos\left(x - \frac{\pi}{3}\right)}{\cos\left(x - \frac{\pi}{6}\right)} \right| + c$
- (C) $2 \log \left| \frac{\sin\left(x - \frac{\pi}{3}\right)}{\sin\left(x - \frac{\pi}{6}\right)} \right| + c$ (D) $2 \log \left| \frac{\cos\left(x - \frac{\pi}{6}\right)}{\cos\left(x - \frac{\pi}{3}\right)} \right| + c$

$$\text{Sol. : } I = \int \operatorname{cosec} \left(x - \frac{\pi}{6} \right) \operatorname{cosec} \left(x - \frac{\pi}{3} \right) dx$$

$$= \int \frac{1}{\sin \left(x - \frac{\pi}{6} \right) \sin \left(x - \frac{\pi}{3} \right)} dx$$

$$= \frac{1}{\sin \frac{\pi}{6}} \int \frac{\sin \left[\left(x - \frac{\pi}{6} \right) - \left(x - \frac{\pi}{3} \right) \right]}{\sin \left(x - \frac{\pi}{6} \right) \sin \left(x - \frac{\pi}{3} \right)} dx$$

$$= \frac{1}{\sin \frac{\pi}{6}} \int \frac{\sin \left(x - \frac{\pi}{6} \right) \cos \left(x - \frac{\pi}{3} \right) - \cos \left(x - \frac{\pi}{6} \right) \sin \left(x - \frac{\pi}{3} \right)}{\sin \left(x - \frac{\pi}{6} \right) \sin \left(x - \frac{\pi}{3} \right)} dx$$

$$= \frac{1}{\left(\frac{1}{2} \right)} \int \left[\cot \left(x - \frac{\pi}{3} \right) - \cot \left(x - \frac{\pi}{6} \right) \right] dx$$

$$= 2 \left[\log \left| \sin \left(x - \frac{\pi}{3} \right) \right| - \log \left| \sin \left(x - \frac{\pi}{6} \right) \right| \right] + c$$

$$= 2 \log \left| \frac{\sin \left(x - \frac{\pi}{3} \right)}{\sin \left(x - \frac{\pi}{6} \right)} \right| + c$$

Ans. : (C)

$$(59) \int \left(\sqrt{\tan x} + \sqrt{\cot x} \right) dx = \dots\dots$$

$$(A) \quad \sqrt{2} \sin^{-1} (\sin x - \cos x) + c$$

$$(B) \quad \sqrt{2} \sin^{-1} (\sin x + \cos x) + c$$

$$(C) \quad \sin^{-1} (\sin x - \cos x) + c$$

$$(D) \quad \sin^{-1} (\sin x + \cos x) + c \quad (\text{IIT : 1988})$$

$$\text{Sol. : } I = \int \left(\sqrt{\tan x} + \sqrt{\cot x} \right) dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$$

$$= \sqrt{2} \int \frac{(\sin x + \cos x)}{\sqrt{2 \sin x \cos x}} dx$$

$$= \sqrt{2} \int \frac{(\sin x + \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}} dx$$

$$\text{Let, } \sin x - \cos x = t.$$

$$\text{So, } (\cos x + \sin x) dx = dt$$

$$I = \sqrt{2} \int \frac{dt}{\sqrt{1 - t^2}}$$

$$= \sqrt{2} \sin^{-1} (t) + c$$

$$= \sqrt{2} \sin^{-1} (\sin x - \cos x) + c$$

Ans. : (A)

$$(60) \int \tan x \tan 2x \tan 3x \, dx = \dots$$

$$(A) \log \left| \frac{\cos x \sqrt{\cos 2x}}{\sqrt[3]{\cos 3x}} \right| + c$$

$$(B) \log \left| \frac{3\sqrt{\cos 3x}}{\cos x \sqrt{\cos 2x}} \right| + c$$

$$(C) \log \left| \frac{\sin x \sqrt{\sin 2x}}{\sqrt[3]{\sin 3x}} \right| + c$$

$$(D) \log \left| \frac{3\sqrt{\sin 3x}}{\sin x \sqrt{\sin 2x}} \right| + c$$

$$\text{Sol. : } \tan 3x = \tan (2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$\therefore \tan 3x (1 - \tan 2x \tan x) = \tan 2x + \tan x$$

$$\therefore \tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$$

$$\therefore I = \int (\tan 3x \tan 2x \tan x) \, dx$$

$$= \int (\tan 3x - \tan 2x - \tan x) \, dx$$

$$= \frac{-1}{3} \log |\cos 3x| + \frac{1}{2} \log |\cos 2x| + \log |\cos x| + c$$

$$= \log \left| \frac{\cos x \sqrt{\cos 2x}}{\sqrt[3]{\cos 3x}} \right| + c$$

Ans. : (A)

$$(61) \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} \cdot \frac{dx}{x} = \dots$$

$$(A) \log \left| \frac{1-\sqrt{1-x}}{\sqrt{x}} \right| - \sin^{-1} \sqrt{x} + c$$

$$(B) 2 \log \left| \frac{1-\sqrt{1-x}}{\sqrt{x}} \right| - 2 \sin^{-1} \sqrt{x} + c$$

$$(C) 2 \log \left| \frac{1-\sqrt{1-x}}{\sqrt{x}} \right| - \cos^{-1} \sqrt{x} + c$$

$$(D) 2 \log \left| \frac{1-\sqrt{1-x}}{\sqrt{x}} \right| - 2 \cos^{-1} \sqrt{x} + c$$

(IIT : 1997)

$$\text{Sol. : } I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} \cdot \frac{dx}{x}$$

$$\text{Substituting, } x = \sin^2 \theta, \text{ we get, } dx = 2 \sin \theta \cos \theta \, d\theta \quad (0 < \theta < \frac{\pi}{2})$$

$$I = \int \sqrt{\frac{1-\sin \theta}{1+\sin \theta}} \times \frac{2 \sin \theta \cos \theta \, d\theta}{\sin^2 \theta}$$

$$= 2 \int \sqrt{\frac{1-\sin \theta}{1+\sin \theta}} \times \frac{1-\sin \theta}{1-\sin \theta} \times \frac{\cos \theta}{\sin \theta} \, d\theta$$

$$= 2 \int \frac{(1-\sin \theta)}{\sin \theta} \, d\theta \quad (0 < \theta < \frac{\pi}{2})$$

$$\begin{aligned}
&= 2 \int (\operatorname{cosec} \theta - 1) d\theta \\
&= 2 (\log |\operatorname{cosec} \theta - \cot \theta| - \theta) + c \\
&= 2 \log \left| \frac{1}{\sqrt{x}} - \frac{\sqrt{1-x}}{\sqrt{x}} \right| - 2 \sin^{-1} \sqrt{x} + c \\
&= 2 \log \left| \frac{1 - \sqrt{1-x}}{\sqrt{x}} \right| - 2 \sin^{-1} \sqrt{x} + c
\end{aligned}$$

Ans. : (B)

(62) If $\int \frac{x^2 dx}{\sqrt{1-x^2}} = P \cos^{-1} x + Qx \sqrt{1-x^2} + c$, then $P + Q = \dots$

- (A) 0 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) -1

Sol. : $I = \int \frac{x^2 dx}{\sqrt{1-x^2}}$

Substituting, $x = \sin \theta$, we get $dx = \cos \theta d\theta$ ($0 < \theta < \frac{\pi}{2}$)

$$\begin{aligned}
I &= \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \\
&= \int \sin^2 \theta d\theta \\
&= \int \left(\frac{1 - \cos 2\theta}{2} \right) d\theta \\
&= \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right] + c' \\
&= \frac{1}{2} \theta - \frac{1}{2} \sin \theta \cos \theta + c' \\
&= \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + c' \\
&= -\frac{1}{2} \cos^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + c
\end{aligned}$$

$$(\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x, c' + \frac{\pi}{4} = c)$$

$$P = -\frac{1}{2}, Q = -\frac{1}{2}$$

$$\therefore P + Q = \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) = -1$$

Ans. : (D)

(63) $\int \frac{e^x (2-x^2)}{(1-x)\sqrt{1-x^2}} dx = \dots$

- (A) $e^x \sqrt{\frac{1+x}{1-x}} + c$ (B) $\frac{e^x}{\sqrt{1-x}} + c$ (C) $\sqrt{\frac{1+x}{1-x^2}} + c$ (D) $\frac{x e^x}{\sqrt{1-x^2}} + c$

$$\begin{aligned}
\text{Sol. : I} &= \int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}} dx \\
&= \int \frac{e^x[(1-x^2)+1]}{(1-x)\sqrt{1-x^2}} dx \\
&= \int \left[\frac{e^x(1-x^2)}{(1-x)\sqrt{1-x^2}} + \frac{e^x}{(1-x)\sqrt{1-x^2}} \right] dx \\
&= \int e^x \left\{ \frac{\sqrt{1-x^2}}{1-x} + \frac{1}{(1-x)\sqrt{1-x^2}} \right\} dx \\
&= \int e^x \left\{ \frac{\sqrt{1+x}}{\sqrt{1-x}} + \frac{d}{dx} \left(\frac{\sqrt{1+x}}{\sqrt{1-x}} \right) \right\} dx \\
&= e^x \sqrt{\frac{1+x}{1-x}} + c
\end{aligned}$$

Ans. : (A)

$$(64) \int \frac{(\log x - 1)^2}{(1 + (\log x)^2)^2} dx = \dots$$

$$(A) \frac{xe^x}{1+x^2} + c \quad (B) \frac{\log x}{1 + (\log x)^2} + c \quad (C) \frac{x}{x^2+1} + c \quad (D) \frac{x}{1 + (\log x)^2} + c$$

$$\text{Sol. : I} = \int \frac{(\log x - 1)^2}{(1 + (\log x)^2)^2} dx$$

Substituting $\log x = t$, we get $x = e^t$ So, $dx = e^t dt$

$$\begin{aligned}
&= \int \left(\frac{t-1}{1+t^2} \right)^2 e^t dt \\
&= \int \frac{t^2 - 2t + 1}{(1+t^2)^2} e^t dt \\
&= \int \left[\frac{t^2+1}{(t^2+1)^2} - \frac{2t}{(t^2+1)^2} \right] e^t dt \\
&= \int \left[\frac{1}{t^2+1} - \frac{2t}{(1+t^2)^2} \right] e^t dt \\
&= \int \left\{ \frac{1}{t^2+1} + \frac{d}{dt} \frac{1}{(t^2+1)} \right\} e^t dt \\
&= \frac{e^t}{t^2+1} + c = \frac{x}{1 + (\log x)^2} + c
\end{aligned}$$

Ans. : (D)

(65) If $\int \frac{xe^x}{\sqrt{1+e^x}} dx = (Px + Q)\sqrt{1+e^x} + R \log \left(\frac{\sqrt{1+e^x} + 1}{\sqrt{1+e^x} - 1} \right) + c$, then $P + Q + R = \dots$

(A) 2

(B) -2

(C) 0

(D) 4

Sol. : $I = \int \frac{xe^x}{\sqrt{1+e^x}} dx$

Substituting $1 + e^x = t^2$

$t > 0$

we get, $e^x dx = 2t dt$

$e^x = t^2 - 1$

$\therefore x = \log_e (t^2 - 1)$

$I = \int \frac{\left[\log(t^2 - 1) \right] 2t}{t} dt$

$= 2 \int \log(t^2 - 1) dt$

$= 2 \left[\log(t^2 - 1) \int 1 dt - \int \left(\frac{d}{dt} \log(t^2 - 1) \right) \int 1 dt \right]$

$= 2 \left[t \log(t^2 - 1) - 2 \int \frac{t^2}{t^2 - 1} dt \right]$

$= 2t \log(t^2 - 1) - 4 \int \left(1 + \frac{1}{t^2 - 1} \right) dt$

$= 2t \log(t^2 - 1) - 4t - 4 \times \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + c$

$= 2\sqrt{1+e^x} \cdot x - 4\sqrt{e^x+1} - 2 \log \left| \frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1} \right| + c$

$= \sqrt{1+e^x} (2x - 4) + 2 \log \left| \frac{\sqrt{e^x+1}+1}{\sqrt{e^x+1}-1} \right| + c$

$\therefore P = 2, Q = -4, R = 2$

$\therefore P + Q + R = 0$

Ans. : (C)

(66) $\int \tan^{-1} \sqrt{x} dx = \dots$

(A) $x \tan^{-1} \sqrt{x} - \sqrt{x} + c$

(B) $(x + 1) \tan^{-1} \sqrt{x} - \sqrt{x} + c$

(C) $\tan^{-1} \sqrt{x} - \sqrt{x} + c$

(D) $x \tan^{-1} \sqrt{x} + \sqrt{x} + c$

Sol. : $I = \int \tan^{-1} \sqrt{x} \, dx$

Let $\theta = \tan^{-1} \sqrt{x}$ i.e. $x = \tan^2 \theta$ ($0 < \theta < \frac{\pi}{2}$)

$\therefore dx = 2 \tan \theta \sec^2 \theta \, d\theta$

$I = 2 \int \theta \cdot \tan \theta \cdot \sec^2 \theta \, d\theta$

Substituting, $\tan \theta = t$, we get, $\sec^2 \theta \, d\theta = dt$ ($0 < \theta < \frac{\pi}{2} \Rightarrow t > 0$)

$I = 2 \int t \cdot \tan^{-1} t \, dt$

$= 2 \left[\frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \frac{t^2}{t^2 + 1} \, dt \right]$

$= t^2 \tan^{-1} t - \int \frac{(t^2 + 1) - (1)}{t^2 + 1} \, dt$

$= t^2 \tan^{-1} t - t + \tan^{-1} t + c$

$= (t^2 + 1) \tan^{-1} t - t + c$

$= \theta \sec^2 \theta - \tan \theta + c$

$= (x + 1) \tan^{-1} \sqrt{x} - \sqrt{x} + c$

Ans. : (B)

(67) If $I = \int \frac{e^x dx}{e^{4x} + e^{2x} + 1}$, $J = \int \frac{e^{-x} dx}{e^{-4x} + e^{-2x} + 1}$, then $J - I = \dots$

(A) $\frac{1}{2} \log \left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1} \right) + c$

(B) $\frac{1}{2} \log \left(\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1} \right) + c$

(C) $\frac{1}{2} \log \left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right) + c$

(D) $\frac{1}{2} \log \left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1} \right) + c$ (IIT : 2008)

Sol. : $J - I = \int \left(\frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} - \frac{e^x}{e^{4x} + e^{2x} + 1} \right) dx$

$= \int \frac{\frac{1}{e^x}}{\frac{1}{e^{4x}} + \frac{1}{e^{2x}} + 1} - \int \frac{e^x}{e^{4x} + e^{2x} + 1} \, dx$

$= \int \frac{e^{3x} \, dx}{1 + e^{2x} + e^{4x}} - \int \frac{e^x}{e^{4x} + e^{2x} + 1} \, dx$

$= \int \frac{e^{3x} - e^x}{e^{4x} + e^{2x} + 1} \, dx$

$= \int \frac{(e^{2x} - 1) e^x}{e^{4x} + e^{2x} + 1} \, dx$

$$= \int \frac{t^2 - 1}{t^4 + t^2 + 1} dt \quad (\text{Let } e^x = t. \text{ So, } e^x dx = dt, t > 0)$$

$$= \int \frac{\left(1 - \frac{1}{t^2}\right)}{t^2 + 1 + \frac{1}{t^2}} dt$$

Substituting $t + \frac{1}{t} = u$ we get, $\left(1 - \frac{1}{t^2}\right) dt = du$

$$\therefore J - I = \int \frac{du}{u^2 - 1}$$

$$= \frac{1}{2} \log \left(\frac{u-1}{u+1} \right) + c \quad (u > 1)$$

$$= \frac{1}{2} \log \left(\frac{t^2 - t + 1}{t^2 + t + 1} \right) + c$$

$$= \frac{1}{2} \log \left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right) + c \quad \text{Ans. : (C)}$$

(68) If $f(t) = \int \frac{t^2 dt}{(1+t^2)(1+\sqrt{1+t^2})}$ and $f(0) = 0$, then $f(1) = \dots$

(A) $\log(\sqrt{2} + 1)$ (B) $\log(\sqrt{2} + 1) - \frac{\pi}{4}$

(C) $\log(\sqrt{2} + 1) + \frac{\pi}{2}$ (D) $\log(\sqrt{2} + 1) + \frac{\pi}{4}$

Sol. : $f(t) = \int \frac{t^2 dt}{(1+t^2)(1+\sqrt{1+t^2})}$

Substituting $t = \tan \theta$, we get, $dt = \sec^2 \theta d\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$

$$f(t) = \int \frac{\tan^2 \theta \sec^2 \theta d\theta}{(1 + \tan^2 \theta)(1 + \sqrt{1 + \tan^2 \theta})}$$

$$= \int \frac{\tan^2 \theta \sec^2 \theta d\theta}{\sec^2 \theta (1 + \sec \theta)}$$

$$= \int \frac{\tan^2 \theta d\theta}{1 + \sec \theta}$$

$$= \int \frac{\sin^2 \theta}{\cos \theta (1 + \cos \theta)} d\theta$$

$$= \int \frac{(1 - \cos^2 \theta) d\theta}{\cos \theta (1 + \cos \theta)}$$

$$\begin{aligned}
&= \int \frac{1 - \cos \theta}{\cos \theta} d\theta \\
&= \int (\sec \theta - 1) d\theta \\
&= \log |\sec \theta + \tan \theta| - \theta + c \\
&= \log |\sqrt{1+t^2} + t| - \tan^{-1} t + c \\
f(0) &= \log (\sqrt{1+0} + 0) - \tan^{-1} 0 + c \\
\therefore c &= 0 \\
f(t) &= \log |t + \sqrt{1+t^2}| - \tan^{-1} t \\
f(1) &= \log |1 + \sqrt{2}| - \tan^{-1} 1 \\
f(1) &= \log (\sqrt{2} + 1) - \frac{\pi}{4}
\end{aligned}$$

Ans. : (B)

$$(69) \quad \int \frac{2 - \sin x}{2 + \cos x} dx = \dots$$

- (A) $\frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + \log (2 + \cos x) + c$
- (B) $\frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + \log (2 + \cos x) + c$
- (C) $\frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) - \log (2 + \cos x) + c$
- (D) 0

$$\text{Sol. : } I = \int \frac{2 - \sin x}{2 + \cos x} dx$$

$$\therefore I = \int \frac{2}{2 + \cos x} dx - \int \frac{\sin x}{2 + \cos x} dx$$

$$\therefore I = I_1 - I_2$$

$$\text{where } I_1 = \int \frac{2}{2 + \cos x} dx,$$

$$\text{Substituting } \tan \frac{x}{2} = t \text{ we get, } \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2dt}{1+t^2}$$

$$\begin{aligned}
\therefore I_1 &= \int \frac{2}{2 + \left(\frac{1-t^2}{1+t^2} \right)} \cdot \frac{2dt}{1+t^2} \\
&= \int \frac{2(1+t^2)}{2+2t^2+1-t^2} \times \frac{2dt}{1+t^2}
\end{aligned}$$

$$= 4 \int \frac{1}{t^2 + 3} dt$$

$$I_1 = 4 \times \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) + c_1$$

$$= \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + c_1$$

$$I_2 = \int \frac{\sin x}{2 + \cos x} dx$$

$$= - \int \frac{\frac{d}{dx}(2 + \cos x)}{(2 + \cos x)} dx$$

$$I_2 = - \log |2 + \cos x| + c_2$$

$$\text{Now, } I = I_1 - I_2$$

$$= \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + \log |2 + \cos x| + c_1 - c_2$$

$$= \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + \log (2 + \cos x) + c, \quad \text{where, } c_1 - c_2 = c$$

Ans. : (B)

$$(70) \int \frac{\sec^2 x}{(\sec x + \tan x)^{\frac{9}{2}}} dx = \dots\dots$$

$$(A) \frac{-1}{(\sec x + \tan x)^{\frac{11}{2}}} \left\{ \frac{1}{11} - \frac{1}{7} (\sec x + \tan x)^2 \right\} + c$$

$$(B) \frac{1}{(\sec x + \tan x)^{\frac{11}{2}}} \left\{ \frac{1}{11} - \frac{1}{7} (\sec x + \tan x)^2 \right\} + c$$

$$(C) \frac{-1}{(\sec x + \tan x)^{\frac{11}{2}}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + c$$

$$(D) \frac{1}{(\sec x + \tan x)^{\frac{11}{2}}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + c$$

(IIT : 2012)

$$\text{Sol. : } I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{\frac{9}{2}}} dx$$

$$\text{Let } \sec x + \tan x = t$$

$$\therefore (\sec x \tan x + \sec^2 x) dx = dt$$

$$\therefore \sec x (\tan x + \sec x) dx = dt$$

$$\therefore \sec x (t) dx = dt$$

$$\therefore \sec x dx = \frac{1}{t} dt$$

$$\text{Now, } \sec x + \tan x = t$$

$$\therefore \sec x - \tan x = \frac{1}{t}$$

$$\therefore 2\sec x = t + \frac{1}{t}$$

$$\therefore \sec x = \frac{1}{2} \left(t + \frac{1}{t} \right)$$

$$\begin{aligned}
\therefore I &= \int \frac{\sec x \cdot (\sec x \, dx)}{(\sec x + \tan x)^{\frac{9}{2}}} \\
&= \frac{1}{2} \int \frac{\left(t + \frac{1}{t}\right) \cdot dt}{\frac{9}{t^2} \cdot t} \\
&= \frac{1}{2} \int \frac{(t^2 + 1)}{\frac{9}{t^2} \cdot t^2} dt \\
&= \frac{1}{2} \int \left(\frac{1}{t^2} + \frac{1}{t^2} \right) dt \\
&= -\frac{1}{2} \left[\frac{2}{7t^2} + \frac{2}{11t^2} \right] \\
&= -\frac{1}{t^2} \left(\frac{t^2}{7} + \frac{1}{11} \right) + c \\
&= \frac{-1}{(\sec x + \tan x)^{\frac{11}{2}}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + c
\end{aligned}$$

Ans. : (C)

(71) If $\int \left(\frac{2 \sin \theta + \cos \theta}{7 \sin \theta - 5 \cos \theta} \right) d\theta = A\theta + B \log |7 \sin \theta - 5 \cos \theta| + c$, then $A + B = \dots$

(A) $\frac{1}{17}$ (B) $\frac{3}{37}$ (C) $\frac{13}{37}$ (D) $\frac{23}{47}$

Sol. : $I = \int \left(\frac{2 \sin \theta + \cos \theta}{7 \sin \theta - 5 \cos \theta} \right) d\theta$

Let, $2 \sin \theta + \cos \theta = P(7 \sin \theta - 5 \cos \theta) + Q \frac{d}{d\theta}(7 \sin \theta - 5 \cos \theta)$

$$= 7P \sin \theta - 5P \cos \theta + Q(7 \cos \theta + 5 \sin \theta)$$

$$2 \sin \theta + \cos \theta = (7P + 5Q) \sin \theta + (7Q - 5P) \cos \theta$$

$$\therefore 7P + 5Q = 2, \quad 7Q - 5P = 1$$

Solving, $P = \frac{9}{74}$, $Q = \frac{17}{74}$

$$\begin{aligned}
I &= \int \left[\frac{9}{74} + \frac{17}{74} \left(\frac{7 \cos \theta + 5 \sin \theta}{7 \sin \theta - 5 \cos \theta} \right) \right] d\theta \\
&= \frac{9\theta}{74} + \frac{17}{74} \log |7 \sin \theta - 5 \cos \theta| + c
\end{aligned}$$

$$\therefore A = \frac{9}{74}, \quad B = \frac{17}{74}$$

$$\therefore A + B = \frac{26}{74} = \frac{13}{37}$$

Ans. : (C)

$$(72) \quad \int \frac{\sin x - 8 \cos x}{2 \sin x - \cos x} dx = \dots$$

$$(A) \quad 2x + 3 \log |2 \sin x - \cos x| + c$$

$$(B) \quad 2x - 3 \log |2 \sin x - \cos x| + c$$

$$(C) \quad x + 3 \log |\sin x - 8 \cos x| + c$$

$$(D) \quad 2x - 3 \log |\sin x - 8 \cos x| + c$$

$$\text{Sol. : } I = \int \frac{\sin x - 8 \cos x}{2 \sin x - \cos x} dx$$

$$\text{Let, } \sin x - 8 \cos x = m(2 \sin x - \cos x) + n \left[\frac{d}{dx} (2 \sin x - \cos x) \right]$$

$$\sin x - 8 \cos x = 2m \sin x - m \cos x + 2n \cos x + n \sin x$$

$$\sin x - 8 \cos x = (2m + n) \sin x + (-m + 2n) \cos x$$

$$2m + n = 1, -m + 2n = -8$$

$$\text{Solving, } m = 2, n = -3$$

$$\therefore I = \int \left[2 - 3 \left(\frac{2 \cos x + \sin x}{2 \sin x - \cos x} \right) \right] dx$$

$$= 2x - 3 \log |2 \sin x - \cos x| + c$$

Ans. : (B)

$$(73) \quad \int \left(\log (\log x) + \frac{1}{(\log x)^2} \right) dx = \dots$$

$$(A) \quad \frac{x}{(\log x)^2} + x \log (\log x) + c$$

$$(B) \quad \frac{1}{\log x} + \log (\log x) + c$$

$$(C) \quad \log (\log x) - \frac{1}{\log x} + c$$

$$(D) \quad x \log (\log x) - \frac{x}{\log x} + c$$

$$\text{Sol. : } I = \int \left(\log (\log x) + \frac{1}{(\log x)^2} \right) dx$$

$$\text{Substituting, } \log x = t, \text{ we get, } x = e^t \text{ i.e. } dx = e^t dt$$

$$\therefore I = \int \left\{ \log (t) + \frac{1}{t^2} \right\} e^t dt$$

$$= \int \left(\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2} \right) e^t dt$$

$$= \int \left(\log t + \frac{1}{t} \right) e^t dt - \int \left\{ \frac{1}{t} + \left(-\frac{1}{t^2} \right) \right\} e^t dt$$

$$= e^t \log t - e^t \frac{1}{t} + c$$

$$= x \log (\log x) - \frac{x}{\log x} + c$$

Ans. : (D)

$$(74) \quad \int e^{3x} \left(\frac{2 + 3 \sin 2x}{1 + \cos 2x} \right) dx = \dots$$

- (A) $\frac{2}{3} e^{3x} \sin 2x + c$ (B) $\frac{3}{2} e^{3x} \tan x + c$ (C) $e^{3x} \tan x + c$ (D) $2e^{3x} \tan x + c$

$$\begin{aligned} \text{Sol. : } I &= \int e^{3x} \left(\frac{2 + 3 \sin 2x}{1 + \cos 2x} \right) dx \\ &= \int e^{3x} \left(\frac{2 + 6 \sin x \cos x}{2 \cos^2 x} \right) dx \\ &= \int e^{3x} (\sec^2 x + 3 \tan x) dx \\ &= \int e^{3x} (3 \tan x + \sec^2 x) dx \\ &= e^{3x} \tan x + c \end{aligned}$$

$$\left[\int e^{kx} (kf(x) + f'(x)) dx = e^{kx} \cdot f(x) + c \right]$$

Ans. : (C)

$$(75) \quad \int \frac{e^{-\frac{x}{2}} \sqrt{1 - \sin x}}{(1 + \cos x)} dx = \dots$$

$$\left(\frac{\pi}{2} < x < \pi \right)$$

$$(A) \quad -e^{\frac{-x}{2}} \tan \frac{x}{2} + c$$

$$(B) \quad e^{\frac{-x}{2}} \sec \frac{x}{2} + c$$

$$(C) \quad -\frac{1}{2} e^{\frac{-x}{2}} \sec \frac{x}{2} + c$$

$$(D) \quad -2e^{\frac{-x}{2}} \sec \frac{x}{2} + c$$

$$\begin{aligned} \text{Sol. : } I &= \int \frac{e^{\frac{-x}{2}} \sqrt{1 - \sin x}}{(1 + \cos x)} dx \\ &= \int \frac{e^{\frac{-x}{2}} \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}}}{1 + \cos x} dx \end{aligned}$$

$$= \int \frac{e^{\frac{-x}{2}} \left(\sin \frac{x}{2} - \cos \frac{x}{2} \right)}{2 \cos^2 \frac{x}{2}} dx$$

$$\left(\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2} \right)$$

$$= \int e^{\frac{-x}{2}} \left(\frac{1}{2} \sec \frac{x}{2} \tan \frac{x}{2} - \frac{1}{2} \sec \frac{x}{2} \right) dx$$

$$= \int e^{\frac{-x}{2}} \left(-\frac{1}{2} \sec \frac{x}{2} + \frac{1}{2} \sec \frac{x}{2} \tan \frac{x}{2} \right) dx$$

$$= e^{\frac{-x}{2}} \sec \frac{x}{2} + c$$

$$\left[\int e^{kx} (kf(x) + f'(x)) dx = e^{kx} \cdot f(x) + c \right] \quad \text{Ans. : (B)}$$

$$(76) \quad \int \frac{dx}{\sqrt{2ax - x^2}} = \dots \quad (a > 0)$$

$$(A) \quad \frac{1}{a} \sin^{-1} \left(\frac{x-a}{a} \right) + c$$

$$(B) \quad \frac{1}{a} \log |(x-a) + \sqrt{2ax - x^2}| + c$$

$$(C) \quad \sin^{-1} \left(\frac{x-a}{a} \right) + c$$

$$(D) \quad \log |(x-a) + \sqrt{2ax - x^2}| + c$$

$$\begin{aligned} \text{Sol. : I} &= \int \frac{dx}{\sqrt{2ax - x^2}} \\ &= \int \frac{dx}{\sqrt{a^2 - a^2 + 2ax - x^2}} \\ &= \int \frac{dx}{\sqrt{(a)^2 - (x-a)^2}} \\ &= \sin^{-1} \left(\frac{x-a}{a} \right) + c \end{aligned}$$

Ans. : (C)

$$(77) \quad \int \sin^{-1} \left(\frac{2x+2}{\sqrt{4x^2 + 8x + 13}} \right) dx = \dots$$

$$(A) \quad 2(x+1) \tan^{-1} \left\{ \frac{2}{3}(x+1) \right\} + \frac{3}{4} \log \left| \sqrt{4 + 9(x+1)^2} \right| + c$$

$$(B) \quad (x+1) \tan^{-1} \left\{ \frac{2}{3}(x+1) \right\} - \frac{3}{2} \log \left| \sqrt{9 + 4(x+1)^2} \right| + c$$

$$(C) \quad (x+1) \tan^{-1} \left\{ \frac{3}{2(x+1)} \right\} + \frac{2}{3} \log \left| \sqrt{4 + 9(x+1)^2} \right| + c$$

$$(D) \quad (x+1) \tan^{-1} \left\{ \frac{3}{2(x+1)} \right\} - \frac{4}{3} \log \left| \sqrt{9 + 4(x+1)^2} \right| + c$$

(IIT : 2000)

$$\begin{aligned} \text{Sol. : I} &= \int \sin^{-1} \left(\frac{2x+2}{\sqrt{4x^2 + 8x + 13}} \right) dx \\ &= \int \sin^{-1} \left(\frac{x+1}{\sqrt{x^2 + 2x + \frac{13}{4}}} \right) dx \\ &= \int \sin^{-1} \left(\frac{x+1}{\sqrt{(x+1)^2 + \left(\frac{3}{2}\right)^2}} \right) dx \end{aligned}$$

Substituting, $x + 1 = \frac{3}{2} \tan \theta$. we get, $dx = \frac{3}{2} \sec^2 \theta d\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$

$$\begin{aligned}
 I &= \int \sin^{-1} \left(\frac{\frac{3}{2} \tan \theta}{\frac{3}{2} \sec \theta} \right) \cdot \frac{3}{2} \sec^2 \theta d\theta \\
 &= \int \sin^{-1} (\sin \theta) \cdot \frac{3}{2} \sec^2 \theta d\theta \\
 &= \frac{3}{2} \int \theta \cdot \sec^2 \theta d\theta \\
 &= \frac{3}{2} [\theta \tan \theta - \int 1 \cdot \tan \theta d\theta] + c' \\
 &= \frac{3}{2} [\theta \tan \theta - \log | \sec \theta |] + c' \\
 &= \frac{3}{2} \left[\tan^{-1} \left\{ \frac{2}{3} (x+1) \right\} \cdot \frac{2}{3} (x+1) - \log \left| \sqrt{1 + \left\{ \frac{2}{3} (x+1) \right\}^2} \right| \right] + c' \\
 &= (x+1) \tan^{-1} \left\{ \frac{2}{3} (x+1) \right\} - \frac{3}{2} \log \left| \sqrt{9 + 4(x+1)^2} \right| + c \quad \text{where, } c = c' + \frac{3}{2} \log 3
 \end{aligned}$$

Ans. : (B)

(78) $\int x \sin^{-1} \left(\sqrt{\frac{2a-x}{4a}} \right) dx = \dots$ $(a > 0)$

(A) $\left(\frac{x^2 - 2a^2}{4} \right) \cos^{-1} \left(\frac{x}{2a} \right) - \frac{x}{8} \sqrt{4a^2 - x^2} + c$

(B) $\left(\frac{x^2 - 2a^2}{2} \right) \cos^{-1} \left(\frac{x}{2a} \right) + \frac{x}{4} \sqrt{2a^2 - x^2} + c$

(C) $\left(\frac{x^2 - 2a^2}{4} \right) \sin^{-1} \left(\frac{x}{2a} \right) - \frac{x}{2} \sqrt{4a^2 - x^2} + c$

(D) $\left(\frac{x^2 - 4a^2}{2} \right) \sin^{-1} \left(\frac{x}{2a} \right) + \frac{x}{8} \sqrt{2a^2 - x^2} + c$

Sol. : $I = \int x \sin^{-1} \left(\sqrt{\frac{2a-x}{4a}} \right) dx$ $(a > 0)$

Substituting, $x = 2a \cos \theta$, we get, $dx = -2a \sin \theta d\theta$ $(0 < \theta < \pi)$

$$\sqrt{\frac{2a-x}{4a}} = \sqrt{\frac{2a-2a \cos \theta}{4a}} = \sqrt{\frac{2 \sin^2 \frac{\theta}{2}}{2}} = \sin \frac{\theta}{2} \quad (0 < \frac{\theta}{2} < \frac{\pi}{2})$$

$$\begin{aligned}
 \therefore I &= \int 2a \cos \theta \cdot \sin^{-1} \left(\sin \frac{\theta}{2} \right) (-2a \sin \theta) d\theta \\
 &= -a^2 \int \theta \cdot \sin 2\theta d\theta \\
 &= -a^2 \left[\theta \cdot \left(\frac{-\cos 2\theta}{2} \right) - \int 1 \cdot \left(\frac{-\cos 2\theta}{2} \right) d\theta \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{a^2}{2} \left[\theta \cdot \cos 2\theta - \frac{\sin 2\theta}{2} \right] + c \\
&= \frac{a^2}{2} \left[\cos^{-1} \left(\frac{x}{2a} \right) \left(\frac{x^2 - 2a^2}{2a^2} \right) - \frac{x}{4a^2} \sqrt{4a^2 - x^2} \right] + c \\
&\left(\begin{array}{l} \because \cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(\frac{x}{2a} \right)^2 - \left(1 - \frac{x^2}{4a^2} \right) = \frac{x^2 - 2a^2}{2a^2} \\ \text{and } \sin 2\theta = 2 \sin \theta \cdot \cos \theta = 2 \sqrt{1 - \frac{x^2}{4a^2}} \cdot \frac{x}{2a} = \frac{x}{2a^2} \sqrt{4a^2 - x^2} \end{array} \right) \\
&= \left(\frac{x^2 - 2a^2}{4} \right) \cos^{-1} \left(\frac{x}{2a} \right) - \frac{x}{8} \sqrt{4a^2 - x^2} + c
\end{aligned}$$

Ans. : (A)

$$(79) \quad \int \cos 2\theta \log \left| \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right| d\theta = \dots$$

$$(A) \quad \frac{\cos 2\theta}{2} \log \left| \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right| - \frac{1}{2} \log |\sin 2\theta| + c$$

$$(B) \quad \frac{\cos 2\theta}{2} \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| + \frac{1}{2} \log |\sin 2\theta| + c$$

$$(C) \quad \frac{\sin 2\theta}{2} \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| + \frac{1}{2} \log |\cos 2\theta| + c$$

$$(D) \quad \frac{\sin 2\theta}{2} \log \left| \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right| - \frac{1}{2} \log |\cos 2\theta| + c$$

(IIT : 1994)

$$\begin{aligned}
\text{Sol. : I} &= \int \cos 2\theta \log \left| \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right| d\theta \\
&= \int \cos 2\theta \cdot \log \left| \frac{1 + \tan \theta}{1 - \tan \theta} \right| d\theta \\
&= \int \cos 2\theta \cdot \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| d\theta \\
&= \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| \cdot \frac{\sin 2\theta}{2} - \int 2 \sec 2\theta \cdot \left(\frac{\sin 2\theta}{2} \right) d\theta \\
&\left[\int \sec \theta d\theta = \log \left| \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right| + c \right] \\
\therefore \int \sec 2\theta d\theta &= \frac{1}{2} \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| + c \\
\therefore \frac{d}{d\theta} \left\{ \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| + c \right\} &= 2 \sec 2\theta \\
\therefore I &= \frac{\sin 2\theta}{2} \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| - \int \tan 2\theta d\theta \\
&= \frac{1}{2} \sin 2\theta \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| + \frac{1}{2} \log |\cos 2\theta| + c
\end{aligned}$$

Ans. : (C)

$$(80) \int e^{\sin x} \left(\frac{x \cos^3 x - \sin x}{\cos^2 x} \right) dx = \dots$$

$$(A) e^{\sin x} (x - \cos x) + c$$

$$(B) e^{\sin x} (2x + \cot x) + c$$

$$(C) e^{\sin x} (x - \sec x) + c$$

$$(D) e^{\sin x} (2x + \tan x) + c$$

$$\text{Sol. : } I = \int e^{\sin x} \left(\frac{x \cos^3 x - \sin x}{\cos^2 x} \right) dx$$

$$= \int e^{\sin x} (x \cos x - \sec x \tan x) dx$$

$$= \int e^{\sin x} x \cos x dx - \int e^{\sin x} \sec x \tan x dx$$

$$I = I_1 - I_2$$

$$\text{Now, } I_1 = \int x \cdot (e^{\sin x} \cos x) dx$$

$$= x \int e^{\sin x} \cos x dx - \int (1 \cdot \int e^{\sin x} \cdot \cos x) dx$$

$$= x e^{\sin x} - \int e^{\sin x} dx + c_1 \quad \left(\int e^{\sin x} \cos x dx = e^{\sin x} \right)$$

$$\text{and } I_2 = \int e^{\sin x} (\sec x \tan x) dx$$

$$= e^{\sin x} \cdot \sec x - \int (e^{\sin x} \cos x \sec x) dx$$

$$I_2 = e^{\sin x} \cdot \sec x - \int e^{\sin x} dx + c_2$$

$$I = I_1 - I_2$$

$$= x \cdot e^{\sin x} - \int e^{\sin x} dx + c_1 - e^{\sin x} \cdot \sec x + \int e^{\sin x} dx - c_2$$

$$= x e^{\sin x} - e^{\sin x} \sec x + c \quad (c_1 - c_2 = c)$$

$$= e^{\sin x} (x - \sec x) + c$$

Ans. : (C)

$$(81) \text{ If } I = \int \sin^{-1} x dx \text{ and } J = \int \sin^{-1} \sqrt{1-x^2} dx, \text{ then } \dots \quad (x > 0)$$

$$(A) J = \frac{\pi}{2} I$$

$$(B) I + J = \frac{\pi}{2}$$

$$(C) I + J = \frac{\pi}{2} x$$

$$(D) I - J = \frac{\pi}{2} x$$

$$\text{Sol. : } J = \int \sin^{-1} \sqrt{1-x^2} dx = \int \cos^{-1} x dx$$

$$\therefore I + J = \int (\sin^{-1} x + \cos^{-1} x) dx = \frac{\pi}{2} \int dx = \frac{\pi}{2} x$$

Ans. : (C)

Note : If $x < 0$ then $I - J = \frac{\pi}{2} x$. So, if the condition is not given, then answer is both (C) and (D).

$$(82) \text{ If } \int \frac{dt}{t^n(1+t^n)^{\frac{1}{n}}} = a \left(\frac{1}{t^n} + 1 \right)^b + c, \text{ then } a \cdot b = \dots$$

$$(A) 1$$

$$(B) -1$$

$$(C) \frac{1}{n}$$

$$(D) -\frac{1}{n}$$

(IIT : 2007)

$$\begin{aligned}\text{Sol. : I} &= \int \frac{dt}{t^n(1+t^n)^{\frac{1}{n}}} \\ &= \int \frac{dt}{t^{n+1}\left(\frac{1}{t^n}+1\right)^{\frac{1}{n}}}\end{aligned}$$

$$\text{Let, } \frac{1}{t^n} + 1 = x$$

$$\therefore -n \cdot t^{-n-1} dt = dx$$

$$-n \left(\frac{1}{t^{n+1}} \right) dt = dx$$

$$\frac{dt}{t^{n+1}} = \frac{-1}{n} dx$$

$$\begin{aligned}\therefore \text{I} &= \frac{-1}{n} \int x^{\frac{-1}{n}} dx \\ &= \frac{-1}{n} \frac{x^{\frac{-1}{n}+1}}{\frac{-1}{n}+1} + c \\ &= \frac{1}{1-n} x^{1-\frac{1}{n}} + c \\ &= \frac{1}{1-n} \left(\frac{1}{t^n} + 1 \right)^{\frac{n-1}{n}} + c\end{aligned}$$

$$\therefore a = \frac{1}{1-n}, \quad b = \frac{n-1}{n}$$

$$\therefore a \cdot b = \left(\frac{1}{1-n} \right) \left(\frac{n-1}{n} \right) = -\frac{1}{n}$$

$$\therefore ab = -\frac{1}{n}$$

Ans. : (D)

$$(83) \quad \text{If } \int \frac{(1+x^n)^{\frac{1}{n}}}{x^{n+2}} dx = a \left(1 + \frac{1}{x^n} \right)^b + c, \text{ then } a + b = \dots \text{ (where } n = 4 \text{)}$$

$$(A) \quad \frac{6}{5}$$

$$(B) \quad \frac{16}{15}$$

$$(C) \quad \frac{21}{20}$$

$$(D) \quad \frac{11}{10}$$

$$\text{Sol. : I} = \int \frac{(1+x^n)^{\frac{1}{n}}}{x^{n+2}} dx$$

$$= \int \frac{\left(\frac{1}{x^n} + 1 \right)^{\frac{1}{n}}}{x^{n+1}} dx$$

$$\text{Let, } 1 + \frac{1}{x^n} = t$$

$$-n \cdot x^{-n-1} dx = dt$$

$$\frac{1}{x^{n+1}} dx = \frac{-1}{n} dt$$

$$I = \frac{-1}{n} \int t^{\frac{1}{n}} dt$$

$$= \frac{-1}{n} \frac{t^{\frac{1}{n}+1}}{\frac{1}{n}+1} + c$$

$$= \frac{-1}{n+1} (t)^{\frac{n+1}{n}} + c$$

$$= \frac{-1}{n+1} \left(1 + \frac{1}{x^n}\right)^{\frac{n+1}{n}} + c$$

$$\therefore n = 4, a = \frac{-1}{n+1} = \frac{-1}{5}, b = \frac{n+1}{n} = \frac{5}{4}$$

$$\therefore a + b = \frac{21}{20}$$

Ans. : (C)

$$(84) \text{ If } \int \left(x + \sqrt{x^2 + 1}\right)^{-5} dx = P \left(x + \sqrt{x^2 + 1}\right)^{-4} + Q \left(x + \sqrt{x^2 + 1}\right)^{-6} + c, \text{ then } P + Q = \dots..$$

$$(A) \quad \frac{-5}{3}$$

$$(B) \quad \frac{-5}{12}$$

$$(C) \quad \frac{-5}{6}$$

$$(D) \quad \frac{-5}{24}$$

$$\text{Sol. : } I = \int \left(x + \sqrt{x^2 + 1}\right)^{-5} dx$$

$$\text{Let, } x + \sqrt{x^2 + 1} = t$$

$$\frac{1}{t} = \frac{1}{x + \sqrt{x^2 + 1}} \times \frac{x - \sqrt{x^2 + 1}}{x - \sqrt{x^2 + 1}}$$

$$= \frac{x - \sqrt{x^2 + 1}}{x^2 - x^2 - 1}$$

$$\frac{1}{t} = -x + \sqrt{x^2 + 1}$$

$$t + \frac{1}{t} = 2\sqrt{x^2 + 1}.$$

$$\text{So, } \sqrt{x^2 + 1} = \frac{1}{2} \left(t + \frac{1}{t}\right)$$

$$\text{Now, } x + \sqrt{x^2 + 1} = t$$

$$\therefore \left(1 + \frac{2x}{2\sqrt{x^2 + 1}}\right) dx = dt$$

$$\left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right) dx = dt$$

$$t dx = \sqrt{x^2 + 1} dt$$

$$dx = \frac{\sqrt{x^2 + 1}}{t} dt$$

$$\therefore dx = \frac{1}{2} \left(t + \frac{1}{t} \right) \frac{dt}{t}$$

$$I = \frac{1}{2} \int \left(t + \frac{1}{t} \right) t^{-6} dt$$

$$= \frac{1}{2} \int (t^{-5} + t^{-7}) dt$$

$$= \frac{1}{2} \left[\frac{t^{-4}}{-4} + \frac{t^{-6}}{-6} \right] + c$$

$$= \frac{-1}{8} t^{-4} - \frac{1}{12} t^{-6} + c$$

$$= \frac{-1}{8} (x + \sqrt{x^2 + 1})^{-4} - \frac{1}{12} (x + \sqrt{x^2 + 1})^{-6} + c$$

$$\therefore P = \frac{-1}{8}, Q = \frac{-1}{12}$$

$$\therefore P + Q = \left(\frac{-1}{8} \right) + \left(\frac{-1}{12} \right) = \frac{-5}{24}$$

Ans. : (D)

(85) If $\int x^3 \sqrt{\frac{1+x^2}{1-x^2}} dx = A \cos^{-1} x^2 + B \sqrt{1-x^4} + C x^2 \sqrt{1-x^4} + D$, then $A + B + C = \dots$

(A) 0

(B) $\frac{-1}{2}$

(C) $\frac{1}{2}$

(D) -1

Sol. : $I = \int x^3 \sqrt{\frac{1+x^2}{1-x^2}} dx$

Let, $x^2 = \sin \theta$

$(0 < \theta < \frac{\pi}{2})$

$\therefore 2x dx = \cos \theta d\theta.$

So, $x dx = \frac{1}{2} \cos \theta d\theta$

$$I = \int x^3 \frac{(1+x^2)}{\sqrt{1-x^4}} dx$$

$$= \int \frac{x^2(1+x^2)}{\sqrt{1-x^4}} (x dx)$$

$$= \frac{1}{2} \int \frac{\sin \theta (1 + \sin \theta)}{\cos \theta} \cos \theta d\theta$$

$$\begin{aligned}
I &= \frac{1}{2} \int (\sin\theta + \sin^2\theta) d\theta \\
&= \frac{1}{2} \int \left(\sin\theta + \frac{1 - \cos 2\theta}{2} \right) d\theta \\
&= \frac{-1}{2} \cos\theta + \frac{1}{4} \int (1 - \cos 2\theta) d\theta \\
&= \frac{-1}{2} \cos\theta + \frac{1}{4} \theta - \frac{\sin 2\theta}{4 \times 2} + c \\
&= \frac{-1}{2} \cos\theta + \frac{\theta}{4} - \frac{\sin\theta \cos\theta}{4} + c \\
&= \frac{-1}{2} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1}x^2 - \frac{1}{4} x^2 \sqrt{1-x^2} + c
\end{aligned}$$

$$A = \frac{-1}{4}, B = \frac{-1}{2}, C = \frac{-1}{4} \quad (\sin^{-1}x^2 = \frac{\pi}{2} - \cos^{-1}x^2)$$

$$A + B + C = \left(\frac{-1}{4}\right) + \left(\frac{-1}{2}\right) + \left(\frac{-1}{4}\right) = -1 \quad \text{Ans. : (D)}$$

$$(86) \int \frac{dx}{(4+3x^2)\sqrt{3-4x^2}} = \dots\dots$$

$$(A) \quad \frac{1}{5} \tan^{-1} \frac{2x}{\sqrt{3-4x^2}} + c$$

$$(B) \quad \frac{1}{10} \tan^{-1} \left(\frac{5x}{2\sqrt{3-4x^2}} \right) + c$$

$$(C) \quad \frac{1}{5} \tan^{-1} \frac{5x}{2\sqrt{3-4x^2}} + c$$

$$(D) \quad \frac{1}{10} \tan^{-1} \frac{5x}{\sqrt{3-4x^2}} + c$$

$$\text{Sol. : } I = \int \frac{1}{(4+3x^2)\sqrt{3-4x^2}} dx$$

$$\text{Let, } x = \frac{\sqrt{3}}{2} \sin\theta \quad (0 < \theta < \frac{\pi}{2})$$

$$\therefore dx = \frac{\sqrt{3}}{2} \cos\theta d\theta$$

$$\therefore I = \int \frac{\frac{\sqrt{3}}{2} \cos\theta d\theta}{\left(4 + \frac{9}{4} \sin^2\theta\right) \sqrt{3-3\sin^2\theta}}$$

$$I = 2 \int \frac{d\theta}{9\sin^2\theta + 16}$$

$$= 2 \int \frac{\sec^2\theta d\theta}{16 + 25 \tan^2\theta}$$

$$= 2 \times \int \frac{dt}{(5t)^2 + (4)^2}$$

$$(\text{Substituting, } \tan\theta = t \text{ we get, } \sec^2\theta d\theta = dt)$$

$$= 2 \times \frac{1}{5} \times \frac{1}{4} \tan^{-1} \left(\frac{5t}{4} \right) + c$$

$$= \frac{1}{10} \tan^{-1} \left(\frac{5}{4} \tan \theta \right) + c$$

$$= \frac{1}{10} \tan^{-1} \left(\frac{5x}{2\sqrt{3-4x^2}} \right) + c$$

Ans. : (B)

$$(87) \int \frac{dx}{5+4\cos x} = \dots\dots$$

$$(A) \quad \frac{1}{3} \tan^{-1} \left(\frac{1}{3} \tan x \right) + c$$

$$(B) \quad \frac{1}{3} \tan^{-1} \left(\frac{1}{3} \tan \frac{x}{2} \right) + c$$

$$(C) \quad \frac{2}{3} \tan^{-1} \left(\frac{1}{3} \tan x \right) + c$$

$$(D) \quad \frac{2}{3} \tan^{-1} \left(\frac{1}{3} \tan \left(\frac{x}{2} \right) \right) + c$$

$$\text{Sol. : I} = \int \frac{dx}{5+4\cos x}$$

$$\text{Let, } \tan \frac{x}{2} = t$$

$$\therefore dx = \frac{2dt}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$I = \int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{5(1+t^2) + 4(1-t^2)}$$

$$= 2 \int \frac{dt}{(t^2) + (3)^2}$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{t}{3} \right) + c$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{1}{3} \tan \frac{x}{2} \right) + c$$

Ans. : (D)

$$(88) \text{ If } \int \frac{dx}{\sin^4 x + \cos^4 x} = \frac{1}{\sqrt{2}} \tan^{-1} (f(x)) + c, \text{ then } \dots\dots$$

$$(A) \quad f(x) = \tan x - \cot x$$

$$(B) \quad f\left(\frac{\pi}{4}\right) = 0$$

$$(C) \quad f(x) = \tan x + \cot x$$

$$(D) \quad f(x) = \frac{1}{2} (\tan x - \cot x)$$

$$\text{Sol. : I} = \int \frac{dx}{\sin^4 x + \cos^4 x}$$

$$\begin{aligned}
I &= \int \frac{\sec^4 x \, dx}{\tan^4 x + 1} && \text{(dividing numerator and denominator by } \cos^4 x \text{)} \\
&= \int \frac{(1 + \tan^2 x) \sec^2 x}{1 + \tan^4 x} \, dx \\
&= \int \frac{1 + t^2}{1 + t^4} \, dt && \text{(Substituting, } \tan x = t, \text{ we get, } \sec^2 x \, dx = dt \text{)} \\
&= \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t^2 + \frac{1}{t^2}\right)} \, dt \\
&= \int \frac{du}{u^2 + (\sqrt{2})^2} && \text{(Substituting, } t - \frac{1}{t} = u, \text{ we get } \left(1 + \frac{1}{t^2}\right) dt = du \text{)} \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + c \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} \right) + c \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{(\tan x - \cot x)}{\sqrt{2}} \right) + c \\
\therefore f(x) &= \frac{\tan x - \cot x}{\sqrt{2}}. && \text{So, } f\left(\frac{\pi}{4}\right) = 0 \qquad \text{Ans. : (B)}
\end{aligned}$$

(89) If $\int \left[\log (\log x) + \frac{1}{(\log x)^2} \right] dx = x [f(x) - g(x)] + c$, then

(A) $f(x) = \log (\log x)$, $g(x) = \frac{1}{\log x}$

(B) $f(x) = \log x$, $g(x) = \frac{1}{\log x}$

(C) $f(x) = \frac{1}{\log x}$, $g(x) = \log (\log x)$

(D) $f(x) = \frac{1}{x \log x}$, $g(x) = \frac{1}{\log x}$

Sol. : $\int \log (\log x) \, dx$

$$= \log (\log x) \int 1 \, dx - \int \left(\frac{d}{dx} \log (\log x) \int 1 \, dx \right) dx$$

$$= x \cdot \log (\log x) - \int \left(\frac{1}{\log x} \cdot \frac{1}{x} \cdot x \right) dx$$

$$= x \cdot \log (\log x) - \int \frac{1}{\log x} \, dx$$

$$= x \cdot \log (\log x) - x \cdot \frac{1}{\log x} + \int \left(\frac{(-1)}{(\log x)^2} \cdot \frac{1}{x} \cdot x \right) dx$$

$$\therefore \int \log (\log x) dx = x \cdot \log (\log x) - x \cdot \frac{1}{\log x} - \int \frac{1}{(\log x)^2} dx$$

$$\begin{aligned} \therefore \int \log (\log x) dx + \int \frac{1}{(\log x)^2} dx &= x \cdot \log (\log x) - x \cdot \frac{1}{\log x} \\ &= x \left[\log (\log x) - \frac{1}{\log x} \right] \end{aligned}$$

$$\therefore f(x) = \log (\log x), g(x) = \frac{1}{\log x} \quad \text{Ans. : (A)}$$

Note : The answer could be $f(x) = \log (\log x) + \sin x$, $g(x) = \frac{1}{\log x} + \sin x$. Actually, $f(x) = \log (\log x) + h(x)$, $g(x) = \frac{1}{\log x} + h(x)$. But according to given options the answer is (A).

$$(90) \text{ If } \int \frac{x \tan^{-1} x}{\sqrt{1+x^2}} dx = \sqrt{1+x^2} f(x) + k \log (x + \sqrt{x^2+1}) + c, \text{ then } \dots$$

$$(A) f(x) = \tan^{-1} x, k = -1$$

$$(B) f(x) = \tan^{-1} x, k = 1$$

$$(C) f(x) = 2 \tan^{-1} x, k = -1$$

$$(D) f(x) = 2 \tan^{-1} x, k = 1$$

$$\text{Sol. : } \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$$

$$\therefore \int \frac{x}{\sqrt{1+x^2}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{1+x^2}} dx = \frac{1}{2} (2\sqrt{1+x^2}) = \sqrt{1+x^2}$$

$$\text{Now, } I = \int (\tan^{-1} x) \frac{x}{\sqrt{1+x^2}} dx$$

$$= (\tan^{-1} x) \int \frac{x}{\sqrt{1+x^2}} dx - \int \left(\frac{d}{dx} (\tan^{-1} x) \int \frac{x}{\sqrt{1+x^2}} dx \right) dx$$

$$= \sqrt{1+x^2} \cdot \tan^{-1} x - \int \left(\frac{1}{1+x^2} \cdot \sqrt{1+x^2} \right) dx$$

$$= \sqrt{1+x^2} \cdot \tan^{-1} x - \int \frac{dx}{\sqrt{1+x^2}}$$

$$= \sqrt{1+x^2} \cdot \tan^{-1} x - \log |x + \sqrt{1+x^2}| + c$$

$$\text{So, } f(x) = \tan^{-1} x, k = -1$$

$$\text{Ans. : (A)}$$

(91) If $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = F(x)$ and $F(1) = 0$, then $F(x) = \dots$ for $0 < x < 1$.

(A) $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}x}{\sqrt{1+x^2}} \right) + \frac{\pi}{2}$

(B) $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}x}{\sqrt{1-x^2}} \right) - \frac{\pi}{2\sqrt{2}}$

(C) $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}x}{\sqrt{1-x^2}} \right) + \frac{\pi}{2\sqrt{2}}$

(D) $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{1-x^2}}{\sqrt{2}x} \right)$

Sol. : $I = \int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

$$= \int \frac{-t dt}{(1+t^2)\sqrt{t^2-1}}$$

(Let, $x = \frac{1}{t}$. So, $dx = \frac{-1}{t^2} dt$) ($0 < x < 1 \Rightarrow t > 1$)

$$= - \int \frac{u du}{(u^2+2)u}$$

(Let, $t^2 - 1 = u^2$. So, $2t dt = 2u du$, $u > 0$)

$$= - \int \frac{du}{(u)^2 + (\sqrt{2})^2}$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + c$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{t^2-1}}{\sqrt{2}} \right) + c$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \frac{\sqrt{1-x^2}}{x} \right) + c$$

Now, $F(1) = 0$.

So, $c = 0$

$$\therefore F(x) = \frac{-1}{\sqrt{2}} \left[\tan^{-1} \frac{\sqrt{1-x^2}}{\sqrt{2}x} \right]$$

$$\therefore F(x) = \frac{-1}{\sqrt{2}} \left[\frac{\pi}{2} - \cot^{-1} \frac{\sqrt{1-x^2}}{\sqrt{2}x} \right]$$

$$= \frac{-\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{2}x}{\sqrt{1-x^2}}$$

($\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $\cot^{-1}x = \tan^{-1} \left(\frac{1}{x} \right)$, $x > 0$)

Ans. : (B)

$$(92) \int e^x \frac{2 - \sin 2x}{1 - \cos 2x} dx = \dots$$

$$(A) -e^x \tan x + c$$

$$(B) -e^x \cot x + c$$

$$(C) e^x \tan x + c$$

$$(D) e^x \cot x + c$$

$$\begin{aligned} \text{Sol. : I} &= \int e^x \frac{2 - \sin 2x}{1 - \cos 2x} dx \\ &= \int e^x \frac{2 - 2 \sin x \cos x}{2 \sin^2 x} dx \\ &= \int e^x (\operatorname{cosec}^2 x - \cot x) dx \\ &= -\int e^x (\cot x - \operatorname{cosec}^2 x) dx \\ &= -e^x \cot x + c \end{aligned}$$

Ans. : (B)

$$(93) \text{ If } \int \frac{(2x^2 + 3)}{(x^2 - 1)(x^2 - 4)} dx = \log \left[\left(\frac{x-2}{x+2} \right)^p \left(\frac{x+1}{x-1} \right)^q \right] + c, \text{ then } p + q = \dots$$

$$(A) \frac{7}{4}$$

$$(B) \frac{1}{12}$$

$$(C) \frac{-1}{12}$$

$$(D) \frac{-7}{4}$$

$$\text{Sol. : I} = \int \frac{(2x^2 + 3)}{(x^2 - 1)(x^2 - 4)} dx$$

$$\text{Let, } \frac{2x^2 + 3}{(x^2 - 1)(x^2 - 4)} = \frac{A}{x^2 - 1} + \frac{B}{x^2 - 4}$$

$$\therefore 2x^2 + 3 = A(x^2 - 4) + B(x^2 - 1)$$

Comparing the coefficients of x^2 and constant terms on both the sides,

$$\therefore A + B = 2, \quad -4A - B = 3$$

$$\text{Solving, } A = \frac{-5}{3}, \quad B = \frac{11}{3}$$

$$\begin{aligned} \therefore I &= \int \frac{-\frac{5}{3}}{(x^2 - 1)} dx + \int \frac{\frac{11}{3}}{x^2 - 4} dx \\ &= \frac{-5}{3} \times \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + \frac{11}{3} \times \frac{1}{2(2)} \log \left| \frac{x-2}{x+2} \right| + c \\ &= \frac{-5}{6} \log \left| \frac{x-1}{x+1} \right| + \frac{11}{12} \log \left| \frac{x-2}{x+2} \right| + c \\ &= \frac{5}{6} \log \left| \frac{x+1}{x-1} \right| + \frac{11}{12} \log \left| \frac{x-2}{x+2} \right| + c \\ &= \log \left[\left(\frac{x+1}{x-1} \right)^{\frac{5}{6}} \cdot \left(\frac{x-2}{x+2} \right)^{\frac{11}{12}} \right] + c \end{aligned}$$

$$\therefore p = \frac{11}{12} \text{ and } q = \frac{5}{6}.$$

$$\text{So, } p + q = \frac{11}{12} + \frac{10}{12} = \frac{21}{12} = \frac{7}{4}$$

Ans. : (A)

$$(94) \int \frac{3x+2}{(x-2)^2(x-3)} dx = \dots\dots$$

$$(A) \quad 11 \log \left| \frac{x+3}{x+2} \right| - \frac{8}{x-2} + c$$

$$(B) \quad 11 \log \left| \frac{x-3}{x-2} \right| - \frac{8}{x-2} + c$$

$$(C) \quad 11 \log \left| \frac{x-3}{x-2} \right| + \frac{8}{x-2} + c$$

$$(D) \quad 11 \log \left| \frac{x+3}{x+2} \right| + \frac{8}{x-2} + c$$

$$\text{Sol. : } I = \int \frac{3x+2}{(x-2)^2(x-3)} dx$$

$$\text{Let, } \frac{3x+2}{(x-2)^2(x-3)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)}.$$

$$\therefore (3x+2) = A(x-2)(x-3) + B(x-3) + C(x-2)^2$$

$$\text{If } x = 2, \text{ then } 8 = -B \quad \text{So, } B = -8$$

$$\text{If } x = 3, \text{ then } 11 = C \quad \text{So, } C = 11$$

$$\text{If } x = 0, \text{ then } 2 = 6A - 3B + 4C$$

$$2 = 6A - 3(-8) + 4(11)$$

$$\therefore A = -11, B = -8, C = 11$$

$$\int \frac{3x+2}{(x-2)^2(x-3)} dx = \int \frac{-11}{x-2} dx + \int \frac{-8}{(x-2)^2} dx + \int \frac{11}{x-3} dx$$

$$= -11 \log |x-2| + \frac{8}{x-2} + 11 \log |x-3| + c$$

$$= 11 \log \left| \frac{x-3}{x-2} \right| + \frac{8}{x-2} + c$$

Ans. : (C)

$$(95) \int \frac{dx}{x(x^5+1)} = \dots\dots \quad (x > 0)$$

$$(A) \quad \frac{1}{5} \log x^4 (x^5+1) + c$$

$$(B) \quad \frac{1}{5} \log \left(\frac{1+x^5}{x^5} \right) + c$$

$$(C) \quad \frac{1}{5} \log \left(\frac{x^4}{(x^5+2)} \right) + c$$

$$(D) \quad \frac{1}{5} \log \left(\frac{x^5}{x^5+1} \right) + c$$

$$\text{Sol. : } I = \int \frac{dx}{x(x^5+1)} = \int \frac{dx}{x^6 \left(1 + \frac{1}{x^5} \right)}$$

$$\therefore I = \frac{-1}{5} \int \frac{1}{t} dt = \frac{-1}{5} \log |t| + c$$

$$(\text{Let, } 1 + \frac{1}{x^5} = t, \quad \text{So, } \frac{-5}{x^6} dx = dt)$$

$$= \frac{-1}{5} \log \left(1 + \frac{1}{x^5} \right) + c = \frac{-1}{5} \log \left(\frac{x^5+1}{x^5} \right) + c = \frac{1}{5} \log \left(\frac{x^5}{x^5+1} \right) + c$$

Ans. : (D)

Second method : $\int \frac{dx}{x(x^5+1)}$

$$= \int \frac{x^4 dx}{x^5(x^5+1)}$$

$$= \frac{1}{5} \int \frac{dt}{t^2+t} \quad (x^5 = t, 5x^4 dx = dt)$$

$$= \frac{1}{5} \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{5} \cdot \frac{1}{2\left(\frac{1}{2}\right)} \log \left| \frac{t+\frac{1}{2}-\frac{1}{2}}{t+\frac{1}{2}+\frac{1}{2}} \right| + c$$

$$= \frac{1}{5} \log \left(\frac{x^5}{x^5+1} \right) \quad (x > 0)$$

Third method :

$$I = \frac{1}{5} \int \frac{dt}{t(t+1)}$$

$$= \frac{1}{5} \left[\int \frac{dt}{t} - \int \frac{dt}{t+1} \right]$$

$$= \frac{1}{5} \log \left| \frac{t}{t+1} \right|$$

$$= \frac{1}{5} \log \left(\frac{x^5}{x^5+1} \right)$$

(96) $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} dx = \dots\dots$

(A) $\frac{13}{12}x + \frac{5}{13} \log |3\cos x + 2\sin x| + c$ (B) $\frac{12}{13}x + \frac{5}{13} \log |3\cos x + 2\sin x| + c$

(C) $\frac{12}{13}x - \frac{5}{13} \log |3\cos x + 2\sin x| + c$ (D) $\frac{13}{12}x - \frac{5}{13} \log |3\cos x + 2\sin x| + c$

Sol. : $I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} dx$

Let $(3\sin x + 2\cos x) = l[(3\cos x + 2\sin x)] + m\left[\frac{d}{dx}(3\cos x + 2\sin x)\right]$

$$\therefore 3\sin x + 2\cos x = l(3\cos x + 2\sin x) + m(-3\sin x + 2\cos x)$$

$$= (2l - 3m)\sin x + (3l + 2m)\cos x$$

Comparing the coefficients of $\sin x$ and $\cos x$,

$$2l - 3m = 3, \quad 3l + 2m = 2$$

Solving, $l = \frac{12}{13}, \quad m = \frac{-5}{13}$

$$\therefore I = \int \frac{\frac{12}{13}(3\cos x + 2\sin x) - \frac{5}{13}(-3\sin x + 2\cos x)}{(3\cos x + 2\sin x)} dx$$

$$= \frac{12}{13} \int dx - \frac{5}{13} \int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x} dx$$

$$= \frac{12}{13}x - \frac{5}{13} \log |3\cos x + 2\sin x| + c$$

Ans. : (C)

(97) $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx = \dots\dots$

(A) $\sin 2x + c$ (B) $-\frac{1}{2} \sin 2x + c$ (C) $\frac{1}{2} \sin 2x + c$ (D) $-\sin 2x + c$

$$\begin{aligned}
\text{Sol. : I} &= \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx \\
&= \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x} dx \\
&= \int (\sin^4 x - \cos^4 x) dx \quad (\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x) \\
&= \int (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) dx \\
&= \int -\cos 2x dx = -\frac{\sin 2x}{2} + c
\end{aligned}$$

Ans. : (B)

$$(98) \quad \int \frac{x^2}{(a+bx)^2} dx = \dots\dots$$

$$(A) \quad \frac{1}{b^2} \left[x + \frac{2a}{b} \log(a+bx) - \frac{a^2}{b} \cdot \frac{1}{a+bx} \right] + c$$

$$(B) \quad \frac{1}{b^2} \left[x - \frac{2a}{b} \log(a+bx) + \frac{a^2}{b} \cdot \frac{1}{a+bx} \right] + c$$

$$(C) \quad \frac{1}{b^2} \left[x + \frac{2a}{b} \log(a+bx) + \frac{a^2}{b} \cdot \frac{1}{a+bx} \right] + c$$

$$(D) \quad \frac{1}{b^2} \left[x + \frac{a}{b} - \frac{2a}{b} \log(a+bx) - \frac{a^2}{b} \cdot \frac{1}{a+bx} \right] + c \quad (\text{IIT : 1979})$$

$$\text{Sol. : I} = \int \frac{x^2}{(a+bx)^2} dx$$

$$\text{Let, } a+bx = t. \quad \text{So, } b dx = dt. \quad \text{So, } dx = \frac{1}{b} dt. \quad \text{Now, } x = \frac{t-a}{b}$$

$$\therefore I = \int \left(\frac{t-a}{b} \right)^2 \cdot \frac{1}{t^2} \cdot \frac{1}{b} dt$$

$$= \frac{1}{b^3} \int \left(1 - \frac{2a}{t} + a^2 t^{-2} \right) dt$$

$$= \frac{1}{b^3} \left[t - 2a \log |t| - \frac{a^2}{t} \right] + c$$

$$= \frac{1}{b^3} \left[(a+bx) - 2a \log |a+bx| - \frac{a^2}{a+bx} \right] + c$$

$$= \frac{1}{b^2} \left[x + \frac{a}{b} - \frac{2a}{b} \log(a+bx) - \frac{a^2}{b} \cdot \frac{1}{a+bx} \right] + c \quad \text{Ans. : (D)}$$

$$(99) \quad \int \frac{\sqrt{x^2 - a^2}}{x} dx = \dots \quad (a > 0)$$

$$(A) \quad \sqrt{x^2 - a^2} - a \tan^{-1} \left(\frac{\sqrt{x^2 - a^2}}{a} \right) + c \quad (B) \quad \sqrt{x^2 - a^2} + a \tan^{-1} \left(\frac{\sqrt{x^2 - a^2}}{a} \right) + c$$

$$(C) \quad \sqrt{x^2 - a^2} + a \tan^{-1} \left(\frac{\sqrt{x^2 - a^2}}{a} \right) + c \quad (D) \quad \cot^{-1} \left(\frac{x}{a} \right) + c$$

$$\text{Sol. : } I = \int \frac{\sqrt{x^2 - a^2}}{x} dx$$

$$\text{Let, } x^2 - a^2 = t^2.$$

$$\text{So, } x^2 = a^2 + t^2 \quad (t > 0)$$

$$\therefore 2x dx = 2t dt.$$

$$\text{So, } x dx = t dt$$

$$I = \int \frac{\sqrt{x^2 - a^2}}{x} dx$$

$$= \int \frac{\sqrt{x^2 - a^2}}{x^2} x dx$$

$$= \int \frac{t}{a^2 + t^2} t dt$$

$$= \int \frac{t^2 + a^2 - a^2}{t^2 + a^2} dt$$

$$= \left(1 - \frac{a^2}{t^2 + a^2} \right) dt$$

$$= t - a^2 \cdot \frac{1}{a} \tan^{-1} \left(\frac{t}{a} \right) + c$$

$$= \sqrt{x^2 - a^2} - a \tan^{-1} \left(\frac{\sqrt{x^2 - a^2}}{a} \right) + c$$

Ans. : (A)

$$\text{Second method : } I = \int \frac{\sqrt{x^2 - a^2}}{x} dx$$

$$\text{Let } x = a \sec \theta.$$

$$\text{So, } dx = a \sec \theta \tan \theta d\theta \quad (0 < \theta < \frac{\pi}{2}, a > 0)$$

$$= \int \frac{\sqrt{a^2 \sec^2 \theta - a^2}}{a \sec \theta} a \sec \theta \tan \theta d\theta$$

$$= a \int \tan^2 \theta d\theta$$

$$= a \int (\sec^2 \theta - 1) d\theta$$

$$= a \tan \theta - a \theta + c$$

$$= \sqrt{x^2 - a^2} - a \sec^{-1} \frac{x}{a} + c$$

$$= \sqrt{x^2 - a^2} - a \tan^{-1} \frac{\sqrt{x^2 - a^2}}{a} + c$$

Ans. : (A)

$$(100) \int \tan^3 2x \cdot \sec 2x \, dx = \dots\dots$$

$$(A) \quad \frac{1}{6} \sec^3 2x - \frac{1}{2} \sec 2x + c$$

$$(B) \quad \frac{1}{6} \sec^3 2x + \frac{1}{2} \sec 2x + c$$

$$(C) \quad \frac{1}{9} \sec^2 2x - \frac{1}{3} \sec 2x + c$$

$$(D) \quad \frac{1}{9} \sec^2 2x + \frac{1}{3} \sec 2x + c$$

$$\text{Sol. : } I = \int \tan^3 2x \cdot \sec 2x \, dx$$

$$\text{Let, } \sec 2x = t. \text{ Then } \sec 2x \tan 2x \, dx = \frac{1}{2} dt$$

$$I = \frac{1}{2} \int (t^2 - 1) dt$$

$$= \frac{1}{2} \cdot \frac{t^3}{3} - \frac{1}{2} t + c$$

$$= \frac{1}{6} \sec^3 2x - \frac{1}{2} \sec 2x + c$$

Ans. : (A)

$$(101) \int \sqrt{\frac{a-x}{x}} \, dx = \dots\dots$$

$(a > 0)$

$$(A) \quad \sin^{-1} \frac{x}{a} + \frac{x}{a} \sqrt{a^2 - x^2} + c$$

$$(B) \quad a \left[\sin^{-1} \sqrt{\frac{x}{a}} + \sqrt{\frac{x}{a}} \sqrt{\frac{a-x}{a}} \right] + c$$

$$(C) \quad \sin^{-1} \frac{x}{a} - \frac{x}{a} \sqrt{a^2 - x^2} + c$$

$$(D) \quad \sin^{-1} \frac{x}{a} - \frac{x}{a} \sqrt{a^2 - x^2} + c$$

$$\text{Sol. : } I = \int \sqrt{\frac{a-x}{x}} \, dx$$

$$\text{Let, } x = a \sin^2 \theta$$

$$(0 < \theta < \frac{\pi}{2})$$

$$\therefore dx = 2a \sin \theta \cos \theta \, d\theta$$

$$\therefore I = \int \sqrt{\frac{\cos^2 \theta}{\sin^2 \theta}} 2a \sin \theta \cos \theta \, d\theta$$

$$= a \int 2 \cos^2 \theta \, d\theta$$

$$= a \int (1 + \cos 2\theta) \, d\theta$$

$$= a \left[\theta + \frac{\sin 2\theta}{2} \right] + c$$

$$= a [\theta + \sin \theta \cos \theta] + c$$

$$= a \left[\sin^{-1} \sqrt{\frac{x}{a}} + \sqrt{\frac{x}{a}} \sqrt{\frac{a-x}{a}} \right] + c$$

Ans. : (B)

$$(102) \int \left(\frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} \right) dx = \dots$$

$$(A) \quad \frac{2}{\pi} [\sqrt{x-x^2} - (1-2x) \sin^{-1} \sqrt{x}] + x + c$$

$$(B) \quad \frac{2}{\pi} [\sqrt{x-x^2} - (1-2x) \sin^{-1} \sqrt{x}] - x + c$$

$$(C) \quad \frac{2}{\pi} [\sqrt{x-x^2} + (1-2x) \sin^{-1} \sqrt{x}] + x + c$$

$$(D) \quad \frac{2}{\pi} [\sqrt{x-x^2} + (1-2x) \sin^{-1} \sqrt{x}] - x + c$$

(IIT : 1986)

$$\text{Sol. : } I = \int \left(\frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} \right) dx$$

$$= \frac{2}{\pi} \int \left\{ \sin^{-1} \sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1} \sqrt{x} \right) \right\} dx \quad (\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \frac{\pi}{2})$$

$$= \frac{2}{\pi} \int \left(2 \sin^{-1} \sqrt{x} - \frac{\pi}{2} \right) dx$$

$$= \frac{4}{\pi} \int \sin^{-1} \sqrt{x} dx - x$$

$$I = \frac{4}{\pi} I_1 - x, \quad \text{where, } I_1 = \int \sin^{-1} \sqrt{x} dx$$

$$\text{Let, } x = \sin^2 \theta$$

$$(0 < \theta < \frac{\pi}{2})$$

$$\therefore dx = 2 \sin \theta \cos \theta d\theta$$

$$\therefore I_1 = \int \theta \cdot \sin 2\theta d\theta$$

$$= \theta \int \sin 2\theta d\theta - \int \left(\frac{d}{d\theta} (\theta) \int \sin 2\theta d\theta \right) d\theta$$

$$= -\theta \cdot \frac{\cos 2\theta}{2} - \int 1 \cdot \left(\frac{-\cos 2\theta}{2} \right) d\theta$$

$$= -\frac{\theta}{2} \cos 2\theta + \frac{1}{4} \sin 2\theta + c$$

$$= -\frac{1}{2} (1-2x) \sin^{-1} \sqrt{x} + \frac{1}{2} \sqrt{x} \sqrt{1-x} + c$$

$$(\cos 2\theta = 1 - 2\sin^2 \theta = 1 - 2x, \sin 2\theta = 2 \sin \theta \cos \theta = 2\sqrt{x} \sqrt{1-x})$$

$$= -\frac{1}{2} (1-2x) \sin^{-1} \sqrt{x} + \frac{\sqrt{x-x^2}}{2} + c$$

$$\text{Now, } I = \frac{4}{\pi} I_1 - x$$

$$= \frac{4}{\pi} \left[-\frac{1}{2} (1-2x) \sin^{-1} \sqrt{x} + \frac{\sqrt{x-x^2}}{2} \right] - x + c$$

$$= \frac{2}{\pi} [\sqrt{x-x^2} - (1-2x) \sin^{-1} \sqrt{x}] - x + c$$

Ans. : (B)

$$(103) \int \frac{dx}{(2bx+x^2)^{\frac{3}{2}}} = \dots \quad (b > 0)$$

$$(A) \quad \frac{-1}{b^2} \frac{(x+2b)}{\sqrt{2bx+x^2}} + c$$

$$(B) \quad \frac{-1}{b^2} \frac{(x-b)}{\sqrt{2bx+x^2}} + c$$

$$(C) \quad \frac{-1}{b^2} \frac{2x}{\sqrt{2bx+x^2}} + c$$

$$(D) \quad \frac{-1}{b^2} \frac{(x+b)}{\sqrt{2bx+x^2}} + c$$

$$\text{Sol. : I} = \int \frac{1}{(2bx+x^2)^{\frac{3}{2}}} \cdot dx$$

$$= \int \frac{1}{\left[(x^2+2bx+b^2) - (b)^2 \right]^{\frac{3}{2}}} dx$$

$$= \int \frac{dx}{\left[(x+b)^2 - (b)^2 \right]^{\frac{3}{2}}}$$

$$\text{Let, } x+b = b \sec \theta$$

$$(0 < \theta < \frac{\pi}{2}, b > 0)$$

$$\therefore dx = b \sec \theta \tan \theta d\theta$$

$$I = \int \frac{b \sec \theta \tan \theta d\theta}{(b^2 \sec^2 \theta - b^2)^{\frac{3}{2}}}$$

$$= \frac{1}{b^2} \int \frac{\sec \theta \tan \theta d\theta}{\tan^3 \theta}$$

$$= \frac{1}{b^2} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

$$= \frac{1}{b^2} \frac{(\sin \theta)^{-1}}{-1} + c$$

$$= \frac{-1}{b^2 \sin \theta} + c$$

$$= \frac{-1}{b^2} \frac{(x+b)}{\sqrt{x^2+2bx}} + c$$

$$\left(\sec \theta = \frac{x+b}{b} \Rightarrow \cos \theta = \frac{b}{x+b} \text{ So, } \sin \theta = \sqrt{1 - \frac{b^2}{(x+b)^2}} = \frac{\sqrt{x^2+2bx}}{x+b} \right) \text{ Ans. : (D)}$$

(104) If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Px + Q \log |9e^{2x} - 4| + c$, then

(A) $P = \frac{-3}{2}, Q = \frac{29}{36}$

(B) $P = \frac{-3}{2}, Q = \frac{35}{36}$

(C) $P = \frac{-1}{2}, Q = \frac{17}{36}$

(D) $P = \frac{-1}{2}, Q = \frac{23}{36}$

(IIT : 1989)

Sol. : $I = \int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = \int \left(\frac{4e^{2x} + 6}{9e^{2x} - 4} \right) dx$

Let, $4e^{2x} + 6 = m(9e^{2x} - 4) + n \left\{ \frac{d}{dx} (9e^{2x} - 4) \right\}$

$\therefore 4e^{2x} + 6 = m(9e^{2x} - 4) + n 18e^{2x}$

$4e^{2x} + 6 = (9m + 18n)e^{2x} - 4m$

Comparing the coefficients, $9m + 18n = 4$ and $-4m = 6$. So, $m = \frac{-3}{2}, n = \frac{35}{36}$

$\therefore I = \int \frac{\frac{-3}{2}(9e^{2x} - 4) + \frac{35}{36} \left\{ \frac{d}{dx} (9e^{2x} - 4) \right\}}{(9e^{2x} - 4)} dx$

$= \frac{-3}{2} \int dx + \frac{35}{36} \int \frac{\frac{d}{dx} (9e^{2x} - 4)}{(9e^{2x} - 4)} dx$

$= \frac{-3}{2} x + \frac{35}{36} \log (9e^{2x} - 4) + c$

$\therefore P = \frac{-3}{2}, Q = \frac{35}{36}$

Ans. : (B)

Second method : Substituting $e^x = t$, $e^x dx = dt$, $tdx = dt$

$\therefore I = \int \frac{4t^2 + 6}{9t^2 - 4} \left(\frac{dt}{t} \right)$

$= \int \frac{(4t^2 + 6) dt}{t(3t + 2)(3t - 2)}$

$= \int \frac{-\frac{3}{2} dt}{t} + \int \frac{\frac{35}{12} dt}{3t + 2} + \int \frac{\frac{35}{12} dt}{3t - 2}$

$= -\frac{3}{2} \log |t| + \frac{35}{36} \log |(3t + 2)(3t - 2)|$

$= -\frac{3}{2} x + \frac{35}{36} \log (9e^{2x} - 4)$

$\therefore P = \frac{-3}{2}, Q = \frac{35}{36}$

$$(105) \int \frac{2 \tan x + 3}{3 \tan x + 4} dx = \dots$$

$$(A) \quad \frac{24}{25} x + \frac{1}{25} \log |3 \sin x + 4 \cos x| + c \quad (B) \quad \frac{18}{25} x + \frac{1}{25} \log |3 \sin x + 4 \cos x| + c$$

$$(C) \quad \frac{14}{25} x + \frac{2}{25} \log |3 \sin x + 4 \cos x| + c \quad (D) \quad \frac{4x}{25} + \frac{2}{25} \log |3 \sin x + 4 \cos x| + c$$

$$\begin{aligned} \text{Sol. : } I &= \int \frac{2 \tan x + 3}{3 \tan x + 4} dx \\ &= \int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x} dx \end{aligned}$$

$$\text{Now, Let } 2 \sin x + 3 \cos x = A(3 \sin x + 4 \cos x) + B \left[\frac{d}{dx} (3 \sin x + 4 \cos x) \right]$$

$$2 \sin x + 3 \cos x = (3A - 4B) \sin x + (4A + 3B) \cos x$$

$$\therefore 3A - 4B = 2, 4A + 3B = 3. \text{ Solving, } A = \frac{18}{25}, B = \frac{1}{25}$$

$$\therefore I = \frac{18}{25} \int dx + \frac{1}{25} \int \frac{\frac{d}{dx}(3 \sin x + 4 \cos x)}{3 \sin x + 4 \cos x} dx$$

$$= \frac{18}{25} x + \frac{1}{25} \log |3 \sin x + 4 \cos x| + c$$

Ans. : (B)

$$(106) \text{ If } \int \sqrt{\tan x} dx = A \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + B \log \left| \frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right| + c, \text{ then } \dots$$

$$(A) \quad A = \frac{1}{\sqrt{2}}, B = 2\sqrt{2}$$

$$(B) \quad A = \frac{1}{\sqrt{2}}, B = -\frac{1}{\sqrt{2}}$$

$$(C) \quad A = \frac{1}{\sqrt{2}}, B = \frac{1}{2\sqrt{2}}$$

$$(D) \quad A = \frac{1}{\sqrt{2}}, B = -\frac{1}{2\sqrt{2}}$$

$$\text{Sol. : } I = \int \sqrt{\tan x} dx$$

$$\text{Let, } \tan x = t^2$$

$$\therefore \sec^2 x dx = 2t dt$$

$$\therefore dx = \frac{2t}{t^4 + 1} dt$$

$$\therefore I = \int \frac{2t^2}{t^4 + 1} dt$$

$$= \int \frac{(t^2 + 1) + (t^2 - 1)}{(t^4 + 1)} dt$$

$$= \int \frac{t^2 + 1}{t^4 + 1} dt + \int \frac{t^2 - 1}{t^4 + 1} dt$$

$$\begin{aligned}
&= \int \frac{\left(1 + \frac{1}{t^2}\right)}{t^2 + \frac{1}{t^2}} dt + \int \frac{\left(1 - \frac{1}{t^2}\right)}{t^2 + \frac{1}{t^2}} dt \\
&= \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} dt + \int \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 - 2} dt
\end{aligned}$$

Let $t - \frac{1}{t} = u$ for the first integrand and $t + \frac{1}{t} = v$ for the second integrand.

$$\therefore \left(1 + \frac{1}{t^2}\right) dt = du \text{ and } \left(1 - \frac{1}{t^2}\right) dt = dv$$

$$\begin{aligned}
\therefore I &= \int \frac{du}{u^2 + 2} + \int \frac{dv}{v^2 - 2} \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log \left| \frac{v - \sqrt{2}}{v + \sqrt{2}} \right| + c \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t^2 - 1}{\sqrt{2}t} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{t^2 + 1 - \sqrt{2}t}{t^2 + 1 + \sqrt{2}t} \right| + c \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2} \tan x} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2} \tan x + 1}{\tan x + \sqrt{2} \tan x + 1} \right| + c
\end{aligned}$$

$$\therefore A = \frac{1}{\sqrt{2}}, B = \frac{1}{2\sqrt{2}}$$

Ans. : (C)

$$(107) \int \frac{(x^2 - 1) dx}{(x^2 + 1)\sqrt{x^4 + 1}} = \dots \quad (x > 0)$$

$$(A) \quad \frac{1}{\sqrt{2}} \cos^{-1} \left(\frac{\sqrt{2}x}{x^2 + 1} \right) + c$$

$$(B) \quad \frac{x}{\sqrt{2}} \sin^{-1} \left(\frac{\sqrt{2}x}{x^2 - 1} \right) + c$$

$$(C) \quad \frac{x}{\sqrt{2}} \sin^{-1} \left(\frac{x - 1}{x + 1} \right) + c$$

$$(D) \quad \frac{1}{\sqrt{2}} \cos^{-1} \left\{ \frac{1}{\sqrt{2}} \left(\frac{x + 1}{x^2 + 1} \right) \right\} + c$$

$$\text{Sol. : } I = \int \frac{x^2 - 1}{(x^2 + 1)\sqrt{x^4 + 1}} dx$$

$$= \int \frac{(x^2 - 1) dx}{x(x^2 + 1)\sqrt{x^2 + \frac{1}{x^2}}}$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x}\right)\sqrt{\left(x + \frac{1}{x}\right)^2 - 2}}$$

$$= \int \frac{dt}{t\sqrt{t^2 - 2}} \quad \left(\text{Let, } x + \frac{1}{x} = t. \text{ So, } \left(1 - \frac{1}{x^2}\right) dx = dt \right)$$

$$= \frac{1}{\sqrt{2}} \sec^{-1} \left(\frac{t}{\sqrt{2}} \right) + c \quad (t > 2)$$

$$= \frac{1}{\sqrt{2}} \cos^{-1} \left(\frac{\sqrt{2}}{t} \right) + c$$

$$= \frac{1}{\sqrt{2}} \cos^{-1} \left(\frac{\sqrt{2}x}{x^2+1} \right) + c$$

Ans. : (A)

(108) If $\int \frac{dx}{\sin^6 x + \cos^6 x} = \tan^{-1} (f(x)) + c$, then $f(x) = \dots$

- (A) $\tan x - 1$ (B) $\tan^2 x - 1$ (C) $\tan x - \cot x$ (D) $2\tan^2 x - 3\cot^2 x$

Sol. : I $= \int \frac{dx}{\sin^6 x + \cos^6 x}$

$$= \int \frac{dx}{(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)}$$

$$= \int \frac{dx}{\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x}$$

Dividing numerator and denominator by $\cos^4 x$,

$$= \int \frac{\sec^4 x dx}{\tan^4 x + 1 - \tan^2 x}$$

$$= \int \frac{(\tan^2 x + 1) \sec^2 x dx}{\tan^4 x + 1 - \tan^2 x}$$

$$= \int \frac{(t^2 + 1) dt}{t^4 + 1 - t^2} \quad (\text{Let, } \tan x = t. \text{ So, } \sec^2 x dx = dt)$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t^2 - 1 + \frac{1}{t^2}\right)}$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 1}$$

$$= \int \frac{du}{u^2 + 1} \quad (\text{Let, } t - \frac{1}{t} = u. \text{ So, } \left(1 + \frac{1}{t^2}\right) dt = du)$$

$$= \tan^{-1} (u) + c$$

$$= \tan^{-1} \left(t - \frac{1}{t} \right) + c$$

$$= \tan^{-1} (\tan x - \cot x) + c$$

$$\therefore f(x) = \tan x - \cot x$$

Ans. : (C)

$$(109) \quad \int e^{\frac{x}{2}} \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx = \dots$$

(A) $e^{\frac{x}{2}} \sin \frac{x}{2} + c$ (B) $e^{\frac{x}{2}} \cos \frac{x}{2} + c$ (C) $\sqrt{2} e^{\frac{x}{2}} \cos \frac{x}{2} + c$ (D) $\sqrt{2} e^{\frac{x}{2}} \sin \frac{x}{2} + c$

Sol. : $I = \int e^{\frac{x}{2}} \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) dx$

$$= 2 \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) e^{\frac{x}{2}} - \int \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \frac{1}{2} \cdot 2e^{\frac{x}{2}} dx$$

$$= 2 \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) e^{\frac{x}{2}} - \int \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) e^{\frac{x}{2}} dx$$

$$I = 2e^{\frac{x}{2}} \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) - \left[2e^{\frac{x}{2}} \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) + \int \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) e^{\frac{x}{2}} dx \right]$$

$$= 2e^{\frac{x}{2}} \left\{ \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) - \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \right\} - I$$

$$\therefore 2I = 2e^{\frac{x}{2}} \left\{ \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) - \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \right\} + c'$$

$$I = \sqrt{2} e^{\frac{x}{2}} \sin \frac{x}{2} + c \quad \left(\frac{c'}{2} = c\right) \quad \text{Ans. : (D)}$$

Second method : $a = \frac{1}{2}, b = \frac{1}{2} \quad \therefore a^2 + b^2 = \frac{1}{2}$

$$\begin{aligned} I &= \int e^{\frac{x}{2}} \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) dx = \frac{e^{\frac{x}{2}}}{\frac{1}{2}} \left[\frac{1}{2} \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) - \frac{1}{2} \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \right] + c \\ &= e^{\frac{x}{2}} \left[\sin \frac{x}{2} \cos \frac{\pi}{4} + \cos \frac{x}{2} \sin \frac{\pi}{4} - \cos \frac{x}{2} \cos \frac{\pi}{4} + \sin \frac{x}{2} \sin \frac{\pi}{4} \right] + c \\ &= e^{\frac{x}{2}} \left[\sqrt{2} \sin \frac{x}{2} \right] \\ &= \sqrt{2} e^{\frac{x}{2}} \sin \frac{x}{2} + c \end{aligned}$$

$$(110) \quad \int \frac{x^2}{(x \sin x + \cos x)^2} dx = \dots$$

(A) $\frac{\sin x + \cos x}{x \sin x + \cos x}$ (B) $\frac{x \sin x - \cos x}{x \sin x + \cos x}$ (C) $\frac{\sin x - x \cos x}{x \sin x + \cos x}$ (D) $\frac{x \sin x + \cos x}{x \sin x + \cos x}$

Sol. : $I = \int \frac{x^2}{(x \sin x + \cos x)^2} dx$

$$= \int \frac{x \cos x}{(x \sin x + \cos x)^2} \cdot \left(\frac{x}{\cos x} \right) dx$$

$$= \frac{x}{\cos x} \int \frac{x \cos x}{(x \sin x + \cos x)^2} dx - \int \left(\frac{d}{dx} \left(\frac{x}{\cos x} \right) \int \frac{x \cos x}{(x \sin x + \cos x)^2} dx \right) dx$$

$$\begin{aligned}
&= \frac{x}{\cos x} \cdot \frac{-1}{(x \sin x + \cos x)} + \int \left(\frac{\cos x + x \sin x}{\cos^2 x} \times \frac{1}{(x \sin x + \cos x)} \right) dx \\
&\quad \left(\frac{d}{dx} (x \sin x + \cos x) = x \cos x \text{ and } \int \frac{1}{t^2} dt = \frac{-1}{t} \right) \\
&= \frac{-x}{(x \sin x + \cos x) \cos x} + \int \sec^2 x dx \\
&= \frac{-x}{(x \sin x + \cos x) \cos x} + \frac{\sin x}{\cos x} + c \\
&= \frac{-x + x \sin^2 x + \sin x \cos x}{(x \sin x + \cos x) \cos x} + c \\
&= \frac{\sin x \cos x - x(1 - \sin^2 x)}{(x \sin x + \cos x) \cos x} + c \\
&= \frac{\cos x (\sin x - x \cos x)}{(x \sin x + \cos x) \cos x} + c \\
&= \frac{\sin x - x \cos x}{x \sin x + \cos x} + c
\end{aligned}$$

Ans. : (C)

$$(111) \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx = \dots + c$$

(A) $\sin x - 6 \tan^{-1}(\sin x)$

(B) $\sin x - 2(\sin x)^{-1}$

(C) $\sin x - 2(\sin x)^{-1} - 6 \tan^{-1}(\sin x)$

(D) $\sin x - 2(\sin x)^{-1} - 5 \tan^{-1}(\sin x)$

(IIT : 1990)

$$\begin{aligned}
\text{Sol. : I} &= \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx \\
&= \int \frac{(\cos^2 x + \cos^4 x) \cos x dx}{\sin^2 x + \sin^4 x} \\
&= \int \frac{[(1 - \sin^2 x) + (1 - \sin^2 x)^2] \cos x dx}{\sin^2 x + \sin^4 x}
\end{aligned}$$

Let, $\sin x = t$.

So, $\cos x dx = dt$

$$\begin{aligned}
\therefore \text{I} &= \int \frac{(1 - t^2) + (1 - t^2)^2}{t^2 + t^4} dt \\
&= \int \frac{1 - t^2 + 1 - 2t^2 + t^4}{t^4 + t^2} dt \\
&= \int \frac{(t^4 + t^2) - 4t^2 + 2}{(t^4 + t^2)} dt \\
&= \int 1 dt - \int \frac{4t^2 - 2}{t^2(t^2 + 1)} dt
\end{aligned}$$

We can use the method of partial fraction as follows :

$$= t - \int \left(\frac{-2}{t^2} + \frac{6}{t^2 + 1} \right) dt \quad (\text{partial fraction})$$

$$= t - \frac{2}{t} - 6 \tan^{-1} t$$

$$= \sin x - 2(\sin x)^{-1} - 6 \tan^{-1}(\sin x) + c$$

$$= \int 1 dt - 4 \int \frac{1}{t^2 + 1} dt + 2 \int \frac{1}{t^2(t^2 + 1)} dt$$

$$= \int 1 dt - 4 \int \frac{1}{t^2 + 1} dt + 2 \int \frac{(t^2 + 1) - (t^2)}{t^2(t^2 + 1)} dt$$

$$= \int 1 dt - 4 \int \frac{1}{t^2 + 1} dt + 2 \int \frac{1}{t^2} dt - 2 \int \frac{1}{t^2 + 1} dt$$

$$= t - 4 \tan^{-1} t + 2 \frac{t^{-1}}{-1} - 2 \tan^{-1} t + c$$

$$= \sin x - 2(\sin x)^{-1} - 6 \tan^{-1}(\sin x) + c$$

Ans. : (C)

$$(112) \int \frac{dx}{(x-a)\sqrt{(x-a)(x-b)}} = \dots + c.$$

$$(A) \frac{2}{a-b} \sqrt{\frac{x-a}{x-b}} \quad (B) \frac{-2}{a-b} \sqrt{\frac{x-b}{x-a}} \quad (C) \frac{1}{\sqrt{(x-a)(x-b)}} \quad (D) \frac{2}{a-b} \sqrt{\frac{x-b}{x-a}}$$

(IIT : 1996)

$$\text{Sol. : I} = \int \frac{dx}{(x-a)\sqrt{(x-a)(x-b)}}$$

$$= \int \frac{dx}{(x-a)^{\frac{3}{2}}(x-b)^{\frac{1}{2}}}$$

$$= \int \frac{dx}{(x-a)^2 \sqrt{\frac{x-b}{x-a}}}$$

Substituting, $\sqrt{\frac{x-b}{x-a}} = t$, we get $\frac{x-b}{x-a} = t^2$

$$\therefore \frac{(x-a) - (x-b)}{(x-a)^2} dx = 2t dt$$

$$\frac{b-a}{(x-a)^2} dx = 2t dt$$

$$\frac{1}{(x-a)^2} dx = \frac{2t}{b-a} dt$$

$$\begin{aligned}
 \therefore I &= \frac{1}{b-a} \int \frac{2t}{t} dt \\
 &= \frac{2t}{b-a} + c \\
 &= \frac{-2}{a-b} \sqrt{\frac{x-b}{x-a}} + c
 \end{aligned}$$

Ans. : (B)

$$(113) \int \frac{x+1}{x(1+xe^x)^2} dx = \dots + c.$$

(IIT : 1996)

$$(A) \log \left| \frac{xe^x}{xe^x+1} \right|$$

$$(B) \log \left| \frac{xe^x}{xe^x+1} \right| - \frac{1}{xe^x+1}$$

$$(C) \log \left| \frac{xe^x}{xe^x+1} \right| + \frac{1}{xe^x+1}$$

$$(D) \frac{1}{1+xe^x} + \log |xe^x+1|$$

$$\begin{aligned}
 \text{Sol. : } I &= \int \frac{x+1}{x(1+xe^x)^2} dx \\
 &= \int \frac{(x+1)e^x}{xe^x(1+xe^x)^2} dx
 \end{aligned}$$

$$\text{Let, } xe^x = t$$

$$\text{So, } (1+x)e^x dx = dt$$

$$\begin{aligned}
 \therefore I &= \int \frac{dt}{t(t+1)^2} = \int \frac{(t+1)-(t)}{t(t+1)^2} dt \\
 &= \int \frac{1}{t(t+1)} dt - \int \frac{1}{(t+1)^2} dt \\
 &= \int \frac{(t+1)-(t)}{t(t+1)} dt - \int (t+1)^{-2} dt \\
 &= \int \frac{1}{t} dt - \int \frac{1}{t+1} dt - \int (t+1)^{-2} dt \\
 &= \log |t| - \log |t+1| - \frac{(t+1)^{-1}}{-1} + c \\
 &= \log \left| \frac{t}{t+1} \right| + \frac{1}{t+1} + c \\
 &= \log \left| \frac{xe^x}{xe^x+1} \right| + \frac{1}{xe^x+1}
 \end{aligned}$$

Ans. : (C)

$$(114) \int \frac{dx}{(2x-7)\sqrt{x^2-7x+12}} = \dots\dots . \quad \left(x > \frac{7}{2}\right)$$

$$(A) \quad 2 \sec^{-1} (2x-7) + c$$

$$(B) \quad \sec^{-1} (2x-7) + c$$

$$(C) \quad \frac{1}{2} \sec^{-1} (2x-7) + c$$

$$(D) \quad -2 \sec^{-1} (2x-7) + c$$

(IIT : 1997)

$$\begin{aligned} \text{Sol. : I} &= \int \frac{dx}{(2x-7)\sqrt{x^2-7x+12}} \\ &= 2 \int \frac{dx}{(2x-7)\sqrt{4x^2-28x+48}} \\ &= 2 \int \frac{dx}{(2x-7)\sqrt{(2x-7)^2-(1)^2}} \\ &= 2 \cdot \frac{1}{2} \sec^{-1} (2x-7) + c \\ &= \sec^{-1} (2x-7) + c \end{aligned}$$

Ans. : (B)

$$(115) \int \frac{\cos x - \sin x}{\cos x + \sin x} (2 + 2\sin 2x) dx = \dots\dots .$$

$$(A) \quad \sin 2x + c$$

$$(B) \quad \cos 2x + c$$

$$(C) \quad \tan 2x + c$$

$$(D) \quad 2\sin 2x + c$$

(IIT : 1997)

$$\begin{aligned} \text{Sol. : I} &= \int \frac{\cos x - \sin x}{\cos x + \sin x} (2 + 2\sin 2x) dx \\ &= \int \frac{\cos x - \sin x}{\cos x + \sin x} 2(1 + \sin 2x) dx \\ &= 2 \int \frac{(\cos x - \sin x)(\cos x + \sin x)^2}{(\cos x + \sin x)} dx \\ &= 2 \int (\cos^2 x - \sin^2 x) dx \\ &= 2 \int (\cos 2x) dx \\ &= 2 \times \frac{1}{2} (\sin 2x) + c \\ &= \sin 2x + c \end{aligned}$$

Ans. : (A)

$$(116) \int \frac{\cos x + 3\sin x - 5}{3 - 3\sin x - \cos x} dx = \dots\dots .$$

$$(A) \quad x - 2 \log \left| \frac{2\tan \frac{x}{2} - 2}{2\tan \frac{x}{2} - 1} \right| + c$$

$$(B) \quad -x - \frac{1}{2} \log \left| \frac{2\tan \frac{x}{2} - 2}{2\tan \frac{x}{2} + 1} \right| + c$$

$$(C) \quad -x - 2 \log \left| \frac{2\tan \frac{x}{2} - 2}{2\tan \frac{x}{2} - 1} \right| + c$$

$$(D) \quad -x + 2 \log \left| \frac{2\tan \frac{x}{2} - 2}{2\tan \frac{x}{2} - 1} \right| + c$$

$$\text{Sol. : } I = \int \frac{\cos x + 3 \sin x - 5}{3 - 3 \sin x - \cos x} dx$$

$$\text{Let } (\cos x + 3 \sin x - 5) = P(3 - 3 \sin x - \cos x) + Q \left[\frac{d}{dx}(3 - 3 \sin x - \cos x) \right] + R$$

$$= P(3 - 3 \sin x - \cos x) + Q(-3 \cos x + \sin x) + R$$

$$\cos x + 3 \sin x - 5 = (3P + R) + (-3P + Q) \sin x + (-P - 3Q) \cos x$$

Now, comparing the coefficients, $3P + R = -5$, $-3P + Q = 3$, $-P - 3Q = 1$

Solving, $P = -1$, $Q = 0$, $R = -2$

$$\begin{aligned} \therefore I &= \int \frac{-(3 - 3 \sin x - \cos x) - 2}{3 - 3 \sin x - \cos x} dx \\ &= - \int dx - 2 \int \frac{dx}{3 - 3 \sin x - \cos x} = -x - 2I_1 \end{aligned}$$

$$\text{where, } I_1 = \int \frac{dx}{3 - 3 \sin x - \cos x}$$

$$\text{Let, } \tan \frac{x}{2} = t. \quad \text{So, } dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore I_1 = \int \frac{1}{3 - 3 \left(\frac{2t}{1+t^2} \right) - \left(\frac{1-t^2}{1+t^2} \right)} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{3 + 3t^2 - 6t - 1 + t^2}$$

$$= 2 \int \frac{dt}{4t^2 - 6t + 2}$$

$$= \int \frac{dt}{2t^2 - 3t + 1}$$

$$= \frac{1}{2} \int \frac{dt}{\left(t - \frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2}$$

$$= \frac{1}{2} \times \frac{1}{2\left(\frac{1}{4}\right)} \log \left| \frac{t - \frac{3}{4} - \frac{1}{4}}{t - \frac{3}{4} + \frac{1}{4}} \right| + c$$

$$= \log \left| \frac{t-1}{t-\frac{1}{2}} \right| + c$$

$$I_1 = \log \left| \frac{2t-2}{2t-1} \right| + c$$

$$I = -x - 2 \log \left| \frac{2t-2}{2t-1} \right| + c$$

$$= -x - 2 \log \left| \frac{2 \tan \frac{x}{2} - 2}{2 \tan \frac{x}{2} - 1} \right| + c$$

Ans. : (C)

$$(117) \int \frac{(3 \cos x + 2)}{\sin x + 2 \cos x + 3} dx = \dots$$

$$(A) \quad \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2 \cos x + 3| - \frac{16}{5} \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{2} \right) + c$$

$$(B) \quad \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2 \cos x + 3| - \frac{8}{5} \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{2} \right) + c$$

$$(C) \quad \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2 \cos x + 3| - \frac{6}{5} \log \left| \frac{\tan \frac{x}{2} + 3}{\tan \frac{x}{2} - 1} \right| + c$$

$$(D) \quad \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2 \cos x + 3| - \frac{4}{5} \log \left| \frac{\tan \frac{x}{2} - 1}{\tan \frac{x}{2} + 3} \right| + c$$

$$\text{Sol. : } I = \int \frac{(3 \cos x + 2)}{\sin x + 2 \cos x + 3} dx$$

$$\text{Let } 3 \cos x + 2 = P(\sin x + 2 \cos x + 3) + Q \left[\frac{d}{dx}(\sin x + 2 \cos x + 3) \right] + R$$

$$\therefore 3 \cos x + 2 = P(\sin x + 2 \cos x + 3) + Q(\cos x - 2 \sin x) + R$$

$$\therefore 3 \cos x + 2 = (P - 2Q) \sin x + (2P + Q) \cos x + (3P + R)$$

$$\text{Now, comparing the coefficients, } P - 2Q = 0, 2P + Q = 3, 3P + R = 2$$

$$\text{Solving equations, } P = \frac{6}{5}, Q = \frac{3}{5}, R = \frac{-8}{5}$$

$$\therefore I = \int \frac{\frac{6}{5}(\sin x + 2 \cos x + 3) + \frac{3}{5}(\cos x - 2 \sin x) - \frac{8}{5}}{\sin x + 2 \cos x + 3} dx$$

$$= \frac{6}{5} \int dx + \frac{3}{5} \int \frac{\cos x - 2 \sin x}{\sin x + 2 \cos x + 3} dx - \frac{8}{5} \int \frac{dx}{\sin x + 2 \cos x + 3}$$

$$I = \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2 \cos x + 3| - \frac{8}{5} I_1 \quad \dots(1)$$

$$\text{where, } I_1 = \int \frac{dx}{\sin x + 2 \cos x + 3}$$

$$\text{Let, } \tan \frac{x}{2} = t. \text{ So, } dx = \frac{2dt}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}$$

$$\begin{aligned}
\therefore I_1 &= \int \frac{1}{\frac{2t}{1+t^2} + 2\left(\frac{1-t^2}{1+t^2}\right) + 3} \times \frac{2dt}{1+t^2} \\
&= \int \frac{2dt}{2t + 2 - 2t^2 + 3 + 3t^2} \\
&= \int \frac{2dt}{t^2 + 2t + 1 + 4} \\
&= \int \frac{2dt}{(t+1)^2 + (2)^2} = 2 \times \frac{1}{2} \tan^{-1} \left(\frac{t+1}{2} \right) + c = \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{2} \right) + c
\end{aligned}$$

Substituting in equation (1),

$$I = \frac{6}{5}x + \frac{3}{5} \log | \sin x + 2\cos x + 3 | - \frac{8}{5} \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{2} \right) \quad \text{Ans. : (B)}$$

$$(118) \int \frac{\sqrt{\cos x}}{\sin x} dx = \dots + c. \quad (0 < x < \frac{\pi}{2})$$

$$(A) \quad \frac{1}{2} \log \left| \frac{\sqrt{\cos x} - 1}{\sqrt{\cos x} + 1} \right| + \tan^{-1} (\sqrt{\cos x}) \quad (B) \quad \frac{1}{2} \log \left| \frac{\sqrt{\cos x} + 1}{\sqrt{\cos x} - 1} \right| + \tan^{-1} (\sqrt{\cos x})$$

$$(C) \quad \log \left| \frac{\sqrt{\cos x} - 1}{\sqrt{\cos x} + 1} \right| + \tan^{-1} (\sqrt{\cos x}) \quad (D) \quad \log \left| \frac{\sqrt{\cos x} + 1}{\sqrt{\cos x} - 1} \right| + \tan^{-1} (\sqrt{\cos x})$$

$$\text{Sol. : } I = \int \frac{\sqrt{\cos x}}{\sin x} dx \quad (0 < x < \frac{\pi}{2})$$

$$\text{Let, } \cos x = u^2.$$

$$\text{So, } -\sin x dx = 2u du$$

$$(u > 0)$$

$$\therefore \frac{dx}{\sin x} = - \frac{2u du}{1 - \cos^2 x}$$

$$\therefore \frac{dx}{\sin x} = - \frac{2u du}{1 - u^4}$$

$$\begin{aligned}
\therefore I &= \int u \left(\frac{-2u}{1 - u^4} \right) du \\
&= \int \frac{2u^2}{u^4 - 1} du = \int \frac{(u^2 + 1) + (u^2 - 1)}{(u^2 - 1)(u^2 + 1)} du \\
&= \int \frac{du}{u^2 - 1} + \int \frac{du}{u^2 + 1}
\end{aligned}$$

$$= \frac{1}{2} \log \left| \frac{u-1}{u+1} \right| + \tan^{-1}(u) + c$$

$$= \frac{1}{2} \log \left| \frac{\sqrt{\cos x} - 1}{\sqrt{\cos x} + 1} \right| + \tan^{-1}(\sqrt{\cos x}) + c$$

Ans. : (A)

$$(119) \int \frac{\tan x + \tan^3 x}{1 + \tan^3 x} dx = \dots + c.$$

$$(A) \quad \frac{1}{3} \log | \tan x + 1 | + \frac{1}{6} \log | \tan^2 x - \tan x + 1 | + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right)$$

$$(B) \quad \frac{1}{3} \log | \tan x + 1 | + \frac{1}{6} \log | \tan^2 x - \tan x + 1 | + \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right)$$

$$(C) \quad -\frac{1}{3} \log | \tan x + 1 | + \frac{1}{6} \log | \tan^2 x - \tan x + 1 | + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right)$$

$$(D) \quad \frac{1}{3} \log | \tan x + 1 | - \frac{1}{6} \log | \tan^2 x - \tan x + 1 | + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right)$$

$$\begin{aligned} \text{Sol. : } I &= \int \frac{\tan x + \tan^3 x}{1 + \tan^3 x} dx \\ &= \int \frac{\tan x (1 + \tan^2 x)}{1 + \tan^3 x} dx \\ &= \int \frac{\tan x \cdot \sec^2 x}{1 + \tan^3 x} dx \end{aligned}$$

$$\text{Let, } \tan x = t.$$

$$\text{So, } \sec^2 x dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{t dt}{1 + t^3} \\ &= \int \frac{t dt}{(1+t)(t^2 - t + 1)} \end{aligned}$$

$$\text{Let, } \frac{t}{(1+t)(t^2 - t + 1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2 - t + 1}$$

$$\therefore t = A(t^2 - t + 1) + (Bt + C)(t + 1)$$

$$t = -1 \quad \Rightarrow -1 = 3A. \quad \text{So, } A = -\frac{1}{3}$$

$$t = 0 \quad \Rightarrow A + C = 0 \Rightarrow C = -A. \quad \text{So, } C = \frac{1}{3}$$

$$t = 1 \quad \Rightarrow A + 2B + 2C = 1$$

$$-\frac{1}{3} + 2B + \frac{2}{3} = 1. \quad \text{So, } B = \frac{1}{3}$$

$$\therefore I = \int \frac{-\frac{1}{3}}{t+1} dt + \int \frac{\frac{1}{3}t + \frac{1}{3}}{t^2 - t + 1} dt$$

$$\begin{aligned}
&= -\frac{1}{3} \int \frac{1}{t+1} dt + \frac{1}{3} \int \frac{\frac{1}{2}(2t-1) + \frac{3}{2}}{(t^2-t+1)} dt \\
&= -\frac{1}{3} \int \frac{1}{t+1} dt + \frac{1}{6} \int \frac{2t-1}{t^2-t+1} dt + \frac{1}{2} \int \frac{dt}{t^2-t+1} \\
&= -\frac{1}{3} \int \frac{1}{t+1} dt + \frac{1}{6} \int \frac{2t-1}{t^2-t+1} dt + \frac{1}{2} \int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= -\frac{1}{3} \log |t+1| + \frac{1}{6} \log |t^2-t+1| + \frac{1}{2} \times \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + c \\
&= -\frac{1}{3} \log |\tan x + 1| + \frac{1}{6} \log |\tan^2 x - \tan x + 1| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2\tan x - 1}{\sqrt{3}} \right) + c
\end{aligned}$$

Ans. : (C)

$$(120) \int \frac{\cos x}{(1 - \sin x)^3 (2 + \sin x)} dx = \dots + c.$$

$$(A) \quad \frac{1}{27} \log \left| \frac{2 + \sin x}{1 - \sin x} \right| + \frac{1}{9(\sin x - 1)} + \frac{1}{6(\sin x - 1)^2}$$

$$(B) \quad \frac{1}{27} \log \left| \frac{2 + \sin x}{1 - \sin x} \right| - \frac{1}{9(1 - \sin x)} + \frac{1}{6(1 - \sin x)^2}$$

$$(C) \quad \frac{1}{27} \log \left| \frac{1 - \sin x}{2 + \sin x} \right| - \frac{1}{9(1 - \sin x)} - \frac{1}{6(1 - 2\sin x)^2}$$

$$(D) \quad \frac{1}{27} \log \left| \frac{1 - \sin x}{2 + \sin x} \right| + \frac{1}{9(\sin x - 1)^2} - \frac{1}{6(\sin x - 1)^2}$$

$$\text{Sol. : I} = \int \frac{\cos x}{(1 - \sin x)^3 (2 + \sin x)} dx$$

$$\text{Let, } \sin x = t.$$

$$\text{So, } \cos x dx = dt$$

$$= \int \frac{dt}{(2+t)(1-t)^3}$$

$$\text{Let, } \frac{1}{(2+t)(1-t)^3} = \frac{A}{2+t} + \frac{B}{1-t} + \frac{C}{(1-t)^2} + \frac{D}{(1-t)^3}$$

$$1 = A(1-t)^3 + B(2+t)(1-t)^2 + C(2+t)(1-t) + D(2+t)$$

$$\text{If } t = -2, \text{ then } 1 = 27A. \quad \text{So, } A = \frac{1}{27}$$

$$\text{If } t = 1, \text{ then } 1 = 3D. \quad \text{So, } D = \frac{1}{3}$$

$$\text{Now, } 1 = \frac{1}{27} (1 - t^3 - 3t + 3t^2) + B(2 - 3t + t^3) + C(2 - t - t^2) + \frac{1}{3}(2 + t)$$

Now, comparing the coefficients of t^3 , $-\frac{1}{27} + B = 0$. So, $B = \frac{1}{27}$

Comparing the coefficients of t^2 , $\frac{1}{9} - C = 0$. So, $C = \frac{1}{9}$

$$\begin{aligned}\therefore \int \frac{dt}{(2+t)(1-t)^3} &= \frac{1}{27} \int \frac{dt}{t+2} + \frac{1}{27} \int \frac{dt}{1-t} + \frac{1}{9} \int (1-t)^{-2} dt + \frac{1}{3} \int (1-t)^{-3} dt \\ &= \frac{1}{27} \log |t+2| - \frac{1}{27} \log |1-t| + \frac{1}{9} \frac{(1-t)^{-1}}{-1} + \frac{1}{3} \frac{(1-t)^{-2}}{2} + c \\ &= \frac{1}{27} \log \left| \frac{2+\sin x}{1-\sin x} \right| - \frac{1}{9(1-\sin x)} + \frac{1}{6(1-\sin x)^2} + c \quad \text{Ans. : (B)}\end{aligned}$$

$$(121) \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx = \dots + c.$$

$$(A) \quad x + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \frac{3}{2} \tan^{-1} \left(\frac{x}{2} \right) \quad (B) \quad x + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + 3 \tan^{-1} \left(\frac{x}{2} \right)$$

$$(C) \quad x + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) - 3 \tan^{-1} \left(\frac{x}{2} \right) \quad (D) \quad x - \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + 3 \tan^{-1} \left(\frac{x}{2} \right)$$

$$\begin{aligned}\text{Sol. : } I &= \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx \\ &= \int \frac{x^4+3x^2+2}{x^4+7x^2+12} dx \\ &= \int \frac{(x^4+7x^2+12)-4x^2-10}{x^4+7x^2+12} dx\end{aligned}$$

$$I = \int 1 dx - 2 \int \frac{2x^2+5}{(x^2+3)(x^2+4)} dx$$

$$I = x - 2I_1, \text{ where } I_1 = \int \frac{2x^2+5}{(x^2+3)(x^2+4)} dx \quad (1)$$

Now, let $x^2 = t$ for partial fractions.

$$\therefore \frac{2t+5}{(t+3)(t+4)} = \frac{A}{t+3} + \frac{B}{t+4}$$

$$\therefore I_1 = \int \frac{-1}{x^2+3} dx + 3 \int \frac{dx}{x^2+4}$$

$$I_1 = \frac{-1}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \frac{3}{2} \tan^{-1} \left(\frac{x}{2} \right) + c_1$$

Substituting in equation (1),

$$I = x - 2 \left[\frac{-1}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \frac{3}{2} \tan^{-1} \left(\frac{x}{2} \right) + c_1 \right]$$

$$= x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3 \tan^{-1} \left(\frac{x}{2} \right) + c \quad (-2c_1 = c) \quad \text{Ans. : (C)}$$

$$(122) \int \frac{\sqrt{x^2+1} (\log(x^2+1) - 2 \log x)}{x^4} dx = \dots + c$$

$$(A) \quad \frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}} \left[\log \left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right] \quad (B) \quad -\frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}} \left[\log \left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right]$$

$$(C) \quad -\frac{1}{3} \left(1 - \frac{1}{x^2}\right)^{\frac{3}{2}} \left[\log \left(1 - \frac{1}{x^2}\right) - \frac{2}{3} \right] \quad (D) \quad -\frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}} \left[\log \left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right]$$

$$\text{Sol. : I} = \int \frac{\sqrt{x^2+1} (\log(x^2+1) - 2 \log x)}{x^4} dx$$

$$= \int x \sqrt{1 + \frac{1}{x^2}} \cdot \log \left(\frac{x^2+1}{x^2} \right) \cdot \frac{dx}{x^4}$$

$$= \int \sqrt{1 + \frac{1}{x^2}} \log \left(1 + \frac{1}{x^2} \right) \cdot \frac{dx}{x^3}$$

$$\text{Let, } 1 + \frac{1}{x^2} = t$$

$$\therefore \frac{-2}{x^3} dx = dt.$$

$$\text{So, } \frac{1}{x^3} dx = \frac{-1}{2} dt$$

$$I = \frac{-1}{2} \int \sqrt{t} \log t dt$$

$$= \frac{-1}{2} \left[\log t \int \sqrt{t} dt - \int \left(\frac{d}{dt} (\log t) \int \sqrt{t} dt \right) dt \right]$$

$$= \frac{-1}{2} \left[(\log t) \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \left(\frac{1}{t} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) dt \right]$$

$$= \frac{-1}{2} \left[\frac{2}{3} \cdot t^{\frac{3}{2}} \log t - \frac{2}{3} \int t^{\frac{1}{2}} dt \right]$$

$$= \frac{-1}{3} t^{\frac{3}{2}} \log t + \frac{1}{3} \times \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{-1}{3} t^{\frac{3}{2}} \log t + \frac{2}{9} t^{\frac{3}{2}} + c$$

$$= \frac{-1}{3} t^{\frac{3}{2}} \left(\log t - \frac{2}{3} \right) + c$$

$$= -\frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}} \left[\log \left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right] + c$$

Ans. : (B)

$$(123) \int \frac{dx}{x(x^n + 1)} dx = \dots + c. \quad (x > 0)$$

$$(A) \quad \frac{1}{n} \log \left(\frac{x^n}{x^n + 1} \right)$$

$$(B) \quad n \log \left(\frac{x^n}{x^n + 1} \right)$$

$$(C) \quad \log \left(\frac{x^n}{x^n + 1} \right)$$

$$(D) \quad \text{None of these} \quad (\text{AIEEE : 2002})$$

$$\text{Sol. : } I = \int \frac{dx}{x(x^n + 1)}$$

$$\text{Let, } x^n + 1 = t. \quad t > 1$$

$$\text{So, } nx^{n-1} dx = dt$$

$$I = \frac{1}{n} \int \frac{n \cdot x^{n-1} dx}{x^n \cdot (x^n + 1)}$$

$$= \frac{1}{n} \int \frac{dt}{(t-1)t}$$

$$= \frac{1}{n} \left(\frac{1}{t-1} - \frac{1}{t} \right) dt$$

$$= \frac{1}{n} [\log(t-1) - \log(t)] + c$$

$$= \frac{1}{n} \log \left(\frac{t-1}{t} \right) + c = \frac{1}{n} \log \left(\frac{x^n}{x^n + 1} \right) + c$$

Ans. : (A)

$$(124) \int \frac{dx}{\cos x - \sin x} = \dots + c$$

$$(A) \quad \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{3\pi}{8} \right) \right|$$

$$(B) \quad \frac{1}{\sqrt{2}} \log \left| \cot \frac{x}{2} \right|$$

$$(C) \quad \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{8} \right) \right|$$

$$(D) \quad \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{3\pi}{8} \right) \right|$$

(AIEEE : 2003)

$$\text{Sol. : } I = \int \frac{dx}{\cos x - \sin x}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x \right)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} \right)}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} \int \frac{dx}{\cos\left(x + \frac{\pi}{4}\right)} \\
&= \frac{1}{\sqrt{2}} \int \sec\left(x + \frac{\pi}{4}\right) dx \\
&= \frac{1}{\sqrt{2}} \log \left| \tan\left(\frac{\pi}{4} + \left(\frac{x}{2} + \frac{\pi}{8}\right)\right) \right| + c = \frac{1}{\sqrt{2}} \log \left| \tan\left(\frac{x}{2} + \frac{3\pi}{8}\right) \right| + c \quad \text{Ans. : (D)}
\end{aligned}$$

(125) If $\int \frac{\sin x}{\sin(x - \alpha)} dx = Ax + B \log |\sin(x - \alpha)| + c$, then (A, B) =

- (A) $(-\sin \alpha, \cos \alpha)$ (B) $(\cos \alpha, \sin \alpha)$ (C) $(\sin \alpha, \cos \alpha)$ (D) $(-\cos \alpha, \sin \alpha)$

(AIEEE-2004)

$$\begin{aligned}
\text{Sol. : I} &= \int \frac{\sin x}{\sin(x - \alpha)} dx \\
&= \int \frac{\sin[(x - \alpha) + \alpha]}{\sin(x - \alpha)} dx \\
&= \int \frac{\sin(x - \alpha) \cos \alpha + \cos(x - \alpha) \sin \alpha}{\sin(x - \alpha)} dx \\
&= \int (\cos \alpha + \sin \alpha \cdot \cot(x - \alpha)) dx \\
&= \cos \alpha \cdot x + \sin \alpha \log |\sin(x - \alpha)| + c
\end{aligned}$$

$$\therefore A = \cos \alpha, B = \sin \alpha.$$

$$\text{So, (A, B) = } (\cos \alpha, \sin \alpha)$$

Ans. : (B)

(126) If $f(x) = \int \left(\frac{x^2 + \sin^2 x}{1 + x^2} \right) \sec^2 x dx$ and $f(0) = 0$, then $f(1) = \dots$ [Online May 19, 2012]

- (A) $\tan 1 - \frac{\pi}{4}$ (B) $\tan 1 + 1$ (C) $\frac{\pi}{4}$ (D) $1 - \frac{\pi}{4}$

$$\begin{aligned}
\text{Sol. : } f(x) &= \int \left(\frac{x^2 + \sin^2 x}{1 + x^2} \right) \sec^2 x dx \\
&= \int \frac{x^2 \sec^2 x + \tan^2 x}{1 + x^2} dx \\
&= \int \frac{x^2(1 + \tan^2 x) + \tan^2 x}{1 + x^2} dx \\
&= \int \frac{x^2}{1 + x^2} dx + \int \tan^2 x dx \\
&= \int \frac{1 + x^2 - 1}{1 + x^2} dx + \int (\sec^2 x - 1) dx
\end{aligned}$$

$$= \int 1 \, dx - \int \frac{1}{1+x^2} \, dx + \int \sec^2 x \, dx - \int 1 \, dx$$

$$= -\tan^{-1}x + \tan x + c$$

Now, $f(0) = 0$, so $f(0) = -\tan^{-1}0 + \tan 0 + c$

$$\therefore c = 0$$

$$\therefore f(x) = -\tan^{-1}x + \tan x$$

$$\text{Again, } f(1) = -\tan^{-1}1 + \tan 1 = \tan 1 - \frac{\pi}{4}$$

Ans. : (A)

$$(127) \int \frac{dx}{\cos x + \sqrt{3} \sin x} = \dots\dots$$

$$(A) \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

$$(B) \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{12} \right) \right| + c$$

$$(C) \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

$$(D) \frac{1}{2} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{12} \right) \right| + c$$

(AIEEE : 2007)

$$\text{Sol. : I} = \int \frac{dx}{\cos x + \sqrt{3} \sin x}$$

$$\therefore \text{I} = \frac{1}{2} \int \frac{dx}{\frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x}$$

$$= \frac{1}{2} \int \frac{dx}{\sin \left(x + \frac{\pi}{6} \right)}$$

$$= \frac{1}{2} \int \operatorname{cosec} \left(x + \frac{\pi}{6} \right) dx$$

$$= \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

Ans. : (C)

$$(128) \sqrt{2} \int \frac{\sin x \, dx}{\sin \left(x - \frac{\pi}{4} \right)} = \dots\dots + c$$

$$(A) x + \log \left| \sin \left(x - \frac{\pi}{4} \right) \right|$$

$$(B) x - \log \left| \cos \left(x - \frac{\pi}{4} \right) \right|$$

$$(C) x + \log \left| \cos \left(x - \frac{\pi}{4} \right) \right|$$

$$(D) x - \log \left| \sin \left(x - \frac{\pi}{4} \right) \right| \quad (\text{AIEEE : 2008})$$

$$\text{Sol. : I} = \sqrt{2} \int \frac{\sin x \, dx}{\sin \left(x - \frac{\pi}{4} \right)}$$

$$\begin{aligned}
&= \sqrt{2} \int \frac{\sin \left[\left(x - \frac{\pi}{4} \right) + \frac{\pi}{4} \right]}{\sin \left(x - \frac{\pi}{4} \right)} dx \\
&= \sqrt{2} \int \frac{\sin \left(x - \frac{\pi}{4} \right) \cos \frac{\pi}{4} + \cos \left(x - \frac{\pi}{4} \right) \sin \frac{\pi}{4}}{\sin \left(x - \frac{\pi}{4} \right)} dx \\
&= \sqrt{2} \int \cos \frac{\pi}{4} dx + \sqrt{2} \sin \frac{\pi}{4} \int \cot \left(x - \frac{\pi}{4} \right) dx \\
&= x + \log \left| \sin \left(x - \frac{\pi}{4} \right) \right| + c
\end{aligned}$$

Ans. : (A)

(129) If $\int \frac{5 \tan x}{\tan x - 2} dx = x + a \log | \sin x - 2 \cos x | + c$, then $a = \dots$

- (A) 1 (B) 2 (C) -1 (D) -2 (AIEEE : 2009)

Sol. : $I = \int \frac{5 \tan x}{\tan x - 2} dx = x + a \log | \sin x - 2 \cos x | + c$

Differentiating both the sides w.r.t. x ,

$$\frac{5 \tan x}{\tan x - 2} = 1 + \frac{a(\cos x + 2 \sin x)}{\sin x - 2 \cos x}$$

$$\therefore \frac{5 \sin x}{\sin x - 2 \cos x} = \frac{\sin x - 2 \cos x + a \cos x + 2a \sin x}{\sin x - 2 \cos x}$$

$$\therefore 5 \sin x = (1 + 2a) \sin x + (a - 2) \cos x$$

$$\therefore 5 = 1 + 2a, \quad a - 2 = 0$$

$$\therefore a = 2$$

Ans. : (B)

(130) If $\int \frac{5 \tan x}{\tan x - 2} dx = x + k \log | \sin x - 2 \cos x | + c$, then $k = \dots$

- (A) -2 (B) 1 (C) 2 (D) -1 (AIEEE : 2012)

Sol. : $I = \int \frac{5 \tan x}{\tan x - 2} dx$

$$= \int \frac{5 \sin x}{\sin x - 2 \cos x} dx$$

Now, let $5 \sin x = P(\sin x - 2 \cos x) + Q \left[\frac{d}{dx}(\sin x - 2 \cos x) \right]$,

$$\therefore 5 \sin x = P(\sin x - 2 \cos x) + Q(\cos x + 2 \sin x)$$

$$\therefore 5 \sin x = (P + 2Q) \sin x + (Q - 2P) \cos x$$

Comparing the coefficients of $\sin x$ and $\cos x$,

$$P + 2Q = 5, \quad Q - 2P = 0. \text{ Solving } P = 1 \text{ and } Q = 2$$

$$\begin{aligned}
 I &= \int \frac{1(\sin x - 2\cos x) + 2(\cos x + 2\sin x)}{(\sin x - 2\cos x)} dx \\
 &= \int 1 dx + 2 \int \frac{\frac{d}{dx}(\sin x - 2\cos x)}{(\sin x - 2\cos x)} dx \\
 &= x + 2 \log |\sin x - 2\cos x| + c
 \end{aligned}$$

$\therefore k = 2$ (Note : For second method see question (129)) **Ans. : (C)**

(131) If $\int \frac{x^2 - x + 1}{x^2 + 1} e^{\cot^{-1}x} dx = A(x) \cdot e^{\cot^{-1}x} + c$, then $A(x) = \dots$

- (A) $-x$ (B) x (C) $\sqrt{1-x}$ (D) $\sqrt{1+x}$

(JEE : 2013)

Sol. : $I = \int \frac{x^2 - x + 1}{x^2 + 1} e^{\cot^{-1}x} dx,$

Let, $\cot^{-1}x = t.$

So, $x = \cot t$

$\therefore \frac{-1}{1+x^2} dx = dt$ and so $\frac{dx}{1+x^2} = -dt$

$$\begin{aligned}
 \therefore I &= -\int (1 + \cot^2 t - \cot t) e^t dt \\
 &= \int (-\operatorname{cosec}^2 t + \cot t) e^t dt \\
 &= e^t \cot t + c \\
 &= x \cdot e^{\cot^{-1}x} + c \\
 \therefore A(x) &= x
 \end{aligned}$$

Ans. : (B)

(132) $\int \frac{x dx}{2 - x^2 + \sqrt{2 - x^2}} = \dots$

- (A) $\log |1 + \sqrt{2 + x^2}| + c$ (B) $-\log |1 + \sqrt{2 - x^2}| + c$
 (C) $-x \log |1 - \sqrt{2 - x^2}| + c$ (D) $x \log |1 - \sqrt{2 + x^2}| + c$

(JEE : 2013)

Sol. : $I = \int \frac{x dx}{2 - x^2 + \sqrt{2 - x^2}}$

($|x| < \sqrt{2}$)

Let, $2 - x^2 = t^2$. ($t > 0$)

So, $-2x dx = 2t dt$.

So, $x dx = -t dt$

$$\begin{aligned}
 \therefore I &= \int \frac{-t dt}{t^2 + t} \\
 &= -\int \frac{dt}{t + 1}
 \end{aligned}$$

$= -\log |t + 1| + c = -\log |\sqrt{2 - x^2} + 1| + c$

Ans. : (B)

Second method :

$$I = \int \frac{x dx}{\sqrt{2-x^2} (\sqrt{2-x^2} + 1)}$$

$$\text{Let } \sqrt{2-x^2} = t.$$

$$\text{So, } \frac{-2x dx}{2\sqrt{2-x^2}} = dt$$

$$\therefore I = \int \frac{-dt}{t+1}$$

$$= -\log |1 + \sqrt{2-x^2}| + c$$

Ans. : (B)

$$(133) \text{ If } \int \frac{dx}{x+x^7} = P(x), \text{ then } \int \frac{x^6 dx}{x+x^7} = \dots\dots$$

$$(A) \log |x| - P(x) + c$$

$$(B) \log |x| + P(x) + c$$

$$(C) x - P(x) + c$$

$$(D) x + P(x) + c$$

(JEE : 2013)

$$\text{Sol. : Now, } P(x) = \int \frac{1}{x+x^7} dx. \text{ Let, } Q(x) = \int \frac{x^6}{x+x^7} dx$$

$$\text{So, } P(x) + Q(x) = \int \frac{1+x^6}{x(1+x^6)} dx$$

$$= \int \frac{1}{x} dx$$

$$= \log |x| + c$$

$$Q(x) = \log |x| - P(x) + c$$

Ans. : (A)

$$(134) \text{ If } \int f(x) dx = \psi(x), \text{ then } \int x^5 f(x^3) dx = \dots\dots$$

$$(A) \frac{1}{3} x^3 \psi(x^3) - 3 \int x^3 \psi(x^3) dx + c$$

$$(B) \frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + c$$

$$(C) \frac{1}{3} [x^3 \psi(x^3) - \int x^5 \psi(x^3) dx] + c$$

$$(D) \frac{1}{3} [x^3 \psi(x^3) - \int x^2 \psi(x^3) dx] + c$$

(JEE : 2014)

$$\text{Sol. : } I = \int x^5 f(x^3) dx$$

$$= \frac{1}{3} \int t \cdot f(t) dt$$

$$(\text{Let, } x^3 = t, \text{ So, } 3x^2 dx = dt)$$

$$= \frac{1}{3} \left[t \int f(t) dt - \int \left(\frac{d}{dt} (t) \int f(t) dt \right) dt \right]$$

$$= \frac{1}{3} \left[t \cdot \psi(t) - \int 1 \cdot \psi(t) dt \right]$$

$$= \frac{1}{3} t \cdot \psi(t) - \frac{1}{3} \int \psi(t) dt$$

$$= \frac{1}{3} x^3 \psi(x^3) - \frac{1}{3} \int \psi(x^3) 3x^2 dx$$

$$= \frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + c$$

Ans. : (B)

$$(135) \int \left(1 + x - \frac{1}{x}\right) \cdot e^{\left(x + \frac{1}{x}\right)} dx = \dots$$

$$(A) (x - 1) e^{x + \frac{1}{x}} + c$$

$$(B) x \cdot e^{x + \frac{1}{x}} + c$$

$$(C) (x + 1) e^{x + \frac{1}{x}} + c$$

$$(D) -x \cdot e^{x + \frac{1}{x}} + c$$

(JEE : 2014)

$$\begin{aligned} \text{Sol. : } I &= \int \left(1 + x - \frac{1}{x}\right) \cdot e^{\left(x + \frac{1}{x}\right)} dx \\ &= \int \left[e^{x + \frac{1}{x}} + \left(x - \frac{1}{x}\right) e^{x + \frac{1}{x}} \right] dx \\ &= \int \left\{ e^{x + \frac{1}{x}} + x \left(1 - \frac{1}{x^2}\right) e^{x + \frac{1}{x}} \right\} dx \end{aligned}$$

$$\text{Now, let } e^{x + \frac{1}{x}} = f(x), \text{ then } f'(x) = e^{x + \frac{1}{x}} \left(1 - \frac{1}{x^2}\right)$$

$$\begin{aligned} \therefore I &= \int [f(x) + x \cdot f'(x)] dx \\ &= x \cdot f(x) + c \\ &= x \cdot e^{x + \frac{1}{x}} + c \end{aligned}$$

$$\left(\int (f(x) + x \cdot f'(x)) dx = x \cdot f(x) + c \right)$$

Ans. : (B)

$$(136) \int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x} dx = \dots$$

$$(A) \frac{1}{2} \sin 2x + c$$

$$(B) -\frac{1}{2} \sin 2x + c$$

$$(C) -\frac{1}{2} \sin x + c$$

$$(D) -\sin^2 x + c$$

(JEE : 2014)

$$\begin{aligned} \text{Sol. : } I &= \int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x} dx \\ &= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x} dx \\ &= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^4 x + \cos^4 x)} dx \\ &= \int (\sin^4 x - \cos^4 x) dx \\ &= \int (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) dx \\ &= -\int \cos 2x dx = -\frac{\sin 2x}{2} + c \end{aligned}$$

Ans. : (B)

$$(137) \int x \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right) dx = \dots$$

($x > 0$).

$$(A) -x + (1 + x^2) \tan^{-1} x + c$$

$$(B) x - (1 + x^2) \cot^{-1} x + c$$

$$(C) -x + (1 + x^2) \cot^{-1} x + c$$

$$(D) x - (1 + x^2) \tan^{-1} x + c$$

(JEE : 2014)

$$\text{Sol. : } I = \int x \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) dx$$

$$\text{Let } x = \tan \theta.$$

$$\text{So, } dx = \sec^2 \theta d\theta \quad (0 < \theta < \frac{\pi}{2})$$

$$\therefore I = \int \tan \theta \cdot \cos^{-1}(\cos 2\theta) \cdot \sec^2 \theta d\theta \quad (0 < 2\theta < \pi)$$

$$= 2 \int \theta \cdot \tan \theta \sec^2 \theta d\theta$$

$$= 2 \left[\theta \int \tan \theta \sec^2 \theta d\theta - \int \left(\frac{d}{d\theta} (\theta) \int \tan \theta \sec^2 \theta d\theta \right) d\theta \right]$$

$$= 2 \left[\theta \cdot \frac{\tan^2 \theta}{2} - \int \left(1 \cdot \frac{\tan^2 \theta}{2} \right) d\theta \right]$$

$$= \theta \cdot \tan^2 \theta - \int (\sec^2 \theta - 1) d\theta$$

$$= \theta \cdot \tan^2 \theta - \tan \theta + \theta + c$$

$$= x^2 \cdot \tan^{-1} x - x + \tan^{-1} x + c$$

$$= -x + (1 + x^2) \tan^{-1} x + c$$

Ans. : (A)

$$(138) \int \frac{\sin^2 x \cdot \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx = \dots (x > 0).$$

$$(A) \frac{1}{1 + \cot^3 x} + c \quad (B) \frac{-1}{3(1 + \tan^3 x)} + c \quad (C) \frac{\sin^3 x}{1 + \cos^3 x} + c \quad (D) \frac{\cos^3 x}{3(1 + \sin^3 x)} + c$$

(JEE : 2014)

$$\text{Sol. : } I = \int \frac{\sin^2 x \cdot \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$$

$$= \int \frac{\sin^2 x \cos^2 x}{\cos^6 x (\tan^3 x + 1)^2} dx$$

$$= \int \frac{\sec^2 x \tan^2 x}{(\tan^3 x + 1)^2} dx,$$

$$= \frac{1}{3} \int \frac{dt}{(t+1)^2} \quad \text{Let } \tan^3 x = t. \quad \text{So, } 3 \tan^2 x \sec^2 x dx = dt$$

$$= \frac{1}{3} \left[\frac{-1}{t+1} \right] + c$$

$$= \frac{-1}{3(1 + \tan^3 x)} + c$$

Ans. : (B)

$$(139) \text{ If } m \neq 0 \text{ and } \int \frac{x^{5m-1} + 2x^{4m-1}}{(x^{2m} + x^m + 1)^3} dx = f(x) + c, \text{ then } f(x) = \dots$$

$$(A) \frac{x^{5m}}{2m(x^{2m} + x^m + 1)^2}$$

$$(B) \frac{x^{4m}}{2m(x^{2m} + x^m + 1)^2}$$

$$(C) \frac{2m(x^{5m} + x^{4m})}{(x^{2m} + x^m + 1)}$$

$$(D) \frac{x^{5m} + x^{4m}}{2m(x^{2m} + x^m + 1)^2}$$

(JEE : 2014)

$$\begin{aligned}
 \text{Sol. : I} &= \int \frac{x^{5m-1} + 2x^{4m-1}}{(x^{2m} + x^m + 1)^3} dx \\
 &= \int \frac{x^{5m-1} + 2x^{4m-1}}{x^{6m} \left(1 + \frac{1}{x^m} + \frac{1}{x^{2m}}\right)^3} dx \\
 &= \int \frac{(x^{-m-1} + 2x^{-2m-1})}{\left(1 + \frac{1}{x^m} + \frac{1}{x^{2m}}\right)^3} dx
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } 1 + x^{-m} + x^{-2m} &= t \\
 (-m \cdot x^{-m-1} - 2m \cdot x^{-2m-1})dx &= dt \\
 -m(x^{-m-1} + 2x^{-2m-1})dx &= dt
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{ I} &= \frac{-1}{m} \int \frac{dt}{t^3} \\
 &= \frac{-1}{m} \int t^{-3} dt = \frac{-1}{m} \cdot \frac{t^{-2}}{-2} + c = \frac{1}{2m(t)^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{ I} &= \frac{1}{2m \left(1 + \frac{1}{x^m} + \frac{1}{x^{2m}}\right)^2} + c \\
 &= \frac{x^{4m}}{2m(x^{2m} + x^m + 1)^2} + c
 \end{aligned}$$

Ans. : (B)

$$(140) \int \frac{dx}{x^2 (x^4 + 1)^{\frac{3}{4}}} = \dots$$

$$\text{(A) } \left(\frac{x^4 + 1}{x^4}\right)^{\frac{1}{4}} + c \quad \text{(B) } (x^4 + 1)^{\frac{1}{4}} + c \quad \text{(C) } -(x^4 + 1)^{\frac{1}{4}} + c \quad \text{(D) } -\left(\frac{x^4 + 1}{x^4}\right)^{\frac{1}{4}} + c$$

(JEE : 2015)

$$\text{Sol. : I} = \int \frac{dx}{x^2 (x^4 + 1)^{\frac{3}{4}}} = \int \frac{dx}{x^5 \left(1 + \frac{1}{x^4}\right)^{\frac{3}{4}}}$$

$$\text{Let, } 1 + \frac{1}{x^4} = t.$$

$$\text{So, } -\frac{4}{x^5} dx = dt$$

$$\text{I} = \frac{-1}{4} \int \frac{dt}{t^{\frac{3}{4}}}$$

$$= \frac{-1}{4} \left[\frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} \right] + c = \frac{-1}{4} \left[\frac{t^{\frac{1}{4}}}{\frac{1}{4}} \right] + c = -t^{\frac{1}{4}} + c = -\left(\frac{x^4 + 1}{x^4}\right)^{\frac{1}{4}} + c$$

Ans. : (D)

: Questions with More Than One Correct Answers :

(141) Primitive of $\sin 6x$ is

(A) $\frac{1}{3}(\sin^6 x - \sin^3 x) + c$

(B) $\frac{1}{3} \sin^2 3x + c$

(C) $-\frac{1}{3} \cos^2 3x + c$

(D) $\frac{1}{3} \sin\left(3x + \frac{\pi}{7}\right) \sin\left(3x - \frac{\pi}{7}\right) + c$

Sol. : $\int \sin 6x \, dx = -\frac{1}{6} \cos 6x + c'$

$$= -\frac{1}{6} (1 - 2\sin^2 3x) + c'$$

$$= -\frac{1}{6} + \frac{1}{3} \sin^2 3x + c'$$

$$= \frac{1}{3} \sin^2 3x + c$$

$$(c' - \frac{1}{6} = c)$$

(B)

Now, $-\frac{1}{6} \cos 6x + c' = -\frac{1}{6} (2\cos^2 3x - 1) + c'$

$$= -\frac{1}{3} \cos^2 3x + \frac{1}{6} + c'$$

$$= -\frac{1}{3} \cos^2 3x + c$$

$$(\frac{1}{6} + c' = c)$$

(C)

So, $\frac{d}{dx} \left[\frac{1}{3} \sin\left(3x + \frac{\pi}{7}\right) \sin\left(3x - \frac{\pi}{7}\right) \right]$

$$= \frac{1}{6} \frac{d}{dx} \left[2 \sin\left(3x + \frac{\pi}{7}\right) \sin\left(3x - \frac{\pi}{7}\right) \right]$$

$$= \frac{1}{6} \frac{d}{dx} \left(\cos \frac{2\pi}{7} - \cos 6x \right)$$

$$= \frac{1}{6} (0 - (-\sin 6x) 6)$$

$$= \sin 6x$$

$$\therefore \int \sin 6x \, dx = \frac{1}{3} \sin\left(3x + \frac{\pi}{7}\right) \sin\left(3x - \frac{\pi}{7}\right) + c$$

(D) **Ans. : (B, C, D)**

(142) If $P(x) = \lim_{n \rightarrow \infty} e^{x \tan\left(\frac{1}{n}\right) \cdot \log\left(\frac{1}{n}\right)}$ and $\int \frac{P(x)}{\sqrt[3]{\sin^{11} x \cos x}} \, dx = Q(x) + c$, then

(A) $Q\left(\frac{\pi}{4}\right) = \frac{3}{2}$

(B) $Q(x)$ is continuous for all x .

(C) $Q(x)$ is not continuous at infinitely many values of x .

(D) $Q\left(\frac{\pi}{4}\right) = \frac{-15}{8}$

Sol. : $\lim_{n \rightarrow \infty} \tan\left(\frac{1}{n}\right) \cdot \log\left(\frac{1}{n}\right)$

$$= \lim_{n \rightarrow \infty} \frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} \cdot \frac{\log n^{-1}}{n}$$

$$= \lim_{n \rightarrow \infty} -\frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} \cdot \frac{\log n}{n} = -1 \times \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{n}}{\frac{1}{n}}\right) = -1 \times 0 = 0$$

$$\therefore P(x) = e^0 = 1$$

$$\begin{aligned}\therefore \int \frac{P(x)}{\sqrt[3]{\sin^{11} x \cos x}} dx &= \int \frac{1}{\sin^{\frac{11}{3}} x \cos^{\frac{1}{3}} x} dx \\&= \int \sin^{-\frac{11}{3}} x \cdot \cos^{-\frac{1}{3}} x dx \\&= \int (\tan x)^{\frac{-11}{3}} \cdot \cos^{-4} x dx \\&= \int (\tan x)^{\frac{-11}{3}} \cdot \sec^4 x dx \\&= \int (\tan x)^{\frac{-11}{3}} (1 + \tan^2 x) \cdot \sec^2 x dx \\&= \int (\tan x)^{\frac{-11}{3}} \sec^2 x dx + \int (\tan x)^{\frac{-5}{3}} \sec^2 x dx \\&= \frac{(\tan x)^{\frac{-11}{3}+1}}{\frac{-11}{3}+1} + \frac{(\tan x)^{\frac{-5}{3}+1}}{\frac{-5}{3}+1} + c\end{aligned}$$

$$\therefore Q(x) = \frac{-3}{8}(\tan x)^{\frac{-8}{3}} - \frac{3}{2}(\tan x)^{\frac{-2}{3}}$$

$$\therefore Q\left(\frac{\pi}{4}\right) = \frac{-3}{8} - \frac{3}{2} = \frac{-15}{8} \quad (\text{D})$$

$$\text{Also } Q(x) \text{ is not continuous at } x = n\pi + \frac{\pi}{2}. \quad (\text{C})$$

Ans. : (C), (D)

(143) If $\int \sin(\log x) dx = P(x) ((\sin Q(x) - \cos R(x))) + c$, then

$$(A) \lim_{x \rightarrow 2} P(x) = 2 \quad (B) \lim_{x \rightarrow 1} \frac{Q(x)}{R(x)} = 1 \quad (C) Q(e^3) = 3 \quad (D) R(e^5) = 5$$

$$\text{Sol. : } I = \int \sin(\log x) dx$$

$$\text{Let, } \log x = t.$$

$$\therefore x = e^t. \quad \text{So, } dx = e^t dt$$

$$\therefore I = \int e^t \sin t dt$$

$$= \frac{e^t}{2} (\sin t - \cos t) + c$$

$$= \frac{x}{2} [\sin(\log x) - \cos(\log x)] + c$$

$$= P(x) [\sin(Q(x)) - \cos(R(x))] + c$$

$$P(x) = \frac{x}{2}, \quad Q(x) = \log x, \quad R(x) = \log x$$

$$\text{Option A : } \lim_{x \rightarrow 2} P(x) = \lim_{x \rightarrow 2} \frac{x}{2} = \frac{2}{2} = 1 \neq 2. \text{ A is not true.}$$

$$\text{Option B : } \lim_{x \rightarrow 1} \frac{Q(x)}{R(x)} = \lim_{x \rightarrow 1} \frac{\log x}{\log x} = 1. \text{ So, B is true.}$$

$$\text{Option C : } Q(e^3) = \log e^3 = 3 \text{ is true.}$$

$$\text{Option D : } R(e^5) = \log e^5 = 5 \text{ is true.}$$

Ans. : (B), (C), (D)

$$(144) \int (\sin 2x - \cos 2x) dx = \dots + c$$

$$(A) -\frac{\cos 2x}{2} - \frac{\sin 2x}{2}$$

$$(B) -\frac{1}{\sqrt{2}} \sin\left(2x + \frac{\pi}{4}\right)$$

$$(C) \frac{1}{\sqrt{2}} \sin\left(2x + \frac{9\pi}{4}\right)$$

$$(D) \frac{1}{\sqrt{2}} \sin\left(2x + \frac{5\pi}{4}\right)$$

$$\begin{aligned} \text{Sol. : I} &= \int (\sin 2x - \cos 2x) dx \\ &= -\frac{\cos 2x}{2} - \frac{\sin 2x}{2} \end{aligned}$$

Ans. : (A)

$$\begin{aligned} \text{Option B : } -\frac{1}{\sqrt{2}} \sin\left(2x + \frac{\pi}{4}\right) &= -\frac{1}{\sqrt{2}} \left(\sin 2x \cos \frac{\pi}{4} + \cos 2x \sin \frac{\pi}{4} \right) \\ &= -\frac{1}{\sqrt{2}} \left(\sin 2x \cdot \frac{1}{\sqrt{2}} + \cos 2x \cdot \frac{1}{\sqrt{2}} \right) \\ &= -\frac{1}{2} \sin 2x - \frac{1}{2} \cos 2x \end{aligned}$$

Ans. : (B)

$$\begin{aligned} \text{Option C : } \frac{1}{\sqrt{2}} \sin\left(2x + \frac{9\pi}{4}\right) &= \frac{1}{\sqrt{2}} \sin\left(2\pi + \left(2x + \frac{\pi}{4}\right)\right) \\ &= \frac{1}{\sqrt{2}} \sin\left(2x + \frac{\pi}{4}\right) \neq -\frac{1}{\sqrt{2}} \sin\left(2x + \frac{\pi}{4}\right) \end{aligned}$$

$$\begin{aligned} \text{Option D : } \frac{1}{\sqrt{2}} \sin\left(2x + \frac{5\pi}{4}\right) &= \frac{1}{\sqrt{2}} \sin\left(\pi + \left(2x + \frac{\pi}{4}\right)\right) \\ &= -\frac{1}{\sqrt{2}} \sin\left(2x + \frac{\pi}{4}\right) \end{aligned}$$

Ans. : (A), (B), (D)

$$(145) \int \sin^{\frac{-3}{2}} x \sin^{\frac{-1}{2}} (x + \theta) dx = \dots + c$$

$$(A) \frac{-2 \sin x}{\sin \theta \sin (x + \theta)}$$

$$(B) \frac{-2}{\sqrt{\sin \theta}} (\cos \theta + \cot x)^{\frac{1}{2}}$$

$$(C) \frac{-2}{\sin \theta} \left(\frac{\sin (\theta - x)}{\sin x} \right)^{\frac{1}{2}}$$

$$(D) \frac{-2}{\sin \theta} \left(\frac{\sin (\theta + x)}{\sin x} \right)^{\frac{1}{2}}$$

$$\begin{aligned} \text{Sol. : I} &= \int \sin^{\frac{-3}{2}} x \sin^{\frac{-1}{2}} (x + \theta) dx \\ &= \int \frac{1}{\sqrt{\sin^3 x \sin (x + \theta)}} dx \\ &= \int \frac{dx}{\sqrt{\sin^3 x (\sin x \cos \theta + \cos x \sin \theta)}} \\ &= \int \frac{\operatorname{cosec}^2 x dx}{\sqrt{\cos \theta + \cot x \sin \theta}} \\ &= \frac{1}{\sqrt{\sin \theta}} \int \frac{\operatorname{cosec}^2 x dx}{\sqrt{\cot \theta + \cot x}} \end{aligned}$$

Let, $\cot x = t$

$$\therefore -\operatorname{cosec}^2 x \, dx = dt.$$

$$\text{So, } \operatorname{cosec}^2 x \, dx = -dt$$

$$\begin{aligned} \therefore I &= \frac{-1}{\sqrt{\sin \theta}} \int (\cot \theta + t)^{\frac{-1}{2}} dt \\ &= \frac{-1}{\sqrt{\sin \theta}} \times \frac{(\cot \theta + t)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c \\ &= \frac{-2}{\sqrt{\sin \theta}} (\cot \theta + \cot x)^{\frac{1}{2}} + c \quad \text{(B)} \\ &= \frac{-2}{\sin \theta} (\cos \theta + \sin \theta \cdot \cot x)^{\frac{1}{2}} + c \\ &= \frac{-2}{\sin \theta} \frac{(\sin x \cdot \cos \theta + \cos x \cdot \sin \theta)^{\frac{1}{2}}}{\sqrt{\sin x}} + c \\ &= \frac{-2}{\sin \theta} \left(\frac{\sin (\theta + x)}{\sin x} \right)^{\frac{1}{2}} + c \quad \text{(D)} \end{aligned}$$

Ans. : (B, D)

$$(146) \int (\cos^{-1} x + \cos^{-1} \sqrt{1-x^2}) \, dx = Ax + P(x) \sin^{-1} x - 2\sqrt{1-x^2} + c, \forall x \in [-1, 0)$$

$$(A) \quad P(x) = x \quad (B) \quad P(x) = -2x \quad (C) \quad A = \frac{\pi}{4} \quad (D) \quad A = \frac{\pi}{2}$$

Sol. : $x \in [-1, 0) \Rightarrow -1 \leq x < 0$

$$\therefore \cos^{-1} \sqrt{1-x^2} = -\sin^{-1} x$$

$$\begin{aligned} I &= \int (\cos^{-1} x + \cos^{-1} \sqrt{1-x^2}) \, dx \\ &= \int (\cos^{-1} x - \sin^{-1} x) \, dx \\ &= \int \left(\frac{\pi}{2} - 2\sin^{-1} x \right) \, dx \\ &= \int \frac{\pi}{2} dx - 2 \left\{ \sin^{-1} x \int 1 \, dx - \int \left(\frac{d}{dx} \sin^{-1} x \int 1 \, dx \right) dx \right\} \\ &= \frac{\pi}{2} x - 2 \left\{ x \cdot \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx \right\} \\ &= \frac{\pi}{2} x - 2x \sin^{-1} x + 2(-\sqrt{1-x^2}) + c \\ &= \frac{\pi}{2} x - 2x \sin^{-1} x - 2\sqrt{1-x^2} + c \end{aligned}$$

$$\therefore A = \frac{\pi}{2} \text{ and } P(x) = -2x$$

Ans. : (B, D)

$$(147) \int \frac{dx}{\sin x + \cos x} = \dots\dots\dots$$

$$(A) \quad \frac{1}{\sqrt{2}} \log \left| \frac{\tan \frac{x}{2} - 1 - \sqrt{2}}{\tan \frac{x}{2} - 1 + \sqrt{2}} \right| + c$$

$$(B) \quad \frac{1}{\sqrt{2}} \log \left| \frac{\tan \frac{x}{2} - 1 + \sqrt{2}}{\tan \frac{x}{2} - 1 - \sqrt{2}} \right| + c$$

$$(C) \quad \frac{1}{\sqrt{2}} \log \left(\tan \left(\frac{x}{2} + \frac{\pi}{8} \right) \right) + c$$

$$(D) \quad \frac{1}{\sqrt{2}} \log \left| \tan \left(x + \frac{\pi}{4} \right) \right| + c$$

$$\text{Sol. : I} = \int \frac{dx}{\sin x + \cos x}$$

$$\text{Let, } \tan \frac{x}{2} = t. \quad \text{So, } dx = \frac{2dt}{1+t^2} \text{ and } \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore \quad \text{I} = \int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{2t + 1 - t^2}$$

$$= 2 \int \frac{dt}{2 - (t^2 - 2t + 1)}$$

$$= 2 \int \frac{dt}{(\sqrt{2})^2 - (t-1)^2}$$

$$= 2 \times \frac{1}{2\sqrt{2}} \log \left| \frac{t-1+\sqrt{2}}{t-1-\sqrt{2}} \right| + c$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{\tan \frac{x}{2} - 1 + \sqrt{2}}{\tan \frac{x}{2} - 1 - \sqrt{2}} \right| + c$$

(B)

$$\text{Also, I} = \int \frac{dx}{\sin x + \cos x}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sin \left(x + \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sqrt{2}} \int \operatorname{cosec} \left(x + \frac{\pi}{4} \right) dx$$

$$= \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) \right| + c$$

(C)

Ans. : (B), (C)

: Assertion and Reason Type Questions :

Explanation : Each of these questions contains statement 1 (Assertion) and statement 2 (Reason). Each question has four choices, option (A), (B), (C) and (D), which are as follows. You have to select the correct choice.

- (A) Statement 1 is true, statement 2 is true. Statement 2 is the correct explanation for statement 1.
 (B) Statement 1 is true, statement 2 is true. Statement 2 is not a correct explanation for statement 1.
 (C) Statement 1 is true, statement 2 is false.
 (D) Statement 1 is false, statement 2 is true.

(148) **Statement 1 :** $\int \tan 5x \tan 3x \tan 2x \, dx = \frac{1}{5} \log |\sec 5x| - \frac{1}{3} \log |\sec 3x| - \frac{1}{2} \log |\sec 2x| + c$

Statement 2 : $\tan 5x - \tan 3x - \tan 2x = \tan 5x \tan 3x \tan 2x$

Sol. : $5x = 3x + 2x$

$$\therefore \tan 5x = \tan (3x + 2x) = \frac{\tan 3x + \tan 2x}{1 - \tan 3x \tan 2x}$$

$$\therefore \tan 5x - \tan 5x \tan 3x \tan 2x = \tan 3x + \tan 2x$$

$$\therefore \tan 5x - \tan 3x - \tan 2x = \tan 5x \tan 3x \tan 2x$$

$$\therefore \text{Statement 2 is true.}$$

$$\begin{aligned} \text{Now, } \int \tan 5x \tan 3x \tan 2x \, dx &= \int (\tan 5x - \tan 3x - \tan 2x) \, dx \\ &= \frac{1}{5} \log |\sec 5x| - \frac{1}{3} \log |\sec 3x| - \frac{1}{2} \log |\sec 2x| + c \end{aligned}$$

$$\therefore \text{Statement 1 is true.}$$

$$\therefore \text{Statement 1 and statement 2 both are true and statement 2 is the correct explanation of statement 1.}$$

Ans. : (A)

(149) **Statement 1 :** If $F(x) = \int \sin^2 x \, dx$, $\forall x \in \mathbb{R}$, then $F(x + \pi) = F(x)$.

Statement 2 : $\sin^2(\pi + x) = \sin^2 x$, $\forall x \in \mathbb{R}$

Sol. : $F(x) = \int \sin^2 x \, dx$

$$\begin{aligned} &= \int \left(\frac{1 - \cos 2x}{2} \right) dx \\ &= \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right] + c' \\ &= \frac{1}{4} [2x - \sin 2x] + c' \end{aligned}$$

$$\begin{aligned} \text{Now, } F(x + \pi) &= \frac{1}{4} [2(x + \pi) - \sin(2x + 2\pi)] + c' \\ &= \frac{1}{4} [2x + 2\pi - \sin 2x] + c' \\ &= \frac{1}{4} [2x - \sin 2x] + \frac{\pi}{2} + c' \\ &= \frac{1}{4} [2x - \sin 2x] + c \quad c = \left(\frac{\pi}{2} + c' \right) \end{aligned}$$

Another method :

Statement 2 is true as π is the principal period of $\sin^2 x$.

$$\begin{aligned} F(x + \pi) &= \int \sin^2(x + \pi) \, dx \\ &= \int \sin^2 x \, dx \\ &= F(x) \end{aligned}$$

$$\therefore \text{Statement 1 is true.}$$

Ans. : (A)

$$F(x + \pi) = F(x)$$

∴ **Statement 1 is true.**

Also, $\sin^2(\pi + x) = \sin^2 x$, $\forall x \in \mathbb{R}$ because π is the principle period of $\sin^2 x$.

∴ **Statement 2 is true.**

∴ Statement 1 and 2 both are true and statement 2 is the correct explanation of statement 1. **Ans. : (A)**

$$(150) \quad \text{Statement 1 : } \int \frac{(3-2x) dx}{\sqrt{4+2x-x^2}} = 2\sqrt{4+2x-x^2} + \sin^{-1} \left(\frac{x-1}{\sqrt{5}} \right) + c$$

$$\text{Statement 2 : } \int \frac{1}{\sqrt{a^2-x^2}} dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c \quad (a > 0)$$

$$\begin{aligned} \text{Sol. : } F(x) &= \int \frac{(3-2x) dx}{\sqrt{4+2x-x^2}} \\ &= \int \frac{(2-2x)}{\sqrt{4+2x-x^2}} dx + \int \frac{dx}{\sqrt{4+2x-x^2}} \\ &= \int \frac{(2-2x)}{\sqrt{4+2x-x^2}} dx + \int \frac{1}{\sqrt{(\sqrt{5})^2 - (x-1)^2}} dx \\ &= 2\sqrt{4+2x-x^2} + \sin^{-1} \left(\frac{x-1}{\sqrt{5}} \right) + c \end{aligned}$$

∴ **Statement 1 is true.**

$$\text{Now, } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c \quad (a > 0)$$

∴ **Statement 2 is not true.**

Ans. : (C)

$$(151) \quad \text{Statement 1 : } \int \frac{e^x(1+\sin x)}{1+\cos x} dx = e^x \tan \frac{x}{2} + c$$

$$\text{Statement 2 : } \int e^x (f(x) + f'(x)) dx = e^x f(x) + c$$

$$\begin{aligned} \text{Sol. : } I &= \int \frac{e^x(1+\sin x)}{1+\cos x} dx \\ &= \int \frac{e^x(1+2\sin \frac{x}{2} \cos \frac{x}{2})}{2\cos^2 \frac{x}{2}} dx \\ &= \int e^x \left(\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx \\ &= \int e^x \left[\tan \frac{x}{2} + \frac{d}{dx} \left(\tan \frac{x}{2} \right) \right] dx \\ &= e^x \tan \frac{x}{2} \end{aligned}$$

∴ Statement 1 and statement 2 both are true and statement 2 is the correct explanation of statement 1. **Ans. : (A)**

(152) **Statement 1** : If $f(n) = \int \cot^n x \, dx$, then $5[f(6) + f(4)] = -\cot^5 x$

Statement 2 : If $f(n) = \int \cot^n x \, dx$, then $f(n) = \frac{\cot^{n-1} x}{n-1} - f(n-2)$, where $n \geq 2$.

$$\begin{aligned} \text{Sol. : } f(n) &= \int \cot^n x \, dx \\ &= \int \cot^{n-2} x \cot^2 x \, dx \\ &= \int \cot^{n-2} x \cdot (\operatorname{cosec}^2 x - 1) \, dx \\ &= \int (\cot x)^{n-2} \cdot \operatorname{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx \end{aligned}$$

$$\therefore f(n) = -\frac{(\cot x)^{n-1}}{n-1} - f(n-2)$$

$\therefore$ **Statement 2 is not true.**

Now, let $n = 6$,

$$f(6) = -\frac{\cot^5 x}{5} - f(4)$$

$$5(f(6) + f(4)) = -\cot^5 x$$

$\therefore$ **Statement 1 is true.**

Ans. : (C)

(153) **Statement 1** : If $\int \frac{1}{f(x)} \, dx = \log (f(x))^2 + c$, then $f(x) = \frac{x}{2} + c$, where c is an arbitrary constant.

Statement 2 : When $f(x) = \frac{x}{2}$, then $\int \frac{1}{f(x)} \, dx = \frac{1}{2} \log |x| + c$.

$$\text{Sol. : } \int \frac{1}{f(x)} \, dx = \log (f(x))^2 + c$$

Differentiating w.r.t. x on both the sides,

$$\frac{1}{f(x)} = \frac{1}{(f(x))^2} \cdot 2f(x) \cdot f'(x) + 0$$

$$\therefore f'(x) = \frac{1}{2}$$

$$\therefore f(x) = \frac{x}{2} + c$$

$\therefore$ **Statement 1 is true.**

$$\text{Now, } \int \frac{1}{f(x)} \, dx = \int \frac{2}{x} \, dx = 2 \int \frac{1}{x} \, dx = 2 \log |x| + c.$$

$\therefore$ **Statement 2 is not true.**

Ans. : (C)

(154) **Statement 1** : $\int \frac{x^4 + 1}{x^6 + 1} \, dx = \tan^{-1} x + \frac{1}{3} \tan^{-1} x^3 + c$

Statement 2 : $\int \frac{x \sin x + \cos x}{x(x + \cos x)} \, dx = \log (x^2 + x \cos x) + c$

$$\text{Sol. : } \int \frac{x^4 + 1}{x^6 + 1} \, dx = \int \frac{(x^4 - x^2 + 1) + x^2}{(x^2 + 1)(x^4 - x^2 + 1)} \, dx$$

$$\begin{aligned}
&= \int \frac{1}{x^2 + 1} dx + \int \frac{x^2}{x^6 + 1} dx \\
&= \int \frac{1}{x^2 + 1^2} dx + \int \frac{x^2}{(x^3)^2 + 1} dx \\
&= \tan^{-1} x + \frac{1}{3} \tan^{-1} x^3 + c
\end{aligned}$$

∴ Statement 1 is true.

$$\begin{aligned}
\text{Now, } \int \frac{x \sin x + \cos x}{x(x + \cos x)} dx &= \int \frac{(\cos x + x) - x(1 - \sin x)}{x(x + \cos x)} dx \\
&= \int \frac{1}{x} dx - \int \frac{1 - \sin x}{x + \cos x} dx \\
&= \log |x| - \log |x + \cos x| + c \\
&= \log \left| \frac{x}{x + \cos x} \right| + c
\end{aligned}$$

∴ Statement 2 is not true.

Ans. : (C)

(155) **Statement 1** : If $\int \frac{2^x}{\sqrt{1-4^x}} dx = k \sin^{-1}(2^x) + c$, then $k = \frac{1}{\log 2}$

Statement 2 : $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c \quad (a > 0)$

Sol. : I = $\int \frac{2^x}{\sqrt{1-4^x}} dx = \int \frac{2^x}{\sqrt{1-(2^x)^2}} dx$

Let, $2^x = t$. So, $2^x \log 2 dx = dt$. Hence, $2^x dx = \frac{1}{\log_e 2} dt$

$$I = \int \frac{dt}{\log 2 (\sqrt{1-t^2})} = \frac{1}{\log 2} \sin^{-1} t + c = \frac{1}{\log 2} \sin^{-1} (2^x) + c$$

∴ $k = \frac{1}{\log 2}$

∴ Statement 1 and statement 2 both are true and statement 2 is the correct explanation of statement 1.

Ans. : (A)

(156) **Statement 1** : $\int \frac{ax^{a-1} + a^x \log a}{a^x + x^a} dx = \log |a^x + x^a| + c$

Statement 2 : $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$

Sol. : I = $\int \frac{ax^{a-1} + a^x \log a}{a^x + x^a} dx$

Let, $a^x + x^a = t$. So, $(a^x \log a + a \cdot x^{a-1}) dx = dt$

∴ I = $\int \frac{1}{t} dt = \log |t| + c = \log |a^x + x^a| + c$

∴ Statement 1 and statement 2 are both true and statement 2 is the correct explanation of statement 1.

Ans. : (A)

