

$$\begin{aligned}
&= 4 \left\{ \left[\theta \cdot \frac{\sec^4 \theta}{4} \right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \left(1 \cdot \frac{\sec^4 \theta}{4} \right) d\theta \right\} \quad \left(\int (\sec \theta)^3 (\sec \theta \tan \theta) d\theta = \frac{\sec^4 \theta}{4} \right) \\
&= 4 \left[\frac{4\pi}{3} - \frac{1}{4} \int_0^{\frac{\pi}{3}} \sec^2 \theta (1 + \tan^2 \theta) d\theta \right] \\
&= \frac{16\pi}{3} - \int_0^{\frac{\pi}{3}} \sec^2 \theta d\theta - \int_0^{\frac{\pi}{3}} \sec^2 \theta \tan^2 \theta d\theta \\
&= \frac{16\pi}{3} - [\tan \theta]_0^{\frac{\pi}{3}} - \left[\frac{\tan^3 \theta}{3} \right]_0^{\frac{\pi}{3}} \\
&= \frac{16\pi}{3} - (\sqrt{3} - 0) - \left(\frac{3\sqrt{3}}{3} - 0 \right) = \frac{16\pi}{3} - (\sqrt{3} + \sqrt{3}) = \frac{16\pi}{3} - 2\sqrt{3} \quad \text{જવાબ : (C)}
\end{aligned}$$

(40) જો $I_{(m, n)} = \int_0^1 x^m (1-x)^n dx$, જ્યાં $m, n \in \mathbb{N}$, તો $\frac{I_{(m, n)}}{I_{(m+1, n-1)}} = \dots$

(A) $\frac{n}{m+1}$ (B) $\frac{m}{n-1}$ (C) $\frac{m-1}{n+1}$ (D) $\frac{n-1}{m+1}$

ઉત્તર : $I_{(m, n)} = \int_0^1 x^m (1-x)^n dx$

$$\begin{aligned}
&= \left[(1-x)^n \cdot \frac{x^{m+1}}{m+1} \right]_0^1 - \int_0^1 n(1-x)^{n-1} \cdot (-1) \frac{x^{m+1}}{m+1} dx \\
&= 0 + \frac{n}{m+1} \int_0^1 x^{m+1} (1-x)^{n-1} dx \\
I_{(m, n)} &= \frac{n}{m+1} I_{(m+1, n-1)}
\end{aligned}$$

$\therefore \frac{I_{(m, n)}}{I_{(m+1, n-1)}} = \frac{n}{m+1}$ જવાબ : (A)

(41) $\int_0^1 (1 + e^{-x^2}) dx$ નું મૂલ્ય અંતરાલમાં છે.

(A) $(0, 1)$ (B) $\left(\frac{1}{e}, 1\right)$ (C) $\left(\frac{1}{e^2}, 2\right)$ (D) $\left(1 + \frac{1}{e}, 2\right)$

ઉકેલ : $0 < x < 1$

$$\therefore 0 < x^2 < 1$$

$$\therefore e^0 < e^{x^2} < e^1 \quad (e^{x^2} \text{ વધતું વિધેય})$$

$$\therefore 1 < e^{x^2} < e$$

$$\therefore \frac{1}{e} < \frac{1}{e^{x^2}} < 1$$

$$\therefore 1 + \frac{1}{e} < 1 + e^{-x^2} < 2$$

$$\therefore \left(1 + \frac{1}{e}\right) (1 - 0) < \int_0^1 (1 + e^{-x^2}) dx < 2(1 - 0)$$

$$\therefore \left(1 + \frac{1}{e}\right) < \int_0^1 (1 + e^{-x^2}) dx < 2$$

$$\therefore \int_0^1 (1 + e^{-x^2}) dx \text{ નું મૂલ્ય } \left(1 + \frac{1}{e}, 2\right) \text{ અંતરાલમાં છે.}$$

જવાબ : (D)

(42) જો f એ ધન વાસ્તવિક સંખ્યાઓ પર વ્યાખ્યાયિત વિધેય હોય તથા

$$I_1 = \int_{1-k}^k x \cdot f\{x(1-x)\} dx \text{ અને } I_2 = \int_{1-k}^k f\{x(1-x)\} dx \text{ જ્યાં } 2k - 1 > 0, \text{ તો } \frac{I_1}{I_2} = \dots\dots$$

(A) 2

(B) 1

(C) $\frac{1}{2}$

(D) 3

(IIT : 1997)

$$\text{ઉકેલ : } I_1 = \int_{1-k}^k x \cdot f\{x(1-x)\} dx$$

$$= \int_{1-k}^k (1-x) f\{(1-x)(1-(1-x))\} dx$$

$$(a + b - x = 1 - x)$$

$$= \int_{1-k}^k (1-x) f\{(1-x)x\} dx$$

$$I_1 = \int_{1-k}^k f\{x(1-x)\} dx - \int_{1-k}^k x f\{x(1-x)\} dx$$

$$\therefore I_1 = I_2 - I_1$$

$$\therefore 2I_1 = I_2$$

$$\therefore \frac{I_1}{I_2} = \frac{1}{2}$$

જવાબ : (C)

$$(43) \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x + \pi)^3 + \cos^2(x + 3\pi)] dx = \dots\dots$$

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{2} - 1$

(C) $\frac{\pi^4}{32}$

(D) $\frac{\pi^4}{32} + \frac{\pi}{2}$

$$\text{ઉકેલ : } I = \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x + \pi)^3 + \cos^2(x + 3\pi)] dx$$

$$= \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x + \pi)^3 + \cos^2(x + \pi)] dx$$

ધારો કે, $x + \pi = t$. અર્થાત્ $dx = dt$

$$x = -\frac{3\pi}{2} \Rightarrow t = -\frac{\pi}{2} \quad \text{તથા } x = -\frac{\pi}{2} \Rightarrow t = \frac{\pi}{2}$$

$$\therefore I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (t^3 + \cos^2 t) dt$$

$$= 0 + 2 \int_0^{\frac{\pi}{2}} \cos^2 t dt \quad (t^3 \text{ અયુગ્મ વિધેય છે અને } \cos^2 t \text{ યુગ્મ વિધેય છે.)$$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{(1 + \cos 2t)}{2} dt = \left[t + \frac{\sin 2t}{2} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

જવાબ : (A)

$$(44) \int_0^{\frac{\pi}{2}} \frac{\sin x}{(1 + \cos x + \sin x)} dx = \dots\dots$$

(A) $\frac{\pi}{4} - \frac{1}{2} \log 2$

(B) $\frac{\pi}{2} - 2 \log 2$

(C) $\pi - \frac{1}{2} \log 2$

(D) $\frac{\pi}{4} - 2 \log 2$

$$\text{ઉકેલ : } I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{(1 + \cos x + \sin x)} dx$$

(i)

$$= \int_0^{\frac{\pi}{2}} \frac{\sin \left(\frac{\pi}{2} - x \right)}{1 + \cos \left(\frac{\pi}{2} - x \right) + \sin \left(\frac{\pi}{2} - x \right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin x + \cos x} dx$$

(ii)

પરિણામ (i) અને (ii)નો સરવાળો કરતાં,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{1 + \sin x + \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1 + \sin x + \cos x) - (1)}{(1 + \sin x + \cos x)} dx$$

$$= \int_0^{\frac{\pi}{2}} 1 dx - \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x + \cos x} dx$$

$$2I = [x]_0^{\frac{\pi}{2}} - I_1$$

$$2I = \frac{\pi}{2} - I_1 \quad \text{(iii)}$$

$$\text{હવે } I_1 = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x + \cos x} dx$$

$$\text{ધારો કે, } \tan \frac{x}{2} = t. \text{ અર્થાત્, } dx = \frac{2dt}{1+t^2}, \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$x = 0 \Rightarrow t = 0 \text{ તથા } x = \frac{\pi}{2} \Rightarrow t = 1$$

$$\therefore I_1 = \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2dt}{2(1+t)}$$

$$= [\log(1+t)]_0^1$$

$$= \log 2 - \log 1$$

$$\therefore 2I = \frac{\pi}{2} - \log 2 \quad \text{(પરિણામ (iii)માં મૂકતાં)}$$

$$I = \frac{\pi}{4} - \frac{1}{2} \log 2$$

જવાબ : (A)

$$(45) \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{(\sin^4 x + \cos^4 x)} dx = \dots$$

- (A) 0 (B) $\frac{\pi^2 - 1}{4}$ (C) $\frac{\pi^2}{16}$ (D) $\frac{\pi}{4} (\pi - 1)$

$$\text{ઉકેલ : } I = \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{(\sin^4 x + \cos^4 x)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \cos x \sin x}{\sin^4 x + \cos^4 x} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} \sin x \cos x}{\cos^4 x + \sin^4 x} dx - \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

$$I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx - I$$

$$\therefore 2I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\tan x \sec^2 x dx}{\tan^4 x + 1} \quad (\text{અંશ અને છેદને } \cos^4 x \text{ વડે ભાગતી})$$

$$\text{ધારો કે, } \tan^2 x = t. \quad \text{અર્થાત્, } 2 \tan x \sec^2 x dx = dt$$

$$x = 0 \Rightarrow t = 0 \text{ અને } x = \frac{\pi}{2} \Rightarrow t = \infty$$

$$\therefore 2I = \frac{\pi}{4} \int_0^{\infty} \frac{dt}{t^2 + 1} = \frac{\pi}{4} [\tan^{-1} t]_0^{\infty} = \frac{\pi}{4} \left[\frac{\pi}{2} - 0\right]$$

$$\therefore 2I = \frac{\pi^2}{8}. \quad \text{અર્થાત્, } I = \frac{\pi^2}{16} \quad \text{જવાબ : (C)}$$

$$(46) \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx = \dots$$

(A) $-\pi$

(B) $\frac{\pi}{2}$

(C) π

(D) 2π

$$\text{ઉકેલ : } I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx \quad (i)$$

$$= \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}} dx \quad (x \Rightarrow a+b-x)$$

$$= \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{a^x + 1} dx \quad (ii)$$

પરિણામ (i) અને (ii)નો સરવાળો કરતાં,

$$\therefore 2I = \int_{-\pi}^{\pi} \left(\frac{\cos^2 x}{1+a^x} + \frac{a^x \cos^2 x}{1+a^x} \right) dx$$

$$2I = \int_{-\pi}^{\pi} \cos^2 x \, dx = 2 \int_0^{\pi} \left(\frac{1+\cos 2x}{2} \right) dx$$

$$\therefore 2I = \left[x + \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= [\pi - (0)] = \pi$$

$$\therefore I = \frac{\pi}{2}$$

જવાબ : (B)

$$(47) \text{ જો } f(x) = \frac{e^x}{1+e^x}, I_1 = \int_{f(-a)}^{f(a)} x g\{x(1-x)\} dx \text{ અને } I_2 = \int_{f(-a)}^{f(a)} g\{x(1-x)\} dx, \text{ તો } \frac{I_2}{I_1} = \dots$$

(A) $\frac{1}{2}$

(B) 1

(C) $\frac{3}{2}$

(D) 2 (AIEEE : 2004)

$$\text{ઉકેલ : } f(x) = \frac{e^x}{1+e^x} \Rightarrow f(a) = \frac{e^a}{1+e^a} \text{ અને } f(-a) = \frac{e^{-a}}{1+e^{-a}} = \frac{1}{1+e^a}$$

$$\therefore f(a) + f(-a) = 1$$

ધારો કે, $f(a) = k$. તો $f(-a) = 1 - k$

$$\therefore I_1 = \int_{1-k}^k x g \{x(1-x)\} dx$$

$$= \int_{1-k}^k (1-x) g \{(1-x)(1-(1-x))\} dx \quad (x \Rightarrow a + b - x)$$

$$I_1 = \int_{1-k}^k (1-x) g \{x(1-x)\} dx$$

$$\therefore I_1 = \int_{1-k}^k g \{x(1-x)\} dx - \int_{1-k}^k x g \{x(1-x)\} dx$$

$$\therefore I_1 = I_2 - I_1$$

$$\therefore 2I_1 = I_2$$

$$\therefore \frac{I_2}{I_1} = 2$$

જવાબ : (D)

$$(48) \text{ જો } f(x) = \begin{cases} e^{\cos x} \cdot \sin x, & \text{જ્યાં } |x| \leq 2 \\ 2 & , \text{ અન્યથા} \end{cases}, \text{ તો } \int_{-2}^3 f(x) dx = \dots$$

(A) 0

(B) 1

(C) 2

(D) 3

(IIT : 2000)

$$\text{ઉકેલ : } I = \int_{-2}^3 f(x) dx$$

$$= \int_{-2}^2 f(x) dx + \int_2^3 f(x) dx$$

$$= \int_{-2}^2 e^{\cos x} \cdot \sin x dx + \int_2^3 2 dx$$

($e^{\cos x} \cdot \sin x$ અયુગ્મ વિધેય છે.)

$$= 0 + 2 [x]_2^3 = 2[3 - 2] = 2$$

જવાબ : (C)

$$(49) \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{(1 + \cos^2 x)} dx = \dots$$

(A) 0

(B) 2π

(C) $\frac{\pi}{4}$

(D) π^2

$$\text{ઉકેલ : } I = \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{(1 + \cos^2 x)} dx$$

$$= 2 \int_{-\pi}^{\pi} \frac{x}{1 + \cos^2 x} dx + 2 \int_{-\pi}^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$I = 0 + 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \quad \left(\frac{x}{1 + \cos^2 x} \text{ અપુરુષ અને } \frac{x \sin x}{1 + \cos^2 x} \text{ પુરુષ} \right) \quad (1)$$

$$= 4 \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1 + \cos^2 (\pi - x)} dx$$

$$= 4 \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx$$

$$I = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx - 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$I = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx - I \quad (\text{પરિણામ (1)})$$

$$2I = 4\pi \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x}$$

$$\text{ધારે } \cos x = t. \text{ તો, } -\sin x dx = dt$$

$$\text{ત્યજ } x = 0 \Rightarrow t = 1 \text{ અને } x = \pi \Rightarrow t = -1$$

$$2I = 4\pi \int_1^{-1} \frac{-dt}{1 + t^2}$$

$$2I = 4\pi \int_{-1}^1 \frac{dt}{1 + t^2} = 4\pi [\tan^{-1} t]_{-1}^1$$

$$\therefore I = 2\pi \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = 2\pi \left(\frac{\pi}{2} \right) = \pi^2 \quad \text{જવાબ : (D)}$$

$$(50) \int_0^{10\pi} |\sin x| dx = \dots$$

(A) 5

(B) 10

(C) 20

(D) 25

ઉકેલ : $I = \int_0^{10\pi} |\sin x| dx$

$= 10 \int_0^{\pi} |\sin x| dx$ ($|\sin x|$ નું મુખ્ય આવર્તમાન π છે.)

$= 10 \int_0^{\pi} \sin x dx$

$= 10 [-\cos x]_0^{\pi} = -10 [\cos \pi - \cos 0] = -10 (-1 - 1) = 20$ જવાબ : (C)

(51) જો $I_1 = \int_0^1 2x^2 dx$, $I_2 = \int_0^1 2x^3 dx$, $I_3 = \int_1^2 2x^2 dx$ અને $I_4 = \int_1^2 2x^3 dx$ હોય તો

(A) $I_1 > I_2$

(B) $I_3 > I_4$

(C) $I_1 < I_2$

(D) $I_3 = I_4$

(AIEEE : 2005)

ઉકેલ : અહીં $x^2 > x^3$, $\forall x \in [0, 1]$, $x^2 < x^3$, $\forall x \in [1, 2]$

$\therefore 2x^2 > 2x^3$, $\forall x \in [0, 1]$, $2x^2 < 2x^3$, $\forall x \in [1, 2]$

$\therefore \int_0^1 2x^2 dx > \int_0^1 2x^3 dx$ અને $\int_1^2 2x^2 dx < \int_1^2 2x^3 dx$

$\therefore I_1 > I_2$ અને $I_3 < I_4$

જવાબ : (A)

(52) જો $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ અને $F(x) = \int_0^x f(t) dt$ હોય તો $F(x^2) = x^2(1+x)$, તો $f(4) = \dots$

(A) $\frac{5}{4}$

(B) 2

(C) 4

(D) 7

(IIT : 2004)

ઉકેલ : અહીં $F(x) = \int_0^x f(t) dt$

$\therefore F(x^2) = \int_0^{x^2} f(t) dt$

$\therefore x^2(1+x) = \int_0^{x^2} f(t) dt$

$[F(x^2) = x^2(1+x)]$

બંને બાજુ વિકલન કરતાં,

$x^2(1) + 2x(1+x) = f(x^2) \cdot (2x)$

$\therefore 3x^2 + 2x = f(x^2) \cdot (2x)$

$\therefore f(x^2) = \frac{3}{2}x + 1$

હવે, $x = 2$ લેતાં, $f(4) = 4$

જવાબ : (C)

(53) જો $\int_{\sin x}^1 t^2 f(t) dt = 1 - \sin x$, $x \in \left(0, \frac{\pi}{2}\right)$, તો $f\left(\frac{1}{\sqrt{3}}\right) = \dots$

(A) $\frac{1}{\sqrt{3}}$

(B) $\frac{1}{3}$

(C) $\sqrt{3}$

(D) 3

(IIT : 2005)

ઉકેલ : $\int_{\sin x}^1 t^2 f(t) dt = 1 - \sin x$

બંને બાજુ x ને સાપેક્ષ વિકલન કરતાં,

$$0 - \sin^2 x f(\sin x) \cos x = -\cos x$$

$$\therefore f(\sin x) = \frac{1}{\sin^2 x}$$

$$\therefore f(x) = \frac{1}{x^2}$$

$$\therefore f\left(\frac{1}{\sqrt{3}}\right) = (\sqrt{3})^2 = 3$$

જવાબ : (D)

(54) જો $y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos x \cdot \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$, તો $\frac{dy}{dx}$ નું મૂલ્ય $x = \pi$ આગળ $\dots$ છે.

(A) 0

(B) π

(C) π^2

(D) 2π

(IIT : 2004)

ઉકેલ : $y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos x \cdot \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$

$$= \cos x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$$

બંને બાજુ x ને સાપેક્ષ વિકલન કરતાં,

$$\frac{dy}{dx} = -\sin x \cdot \left(\int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta \right) + \left(\frac{\cos x}{1 + \sin^2 x} \cdot 2x \right) \cos x$$

$$\therefore \left(\frac{dy}{dx} \right)_{x=\pi} = [0] + \frac{2\pi(-1)^2}{1+0} = 2\pi$$

$$\therefore \left(\frac{dy}{dx} \right)_{x=\pi} = 2\pi$$

જવાબ : (D)

(55) જો $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, તો $f(x)$ પર વધતું વિધેય છે.

(A) અંતરાલ $[-2, 2]$

(B) અંતરાલ $[0, \infty]$

(C) અંતરાલ $(-\infty, 0)$

(D) x ની કોઈ પણ કિંમત માટે નહિ.

(IIT : 2003)

ઉકેલ : $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$

$\therefore f'(x) = e^{-(x^2+1)^2} (2x) - e^{-(x^2)^2} \cdot 2x$

$$\left(F(x) = \int_{h(x)}^{g(x)} f(t) dt \Rightarrow F'(x) = f[g(x)] \cdot g'(x) - f[h(x)] \cdot h'(x) \right)$$

$\therefore f'(x) = (2x) [e^{-(x^4+2x^2+1)} - e^{-x^4}]$
 $= (2x) (e^{-x^4}) [e^{-(2x^2+1)} - 1]$
 $= (2x) (e^{-x^4}) (1 - e^{2x^2+1}) e^{-(2x^2+1)}$

હવે, $e^{(2x^2+1)} > e > 1 \Rightarrow 1 - e^{(2x^2+1)} < 0$

અને $e^{-x^4} > 0$, $e^{-(2x^2+1)} > 0$

હવે, f વધતું વિધેય છે. આથી, $f'(x) > 0$ જરૂરી છે.

જો $f'(x) > 0$ તો $2x < 0$ એટલે કે $x < 0$ જરૂરી છે.

f એ વધતું વિધેય હોય, તો $x < 0$. આથી, $x \in (-\infty, 0)$

જવાબ : (C)

(56) વિધેય $f(x) = \int_{-1}^x t(e^t - 1)(t - 1)(t - 2)^3(t - 3)^5 dt$, એ x ની કિંમત માટે સ્થાનીય ન્યૂનતમ બને છે.

(A) $x = 0$ અને $x = 1$

(B) $x = 0$ અને $x = 2$

(C) $x = 1$ અને $x = 2$

(D) $x = 1$ અને $x = 3$

(IIT : 1999)

ઉકેલ : $f(x) = \int_{-1}^x t(e^t - 1)(t - 1)(t - 2)^3(t - 3)^5 dt$

$\therefore f'(x) = x(e^x - 1)(x - 1)(x - 2)^3(x - 3)^5$

હવે, સ્થાનીય મહત્તમ કે ન્યૂનતમ કિંમત માટે $f'(x) = 0$

$\therefore x(e^x - 1)(x - 1)(x - 2)^3(x - 3)^5 = 0$

$\therefore x = 0$ અથવા $x = 1$ અથવા $x = 2$ અથવા $x = 3$

$\therefore x = 0, 1, 2$ અથવા 3 આગળ સ્થાનીય ન્યૂનતમ થઈ શકે.

હવે, ખૂબ નાની ધન વાસ્તવિક સંખ્યા h માટે,

(i) $f'(0 - h) < 0$ અને $f'(0 + h) < 0$

$\therefore x = 0$ આગળ મહત્તમ કે ન્યૂનતમ નથી.

(ii) $f'(1 - h) < 0$ અને $f'(1 + h) > 0$

$\therefore x = 1$ આગળ સ્થાનિક ન્યૂનતમ છે.

(iii) $f'(2 - h) > 0$ અને $f'(2 + h) < 0$

$\therefore x = 2$ આગળ સ્થાનિક ન્યૂનતમ નથી.

(iv) $f'(3 - h) < 0$ અને $f'(3 + h) > 0$

$\therefore x = 3$ આગળ સ્થાનિક ન્યૂનતમ છે.

$\therefore x = 1$ અને $x = 3$ આગળ સ્થાનિક ન્યૂનતમ છે.

જવાબ : (D)

(57) $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{(n+r)^2} = \dots\dots$

(A) $\frac{3}{4}$

(B) $\frac{2}{3}$

(C) $\frac{2}{9}$

(D) $\frac{1}{2}$

ઉકેલ : $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{(n+r)^2}$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{n^2 \left(1 + \frac{r}{n}\right)^2}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{1}{\left(1 + \frac{r}{n}\right)^2}$$

$$= \int_0^1 \frac{1}{(1+x)^2} dx \quad \left[\frac{r}{n} \text{ ને સ્થાને } x, \frac{1}{n} \text{ ના સ્થાને } dx \text{ અને } \Sigma \text{ ના સ્થાને } \int \text{ લેતાં} \right]$$

$$= \left[\frac{-1}{1+x} \right]_0^1 = \left(-\frac{1}{2} - (-1) \right) = -\frac{1}{2} + 1 = \frac{1}{2}$$

જવાબ : (D)

(58) જો $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, તો $\lim_{n \rightarrow \infty} n [I_n + I_{n+2}] = \dots\dots$

(A) 0

(B) $\frac{1}{2}$

(C) 1

(D) ∞

ઉકેલ : $I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^n x \, dx + \int_0^{\frac{\pi}{4}} \tan^{n+2} x \, dx$

$$= \int_0^{\frac{\pi}{4}} \tan^n x (\tan^2 x + 1) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x \cdot \sec^2 x \, dx$$

$$= \left[\frac{\tan^{n+1} x}{n+1} \right]_0^{\frac{\pi}{4}}$$

$$\begin{aligned}\therefore \lim_{n \rightarrow \infty} n [I_n + I_{n+2}] &= \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right) \\ &= 1\end{aligned}$$

જવાબ : (C)

$$(59) \lim_{x \rightarrow 0} \int_0^{x^2} \frac{\cot t^2 dt}{x \sin x} = \dots\dots$$

(A) $-\frac{1}{2}$

(B) 1

(C) $\frac{3}{2}$

(D) 2

(IIT : 1997)

ઉકેલ : $\lim_{x \rightarrow 0} \int_0^{x^2} \frac{\cot t^2 dt}{x \sin x}$

હારે છે, $f(x) = \int_0^{x^2} \cos t^2 dt$ અને $g(x) = x \sin x$

હવે, $f(0) = \int_0^0 \cos t^2 dt = 0$ અને $g(0) = 0 \sin 0 = 0$

$$\therefore \lim_{x \rightarrow 0} \int_0^{x^2} \frac{\cos t^2 dt}{x \sin x} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} \quad (\text{લા-પૅટલનો નિયમ})$$

$$= \lim_{x \rightarrow 0} \frac{(\cos x^4) (2x)}{x \cos x + \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{2[-x(\sin x^4)4x^3 + \cos x^4(1)]}{-x \sin x + \cos x + \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{2[-4x^4 \sin x^4 + \cos x^4]}{2\cos x - x \sin x} = \frac{2(0+1)}{2(1)-0} = \frac{2}{2} = 1 \quad \text{જવાબ : (B)}$$

$$(60) \lim_{x \rightarrow 0} \frac{1}{x} \left\{ \int_y^c e^{\sin^2 t} dt - \int_{x+y}^c e^{\sin^2 t} dt \right\} = \dots\dots$$

(A) $\sin y \cdot e^y$

(B) 0

(C) $e^{\sin^2 y}$

(D) $\sin 2y \cdot e^{\sin^2 y}$

ઉકેલ : હારે છે, $X = \lim_{x \rightarrow 0} \frac{1}{x} \left\{ \int_y^c e^{\sin^2 t} dt - \int_{x+y}^c e^{\sin^2 t} dt \right\}$

$$= \lim_{x \rightarrow 0} \frac{1}{x} \left\{ \int_y^c e^{\sin^2 t} dt + \int_c^{x+y} e^{\sin^2 t} dt \right\}$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{\int_y^{x+y} e^{\sin^2 t} dt}{x} \right\}$$

હવે, ધારો કે $f(x) = \int_y^{x+y} e^{\sin^2 t} dt$ અને $g(x) = x$

$$f(0) = 0 \text{ અને } g(0) = 0$$

$$\therefore \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} \quad (\text{લા-પીટલનો નિયમ})$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{e^{\sin^2(x+y)} \frac{d}{dx}(x+y)}{1} \right\}$$

$$= \lim_{x \rightarrow 0} e^{\sin^2(x+y)} = e^{\sin^2 y}$$

જવાબ : (C)

(61) ધારો કે, $f: \mathbb{R} \rightarrow \mathbb{R}$ વિકલનીય વિધેય છે. $f(2) = 6$ અને $f'(2) = \frac{1}{48}$, તો $\lim_{x \rightarrow 2} \frac{\int_6^{f(x)} 4t^3 dt}{x-2} = \dots$

(A) 32

(B) 26

(C) 18

(D) 12 (AIEEE : 2005)

ઉકેલ : ધારો કે, $g(x) = \int_6^{f(x)} 4t^3 dt$ અને $h(x) = x - 2$

$$\text{તો } g(2) = \int_6^{f(2)} 4t^3 dt = \int_6^6 4t^3 dt = 0 \text{ તથા } h(2) = 2 - 2 = 0$$

$$\therefore \lim_{x \rightarrow 2} \frac{g(x)}{h(x)} = \lim_{x \rightarrow 2} \frac{g'(x)}{h'(x)} \quad (\text{લા-પીટલનો નિયમ})$$

$$= \lim_{x \rightarrow 2} \frac{4\{f(x)\}^3 \cdot f'(x)}{1}$$

$$= 4 \{f(2)\}^3 \cdot f'(2) = 4 \times 6^3 \times \frac{1}{48} = 18$$

જવાબ : (C)

(62) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=n+1}^{2n} \log \left(1 + \frac{r}{n} \right) = \dots$

(A) $\log \left(\frac{27}{4e} \right)$

(B) $\log \left(\frac{4}{e} \right)$

(C) $\log \left(\frac{25}{3e^2} \right)$

(D) $\log \left(\frac{27}{e^2} \right)$

ઉકેલ : $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=n+1}^{2n} \log \left(1 + \frac{r}{n} \right)$

$\frac{r}{n}$ ને સ્થાને x , $\frac{1}{n}$ ના સ્થાને dx અને $\sum$ ના સ્થાને $\int$ લેતાં,

$$\text{તથા સંકલનની અધ:સીમા} = \lim_{n \rightarrow \infty} \frac{r}{n} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 1 \quad (r = n+1)$$

$$\text{સંકલનની ઊર્ધ્વ:સીમા} = \lim_{n \rightarrow \infty} \frac{r}{n} = \lim_{n \rightarrow \infty} \frac{2n}{n} = 2 \quad (r = 2n)$$

$$\therefore \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=n+1}^{2n} \log \left(1 + \frac{r}{n} \right)$$

$$\int_1^2 \log (1+x) dx = [x \log (1+x)]_1^2 - \int_1^2 \frac{x dx}{1+x}$$

$$= 2 \log 3 - \log 2 - \int_1^2 \frac{x+1-1}{1+x} dx$$

$$= \log \frac{9}{2} - \int_1^2 dx + \int_1^2 \frac{dx}{1+x}$$

$$= \log \frac{9}{2} - [x]_1^2 + [\log (1+x)]_1^2$$

$$= \log \frac{9}{2} + \log 3 - \log 2 - 1$$

$$= \log \frac{27}{4e} \quad (1 = \log e)$$

જવાબ : (A)

$$(63) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}} = \dots\dots$$

$$(A) \sqrt{2} - 1$$

$$(B) \sqrt{5} - 1$$

$$(C) 2\sqrt{3} - 5$$

$$(D) 5\sqrt{2} - 3$$

$$\text{ઉકેલ : } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{\frac{r}{n}}{\sqrt{1 + \left(\frac{r}{n}\right)^2}}$$

$$= \int_0^2 \frac{x}{\sqrt{1+x^2}} dx$$

$$= \frac{1}{2} \int_0^2 \frac{2x}{\sqrt{1+x^2}} dx = \frac{1}{2} \left[2\sqrt{1+x^2} \right]_0^2 = (\sqrt{5} - 1)$$

જવાબ : (B)

$$(64) \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{4n} \right\} = \dots\dots$$

$$(A) \log 2$$

$$(B) 2 \log 3$$

$$(C) \log 4$$

$$(D) \frac{1}{2} \log 5$$

$$\text{ઉકેલ : } \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{4n} \right\}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=0}^{3n} \frac{1}{n+r}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=0}^{3n} \frac{1}{n} \left(\frac{1}{1+\frac{r}{n}} \right)$$

$$= \int_0^3 \frac{1}{1+x} dx = [\log(1+x)]_0^3 = \log 4$$

જવાબ : (C)

$$(65) \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sec^2 \frac{\pi}{4n} + \sec^2 \frac{2\pi}{4n} + \dots + \sec^2 \frac{n\pi}{4n} \right] = \dots$$

(A) $\frac{2}{\pi}$

(B) $\frac{\pi}{2}$

(C) $\frac{4}{\pi}$

(D) 1

$$\text{ઉકેલ : } \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sec^2 \frac{\pi}{4n} + \sec^2 \frac{2\pi}{4n} + \dots + \sec^2 \frac{n\pi}{4n} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \sec^2 \frac{r\pi}{4n}$$

$$= \int_0^1 \sec^2 \left(\frac{\pi}{4} x \right) dx$$

$$= \left[\frac{\tan \left(\frac{\pi}{4} x \right)}{\left(\frac{\pi}{4} \right)} \right]_0^1 = \frac{4}{\pi} \left[\tan \frac{\pi}{4} - \tan 0 \right] = \frac{4}{\pi}$$

જવાબ : (C)

$$(66) \lim_{n \rightarrow \infty} \left(\frac{1^{99} + 2^{99} + 3^{99} + \dots + n^{99}}{n^{100}} \right) = \dots$$

(A) $\frac{101}{100}$

(B) $\frac{1}{101}$

(C) $\frac{99}{100}$

(D) $\frac{1}{100}$

$$\text{ઉકેલ : } \lim_{n \rightarrow \infty} \left(\frac{1^{99} + 2^{99} + 3^{99} + \dots + n^{99}}{n^{100}} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n} \right)^{99}$$

$$= \int_0^1 x^{99} dx = \left[\frac{x^{100}}{100} \right]_0^1 = \frac{1}{100}$$

જવાબ : (D)

$$(67) \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{2^2}{n^2} + \frac{3}{n^2} \sec^2 \frac{3^2}{n^2} + \dots + \frac{1}{n} \sec^2 1 \right] = \dots$$

(A) $2 \tan 1$

(B) $\frac{1}{2} \tan \frac{1}{2}$

(C) $2 \tan \frac{1}{2}$

(D) $\frac{1}{2} \tan 1$

(AIEEE : 2005)

$$\begin{aligned}
\text{ઉકેલ : } & \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{2^2}{n^2} + \frac{3}{n^2} \sec^2 \frac{3^2}{n^2} + \dots + \frac{1}{n} \sec^2 1 \right] \\
&= \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \frac{1^2}{n^2} + \frac{2}{n^2} \sec^2 \frac{2^2}{n^2} + \frac{3}{n^2} \sec^2 \frac{3^2}{n^2} + \dots + \frac{n}{n^2} \sec^2 \frac{n^2}{n^2} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{r}{n} \sec^2 \left(\frac{r^2}{n^2} \right) \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n} \right) \sec^2 \left(\frac{r}{n} \right)^2 \\
&= \int_0^1 x \sec^2 (x)^2 dx \\
&= \frac{1}{2} \int_0^1 \sec^2 t dt \\
&= \frac{1}{2} [\tan t]_0^1 = \frac{1}{2} (\tan 1 - \tan 0) = \frac{1}{2} \tan (1)
\end{aligned}$$

જવાબ : (D)

(68) જો $f(x) = A \sin \left(\frac{\pi}{2} x \right) + B$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$ અને $\int_0^1 f(x) dx = \frac{2A}{\pi}$ તો અચલ A અને B ની કિંમતો અનુક્રમે.

- (A) $\frac{\pi}{2}$ અને $\frac{\pi}{2}$ (B) $\frac{2}{\pi}$ અને $\frac{3}{\pi}$ (C) 0 અને $\frac{-4}{\pi}$ (D) $\frac{4}{\pi}$ અને 0

(IIT : 1993)

ઉકેલ : અહીં, $f(x) = A \sin \left(\frac{\pi}{2} x \right) + B$

ધારો કે, $x^2 = t$. અર્થાત્, $2x dx = dt$. $x dx = \frac{1}{2} dt$

$$\therefore f'(x) = \frac{A\pi}{2} \cos \left(\frac{\pi}{2} x \right) \quad x = 0 \Rightarrow t = 0, \quad x = 1 \Rightarrow t = 1$$

$$f'\left(\frac{1}{2}\right) = \frac{A\pi}{2} \cos \left(\frac{\pi}{4} \right) = \frac{A\pi}{2\sqrt{2}} = \sqrt{2}. \text{ અર્થાત્, } A = \frac{4}{\pi}$$

નોંધ : અહીંથી જ જવાબ D નક્કી થઈ ગયો !

$$\text{હવે, } \int_0^1 f(x) dx = \frac{2A}{\pi}$$

$$\therefore \int_0^1 \left(A \sin \left(\frac{\pi x}{2} \right) + B \right) dx = \frac{2A}{\pi}$$

$$\therefore \left[\frac{-2A}{\pi} \cos \frac{\pi x}{2} + Bx \right]_0^1 = \frac{2A}{\pi}$$

$$\therefore B + \frac{2A}{\pi} = \frac{2A}{\pi}. \text{ અર્થાત્, } B = 0$$

$$\therefore A = \frac{4}{\pi} \text{ અને } B = 0$$

જવાબ : (D)

(69) જો $g(x) = \int_0^x \cos^4 t \, dt$, તો $g(x + \pi) = \dots\dots$

- (A) $g(x) + g(\pi)$ (B) $g(x) - g(\pi)$ (C) $g(x) g(\pi)$ (D) $\frac{g(x)}{g(\pi)}$ (IIT : 1997)

ઉકેલ : $g(x) = \int_0^x \cos^4 t \, dt$

$$\begin{aligned} \therefore g(x + \pi) &= \int_0^{\pi+x} \cos^4 t \, dt \\ &= \int_0^{\pi} \cos^4 t \, dt + \int_{\pi}^{\pi+x} \cos^4 t \, dt \end{aligned}$$

$\therefore g(x + \pi) = I_1 + I_2$

$$I_1 = \int_0^{\pi} \cos^4 t \, dt = g(\pi)$$

$$I_2 = \int_{\pi}^{\pi+x} \cos^4 t \, dt$$

હવે, $t = \pi + \theta$. આથી, $dt = d\theta$, $t = \pi \Rightarrow \theta = 0$; $t = \pi + x \Rightarrow \theta = x$

$$\therefore I_2 = \int_0^x \cos^4 (\pi + \theta) \, d\theta = \int_0^x \cos^4 \theta \, d\theta = \int_0^x \cos^4 t \, dt = g(x)$$

$\therefore g(x + \pi) = I_1 + I_2 = g(\pi) + g(x)$

જવાબ : (A)

(70) $\int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin (\log 6 - x^2)} \, dx = \dots\dots$

- (A) $\frac{1}{4} \log \frac{3}{2}$ (B) $\frac{1}{2} \log \frac{3}{2}$ (C) $\log \frac{3}{2}$ (D) $\frac{1}{6} \log \frac{3}{2}$ (IIT : 2011)

ઉકેલ : $\int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin (\log 6 - x^2)} \, dx$

હવે, $x^2 = t \Rightarrow x \, dx = \frac{1}{2} \, dt$

$x = \sqrt{\log 2} \Rightarrow t = \log 2$ તથા $x = \sqrt{\log 3} \Rightarrow t = \log 3$

$$\therefore I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin t \, dt}{\sin t + \sin (\log 6 - t)} \quad \dots(i)$$

$$\therefore I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin (\log 3 + \log 2 - t) \, dt}{\sin (\log 3 + \log 2 - t) + \sin (\log 6 - (\log 3 + \log 2 - t))}$$

$$I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin (\log 6-t) d t}{\sin (\log 6-t)+\sin t} \quad \dots(ii)$$

સમીકરણો (i) અને (ii)નો સરવાળો કરતાં,

$$2I = \frac{1}{2} \int_{\log 2}^{\log 3} 1 d t = \frac{1}{2} [t]_{\log 2}^{\log 3} = \frac{1}{2} (\log 3 - \log 2) = \frac{1}{2} \log \left(\frac{3}{2}\right)$$

$$I = \frac{1}{4} \log \left(\frac{3}{2}\right) \quad \text{જવાબ : (A)}$$

$$(71) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^2 + \log \frac{\pi-x}{\pi+x} \right) \cos x \, dx = \dots\dots$$

(A) 0 (B) $\frac{\pi^2}{2} - 4$ (C) $\frac{\pi^2}{2} + 4$ (D) $\frac{\pi^2}{2}$ (IIT : 2013)

$$\text{ઉકેલ : } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^2 + \log \frac{\pi-x}{\pi+x} \right) \cos x \, dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log \left(\frac{\pi-x}{\pi+x} \right) \cdot \cos x \, dx$$

$$\therefore I = 2 \int_0^{\frac{\pi}{2}} x^2 \cos x \, dx + 0$$

($\because x^2 \cos x$ યુગ્મ વિધેય છે અને $\cos x \cdot \log \left(\frac{\pi-x}{\pi+x} \right)$ અયુગ્મ વિધેય છે.)

$$\therefore I = 2 \left\{ [x^2 \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2x \sin x) \, dx \right\}$$

$$= 2 \left\{ \left[\frac{\pi^2}{4} \right] - 2(-x \cos x)_0^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} 1 \cdot (-\cos x) \, dx \right\}$$

$$= 2 \left\{ \frac{\pi^2}{4} - 2[\sin x]_0^{\frac{\pi}{2}} \right\} = 2 \left[\frac{\pi^2}{4} - 2 \right] = \left(\frac{\pi^2}{2} - 4 \right) \quad \text{જવાબ : (B)}$$

(72) જો વક્ર $y = \int_0^x |t| dt$ ને દોરેલ સ્પર્શક એ રેખા $y = 2x$ ને સમાંતર હોય, તો સ્પર્શકથી બનતો x -અંતઃખંડ છે.

- (A) ± 1 (B) ± 2 (C) ± 3 (D) ± 4 (JEE : 2013)

ઉકેલ : અહીં $y = \int_0^x |t| dt$

$$\therefore \frac{dy}{dx} = |x| \cdot 1 = |x|$$

હવે, સ્પર્શક એ રેખા $y = 2x$ ને સમાંતર છે, તેથી સ્પર્શકનો ઢાળ = રેખાનો ઢાળ

$$\therefore |x| = 2$$

$$\therefore x = \pm 2$$

$$\text{હવે, } y = \int_0^{\pm 2} |t| dt = \pm 2 \left(\int_1^2 t dt = \left[\frac{t^2}{2} \right]_0^2 = 2, \int_0^{-2} -t dt = \int_{-2}^0 t dt = \left[\frac{t^2}{2} \right]_{-2}^0 = -2 \right)$$

$$\therefore \text{જરૂરી બિંદુઓ } (2, 2) \text{ તથા } (-2, -2) \text{ થશે અને ઢાળ} = 2$$

$$\therefore \text{સ્પર્શકની સમીકરણો } (y - 2) = 2(x - 2) \text{ અને } y + 2 = 2(x + 2)$$

હવે, x -અંતઃખંડ માટે $y = 0$ લેતાં,

$$\therefore 0 - 2 = 2(x - 2) \text{ અને } 0 + 2 = 2(x + 2)$$

$$\therefore x = \pm 1$$

જવાબ : (A)

$$(73) \int_0^{\pi} \sqrt{1 + 4 \sin^2 \frac{x}{2} - 4 \sin \frac{x}{2}} dx = \dots\dots$$

- (A) $\pi - 4$ (B) $\frac{2\pi}{3} - 4 - 4\sqrt{3}$ (C) $4\sqrt{3} - 4$ (D) $4\sqrt{3} - 4 - \frac{\pi}{3}$

(JEE : 2014)

$$\text{ઉકેલ : } I = \int_0^{\pi} \sqrt{1 + 4 \sin^2 \frac{x}{2} - 4 \sin \frac{x}{2}} dx$$

$$= \int_0^{\pi} \sqrt{\left(1 - 2 \sin \frac{x}{2}\right)^2} dx$$

$$= \int_0^{\pi} \left| 1 - 2 \sin \frac{x}{2} \right| dx$$

$$= \int_0^{\frac{\pi}{3}} \left(1 - 2 \sin \frac{x}{2}\right) dx - \int_{\frac{\pi}{3}}^{\pi} \left(1 - 2 \sin \frac{x}{2}\right) dx$$

$$= \left[x + 4 \cos \frac{x}{2} \right]_0^{\frac{\pi}{3}} - \left[x + 4 \cos \frac{x}{2} \right]_{\frac{\pi}{3}}^{\pi}$$

$$= \left[\left(\frac{\pi}{3} + \frac{4\sqrt{3}}{2} \right) - (0+4) \right] - \left[\left(\pi - \frac{\pi}{3} - \frac{4\sqrt{3}}{2} \right) \right]$$

$$= \left(4\sqrt{3} - 4 - \frac{\pi}{3} \right)$$

જવાબ : (D)

$$(74) \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2 \operatorname{cosec} x)^{17} dx = \dots$$

$$(A) \int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$$

$$(B) \int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17} du$$

$$(C) \int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17} du$$

$$(D) \int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du \quad (\text{IIT : 2014})$$

$$\text{ઉકેલ : } I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2 \operatorname{cosec} x)^{17} dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2^{17} (\operatorname{cosec} x)^{16} \operatorname{cosec} x dx$$

$$\text{ધારો કે, } \operatorname{cosec} x + \cot x = t$$

$$\therefore -\operatorname{cosec} x dx = \frac{dt}{\operatorname{cosec} x + \cot x}$$

$$\therefore \operatorname{cosec} x dx = \frac{-dt}{t}$$

$$\therefore t + \frac{1}{t} = 2 \operatorname{cosec} x$$

$$\text{તથા } x = \frac{\pi}{4} \Rightarrow t = \sqrt{2} + 1 \text{ અને } x = \frac{\pi}{2} \Rightarrow t = 1$$

$$\therefore I = \int_{\sqrt{2}+1}^1 -2^{17} \left(\frac{t+\frac{1}{t}}{2} \right)^{16} \frac{dt}{t}$$

$$\text{ધારો કે, } t = e^u \Rightarrow dt = e^u du$$

$$\text{જ્યારે, } t = 1 \Rightarrow u = 0 \text{ અને } t = \sqrt{2} + 1 \Rightarrow u = \log(\sqrt{2} + 1)$$

$$\therefore I = - \int_{\log(\sqrt{2}+1)}^0 2(e^u + e^{-u})^{16} \frac{e^u du}{e^u}$$

$$= \int_0^{\log(\sqrt{2}+1)} 2(e^u + e^{-u})^{16} du$$

જવાબ : (A)

$$(75) \int_2^4 \frac{\log x^2}{\log x^2 + \log (36 - 12x + x^2)} dx = \dots$$

- (A) 2 (B) 4 (C) 1 (D) 6 (JEE : 2015)

$$\text{ઉકેલ : } I = \int_2^4 \frac{\log x^2}{\log x^2 + \log (36 - 12x + x^2)} dx$$

$$= \int_2^4 \frac{\log x^2}{\log x^2 + \log (6 - x)^2} dx$$

$$= \int_2^4 \frac{2 \log x}{2 \log x + 2 \log (6 - x)} dx$$

$$I = \int_2^4 \frac{\log x}{\log x + \log (6 - x)} dx \quad (i)$$

$$I = \int_2^4 \frac{\log (6 - x)}{\log (6 - x) + \log x} dx \quad (ii)$$

સમીકરણ (i) અને (ii)નો સરવાળો કરતાં,

$$\therefore 2I = \int_2^4 1 dx = [x]_2^4 = 2$$

$$\therefore I = 1$$

જવાબ : (C)

એક કરતાં વધુ સાચા વિકલ્પોવાળા પ્રશ્નો

$$(76) \text{ જો } S_n = \sum_{k=1}^n \frac{n}{n^2 + nk + k^2} \text{ અને } T_n = \sum_{k=1}^{n-1} \frac{n}{n^2 + nk + k^2}, \text{ જ્યાં } n = 1, 2, 3, \dots \text{ તો.}$$

- (A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$ (C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$ (IIT : 2008)

$$\text{ઉકેલ : } S_n < \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n}{n^2 + nk + k^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \left(\frac{1}{1 + \left(\frac{k}{n}\right) + \left(\frac{k}{n}\right)^2} \right)$$

$$= \int_0^1 \frac{dx}{1 + x + x^2}$$

$$= \int_0^1 \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\begin{aligned}
&= \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) \right]_0^1 \\
&= \frac{2}{\sqrt{3}} \left[\tan^{-1} \sqrt{3} - \tan^{-1} \frac{1}{\sqrt{3}} \right] = \frac{\pi}{3\sqrt{3}}
\end{aligned}$$

$$\therefore S_n < \frac{\pi}{3\sqrt{3}}$$

$$\text{એટલે, } h \sum_{k=1}^{n-1} f(kh) > \int_0^1 f(x) dx > h \sum_{k=1}^n f(kh)$$

$$\therefore T_n > \int_0^1 \frac{dx}{x^2 + x + 1}$$

$$\therefore T_n > \frac{\pi}{3\sqrt{3}}$$

$$\therefore S_n < \frac{\pi}{3\sqrt{3}} \text{ એટલે } T_n > \frac{\pi}{3\sqrt{3}}$$

જવાબ : (A), (D)

$$(77) \text{ જો } I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx, \quad n = 0, 1, 2, 3, \dots \text{ તો,}$$

$$(A) \quad I_n = I_{n+2} \quad (B) \quad \sum_{m=1}^{10} I_{2n+1} = 10\pi \quad (C) \quad \sum_{m=1}^{10} I_{2m} = 0 \quad (D) \quad I_n = I_{n+1}$$

(IIT : 2009)

$$\text{ઉકેલ : } I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx \quad (i)$$

$$\therefore I_n = \int_{-\pi}^{\pi} \frac{\sin(-nx)}{(1 + \pi^{-x})(-\sin x)} dx \quad (x \text{ ને બદલે } a + b - x \text{ લેતો}) \quad (ii)$$

$$\therefore 2I_n = \int_{-\pi}^{\pi} \left(\frac{1}{1 + \pi^x} + \frac{1}{1 + \pi^{-x}} \right) \frac{\sin nx}{\sin x} dx \quad (\text{સમીકરણ (i) અને (ii)નો સરવાળો કરતો})$$

$$\therefore 2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx$$

$$\therefore I_n = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx$$

$$\begin{aligned}
\text{હવે, } I_{n+2} - I_n &= \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x} dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \frac{2 \sin x \cdot \cos(n+1)x}{\sin x} dx \\
&= \int_{-\pi}^{\pi} \cos(n+1)x dx \\
&= \left[\frac{\sin(n+1)x}{(n+1)} \right]_{-\pi}^{\pi} = 0
\end{aligned}$$

$$\therefore I_{n+2} - I_n = 0$$

$$\therefore I_{n+2} = I_n$$

$$\text{તેથી, } I_0 = I_2 = I_4 = \dots = I_{10}$$

$$I_1 = I_3 = I_5 = \dots = I_{21}$$

$$\therefore I_0 = \frac{1}{2} \int_{-\pi}^{\pi} 0 dx = 0$$

$$I_1 = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin x}{\sin x} dx = \frac{1}{2} (2\pi) = \pi$$

$$\text{તેથી, } \sum_{m=1}^{10} I_{2m} = 0 \text{ અને } \sum_{n=1}^{10} I_{2m+1} = 10\pi$$

જવાબ : (A), (B), (C)

(78) જો $f(a) > 0$ અને $f(x)$ એ ઘટતું ન હોય તેવું $[a, b]$ માં સતત વિધેય હોય, તો $\frac{1}{b-a} \int_a^b f(x) dx$ ની

(A) મહત્તમ કિંમત $f(b)$ છે.

(B) ન્યૂનતમ કિંમત $f(a)$ છે.

(C) મહત્તમ કિંમત $bf(b)$ છે.

(D) મહત્તમ કિંમત $\frac{f(a)}{b-a}$ છે.

ઉકેલ : $f(x)$ એ સતત અને વધતું વિધેય છે.

$$\text{તેથી, } f(a) \leq f(x) \leq f(b)$$

$$\therefore \int_a^b f(a) dx \leq \int_a^b f(x) dx \leq \int_a^b f(b) dx$$

$$\text{વળી } f(a)(b-a) \leq f(x)(b-a) \leq f(b)(b-a)$$

$$\therefore f(a) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq f(b)$$

જવાબ : (A), (B)

(79) $\int_{\frac{\pi}{2}}^{\alpha} \sin x \, dx = \sin 2\alpha$, $\alpha \in [0, 2\pi]$ નું સમાધાન કરતી α ની કિંમતો

(A) $\frac{\pi}{2}$

(B) $\frac{3\pi}{2}$

(C) $\frac{7\pi}{6}$

(D) $\frac{11\pi}{6}$

ઉકેલ : $\int_{\frac{\pi}{2}}^{\alpha} \sin x \, dx = \sin 2\alpha$

$$\therefore -[\cos x]_{\frac{\pi}{2}}^{\alpha} = \sin 2\alpha$$

$$\therefore -[\cos \alpha - 0] = \sin 2\alpha$$

$$\therefore 2\sin \alpha \cos \alpha + \cos \alpha = 0$$

$$\therefore \cos \alpha (2\sin \alpha + 1) = 0$$

$$\therefore \cos \alpha = 0 \text{ અથવા } \sin \alpha = -\frac{1}{2}$$

$$\therefore \alpha = \frac{\pi}{2}, \frac{3\pi}{2} \text{ અથવા } \alpha = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$

$$\therefore \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

જવાબ : (A), (B), (C), (D)

(80) જો $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, તો

(A) $I_n + I_{n-2} = \frac{1}{n-1}$

(B) $I_{n+1} + I_{n-1} = \frac{1}{n}$

(C) $\frac{1}{n+1} < 2 I_n < \frac{1}{n-1}$, જ્યાં $n \in \mathbb{N}$, $n > 1$

(D) $(I_{n+2} + I_n)(n+1) = 1$

ઉકેલ : $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

$$I_n = \left[\frac{(\tan x)^{n-1}}{n-1} \right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$I_n + I_{n-2} = \frac{1}{n-1}$$

...(i)

(A) સત્ય છે.

હવે, સમીકરણ (i) માં n ની સ્થાને $(n + 1)$ લેતાં,

$$I_{n+1} + I_{n-1} = \frac{1}{n} \quad \dots(ii) \quad \text{(B) સત્ય છે.}$$

હવે, સમીકરણ (ii)માં n ની સ્થાને $(n + 1)$ લેતાં,

$$I_{n+2} + I_n = \frac{1}{n+1} \quad \dots(iii)$$

$$\therefore (n+1)(I_{n+2} + I_n) = 1 \quad \text{(D) સત્ય છે.}$$

$$\text{વળી, } 0 < x < \frac{\pi}{4} \Rightarrow 0 < \tan x < 1$$

$$\therefore \tan^{n+2}x < \tan^n x < \tan^{n-2}x$$

$$\therefore \tan^n x + \tan^{n+2}x < 2 \tan^n x < \tan^n x + \tan^{n-2}x$$

$$\therefore \int_0^{\frac{\pi}{4}} (\tan^n x + \tan^{n+2}x) dx < 2 \int_0^{\frac{\pi}{4}} \tan^n x dx < \int_0^{\frac{\pi}{4}} (\tan^n x + \tan^{n-2}x) dx$$

$$\therefore I_n + I_{n+2} < 2 I_n < I_n + I_{n-2}$$

$$\therefore \frac{1}{n+1} < 2 I_n < \frac{1}{n-1} \quad \text{(i) અને (iii) પરથી)}$$

(C) પણ સત્ય જવાબ છે.

જવાબ : (A), (B), (C), (D)

$$(81) \int_0^{\pi} x f(\sin x) dx = \dots\dots$$

$$(A) \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx \quad (B) \frac{\pi}{4} \int_0^{\pi} f(\sin x) dx \quad (C) \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx \quad (D) \pi \int_0^{\frac{\pi}{2}} f(\cos x) dx$$

$$\text{ઉકેલ : } I = \int_0^{\pi} x f(\sin x) dx$$

$$I = \int_0^{\pi} (\pi - x) f(\sin(\pi - x)) dx$$

$$= \int_0^{\pi} (\pi - x) f(\sin x) dx$$

$$= \int_0^{\pi} \pi f(\sin x) dx - \int_0^{\pi} x f(\sin x) dx$$

$$I = \pi \int_0^{\pi} f(\sin x) dx - I$$

$$\therefore 2I = \pi \int_0^{\pi} f(\sin x) dx$$

$$\therefore I = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx \quad \text{(વિકલ્પ (A))}$$

$$= \frac{\pi}{2} \cdot 2 \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx \quad (\text{વિકલ્પ (C)})$$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} f(\cos x) dx \quad (\text{વિકલ્પ (D)})$$

જવાબ : (A), (C), (D)

નિષ્કર્ષ અને કારક (Assertion and Reason)

સમજૂતી : આ પ્રકારના પ્રશ્નોમાં બે વિધાન હોય છે. વિધાન 1 (નિષ્કર્ષ) અને વિધાન 2 (કારક). આ પ્રકારના પ્રશ્નોમાં ચાર વિકલ્પો હોય છે. વિકલ્પ (A), (B), (C) અને (D) નીચે પ્રમાણે હોય છે.

(A) વિધાન 1 અને વિધાન 2 બંને સત્ય છે અને વિધાન 2 એ વિધાન 1ની યોગ્ય સમજૂતી માટે પર્યાપ્ત છે.

(B) વિધાન 1 અને વિધાન 2 બંને સત્ય છે અને વિધાન 2 એ વિધાન 1ની સમજૂતી માટે પર્યાપ્ત નથી.

(C) વિધાન 1 સત્ય છે અને વિધાન 2 અસત્ય છે.

(D) વિધાન 1 અસત્ય છે અને વિધાન 2 સત્ય છે.

$$(82) \text{ વિધાન 1 : } \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{(3 + 4x^3) dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} = 4\sqrt{3} \tan^{-1} \frac{1}{2}$$

$$\text{વિધાન 2 : } \int_{-a}^a f(x) dx = \begin{cases} 0, & \text{જો } f \text{ અયુગ્મ વિધેય હોય તો,} \\ 2 \int_0^a f(x) dx, & \text{જો } f \text{ યુગ્મ વિધેય હોય તો,} \end{cases}$$

$$\text{ઉકેલ : } I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{(3 + 4x^3) dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$$

$$I = 2 \int_0^{\frac{\pi}{3}} \frac{3}{2 - \cos\left(x + \frac{\pi}{3}\right)} dx + 0$$

$$\therefore I = 2 \int_0^{\frac{\pi}{3}} \frac{3 dx}{2 - \cos\left(x + \frac{\pi}{3}\right)}$$

$$x + \frac{\pi}{3} = t \Rightarrow dx = dt$$

$$x = 0 \Rightarrow t = \frac{\pi}{3} \quad \text{અને} \quad x = \frac{\pi}{3} \Rightarrow t = \frac{2\pi}{3}$$

$$\therefore = 6 \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{dt}{2 - \cos t}$$

ધારો કે, $\tan \frac{t}{2} = u$ અર્થાત્, $dt = \frac{2du}{1+u^2}$, $\cos t = \frac{1-u^2}{1+u^2}$

$t = \frac{\pi}{3} \Rightarrow u = \frac{1}{\sqrt{3}}$ અને $t = \frac{2\pi}{3} \Rightarrow u = \sqrt{3}$

$$\therefore I = 12 \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{du}{1+3u^2}$$

$$= 4\sqrt{3} [\tan^{-1} \sqrt{3} u]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= 4\sqrt{3} [\tan^{-1} 3 - \tan^{-1} 1]$$

$$= 4\sqrt{3} \tan^{-1} \left(\frac{3-1}{1+3} \right)$$

$$= 4\sqrt{3} \tan^{-1} \left(\frac{1}{2} \right)$$

$\therefore$ વિધાન 1 અને 2 બંને સત્ય છે અને વિધાન 2 એ વિધાન 1ની યોગ્ય સમજૂતી આપે છે.

જવાબ : (A)

(83) વિધાન 1 : જો $y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos x \cdot \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$, તો $x = \pi$ આગળ $\frac{dy}{dx} = \pi$ ($x > 0$)

વિધાન 2 : $\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) dt = f(b(x)) \frac{d(b(x))}{dx} - f(a(x)) \frac{d(a(x))}{dx}$

ઉકેલ : $y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos x \cdot \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$

$$\therefore \frac{dy}{dx} = -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta + \cos x \frac{d}{dx} \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$$

$$= -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta + \frac{(\cos x)}{1 + \sin^2 x} \cdot 2x \cdot \cos x$$

હવે, જો $x = \pi$, તો $\frac{dy}{dx} = 0 - 2\pi$

$$\therefore \frac{dy}{dx} = -2\pi$$

$\therefore$ વિધાન 1 અસત્ય છે અને વિધાન 2 સત્ય છે.

જવાબ : (D)

(84) વિધાન 1 : $I_n = \int_0^1 (1 - x^5)^n dx$, તો $\frac{I_{10}}{I_{11}} = \frac{55}{54}$

વિધાન 2 : $u(x)$ અને $v(x)$ વિકલનીય વિધાનો હોય, તો $\int u dv = uv - \int v du$

$$\begin{aligned} \text{ઉકેલ : } I_{11} &= \int_0^1 (1 - x^5)^{11} dx \\ &= \int_0^1 (1 - x^5)(1 - x^5)^{10} dx \\ &= \int_0^1 (1 - x^5)^{10} dx - \int_0^1 x^5 (1 - x^5)^{10} dx \end{aligned}$$

$$I_{11} = I_{10} + \int_0^1 \frac{x}{5} (1 - x^5)^{10} (-5x^4) dx$$

$$\begin{aligned} I_{11} &= I_{10} + \left[\frac{x}{5} \frac{(1 - x^5)^{11}}{11} \right]_0^1 - \frac{1}{55} \int_0^1 (1 - x^5)^{11} dx \\ &= I_{10} - \frac{1}{55} I_{11} \end{aligned}$$

$$\therefore \left(1 + \frac{1}{55}\right) I_{11} = I_{10}$$

$$\therefore \frac{I_{10}}{I_{11}} = \frac{56}{55}$$

$\therefore$ વિધાન 1 અસત્ય છે જ્યારે વિધાન 2 સત્ય છે.

જવાબ : (D)

(85) વિધાન 1 : $\int_1^3 \left\{ \tan^{-1} \frac{x}{1+x^2} + \tan^{-1} \frac{x^2+1}{x} \right\} dx = \pi$

વિધાન 2 : $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ અને $\tan^{-1}x = \cot^{-1} \frac{1}{x}$, $x > 0$

$$\text{ઉકેલ : } \int_1^3 \left\{ \tan^{-1} \left(\frac{x}{1+x^2} \right) + \tan^{-1} \left(\frac{x^2+1}{x} \right) \right\} dx$$

$$= \int_1^3 \left\{ \tan^{-1} \frac{x}{1+x^2} + \cot^{-1} \frac{x}{1+x^2} \right\} dx$$

$$= \int_1^3 \frac{\pi}{2} dx = \frac{\pi}{2} [x]_1^3 = \frac{\pi}{2} (3 - 1) = \pi$$

$\therefore$ વિધાન 1 સત્ય છે અને વિધાન 2 સત્ય છે તથા વિધાન 2 વિધાન 1ની સમજૂતી આપે છે.

જવાબ : (A)

(86) વિધાન 1 : જો $n > 1$, તો $\int_0^{\infty} \frac{dx}{1+x^n} = \int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}}$

વિધાન 2 : $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

ઉકેલ : ધારો કે $I_1 = \int_0^{\infty} \frac{dx}{1+x^n}$

ધારો કે, $x^n = \tan^2 \theta$. અર્થાત્, $n \cdot x^{n-1} dx = 2 \tan \theta \sec^2 \theta d\theta$

$$dx = \frac{2 \tan \theta \sec^2 \theta d\theta}{n x^{n-1}}$$

$$= x \frac{2 \tan \theta \sec^2 \theta d\theta}{n x^n}$$

$$= \tan^n \theta \frac{2 \tan \theta \sec^2 \theta d\theta}{n \tan^2 \theta}$$

$$= \frac{2}{n} \tan^{\left(\frac{2}{n}-1\right)} \theta \sec^2 \theta d\theta$$

$x = 0 \Rightarrow \theta = 0$ અને $x = \infty \Rightarrow \theta = \frac{\pi}{2}$

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{2}{n} \cdot \tan^{\left(\frac{2}{n}-1\right)} \theta \cdot d\theta$$

$$\therefore I_1 = \frac{2}{n} \int_0^{\frac{\pi}{2}} (\tan \theta)^{\left(\frac{2}{n}-1\right)} d\theta$$

(i)

હવે ધારો કે, $I_2 = \int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}}$

ધારો કે, $x^n = \sin^2 \phi$ અર્થાત્, $1 - x^n = 1 - \sin^2 \phi = \cos^2 \phi$

$$\therefore (1-x^n)^{\frac{1}{n}} = (\cos \phi)^{\frac{2}{n}}$$

તથા જ્યારે $x = 0 \Rightarrow \phi = 0$ તથા $x = 1 \Rightarrow \phi = \frac{\pi}{2}$

$$\therefore dx = \frac{2}{n} \sin^{\left(\frac{2}{n}-1\right)} \phi \cos \phi d\phi$$

$$\therefore I_2 = \int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}}$$

$$= \frac{2}{n} \int_0^{\frac{\pi}{2}} \left(\frac{\sin \phi}{\cos \phi} \right)^{\frac{2}{n}-1} d\phi$$

$$= \frac{2}{n} \int_0^{\frac{\pi}{2}} (\tan \phi)^{\frac{2}{n}-1} d\phi = I_1$$

$$\therefore I_2 = I_1$$

તેથી, વિધાન 1 સત્ય છે તથા વિધાન 2 પણ સત્ય છે. પણ વિધાન 2 એ વિધાન 1ની યોગ્ય સમજૂતી આપતું નથી. જવાબ : (B)

$$(87) \text{ વિધાન 1 : } \int_0^{10\pi} |\cos x| dx = 20$$

$$\text{વિધાન 2 : જો } \int_a^b f(x) dx \geq 0 \text{ તો } f(x) \geq 0, \forall x \in [a, b].$$

$$\begin{aligned} \text{ઉકેલ : } \int_0^{10\pi} |\cos x| dx &= 10 \int_0^{\pi} |\cos x| dx \\ &= 20 \int_0^{\frac{\pi}{2}} \cos x dx = 20 \end{aligned}$$

તેથી, વિધાન 1 સત્ય છે. વિધાન 2 અસત્ય છે. કારણ કે સત્ય વિધાન આ પ્રમાણે બને.

$$\text{નોંધ : જો } [a, b] \text{ માં } f(x) \geq 0 \text{ તો } \int_a^b f(x) dx \geq 0.$$

જવાબ : (C)

$$\text{જો } f(x) = x^2 - 3x + 2 = (x-1)(x-2)$$

$$f(x) < 0, x \in (1, 2) \text{ માટે}$$

$$\begin{aligned} \int_0^3 (x^2 - 3x + 2) dx &= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x \right]_0^3 \\ &= \left(9 - \frac{27}{2} + 6 \right) = \frac{3}{2} > 0 \end{aligned}$$

$$(88) \text{ વિધાન 1 : જો } g(x) = \int_0^{\frac{\pi}{2}} (\sin^6 x + \cos^6 x) dx, \text{ તો } \frac{\pi}{8} < g(x) < \frac{\pi}{2}$$

વિધાન 2 : વિધેય $f(x)$ ની ન્યૂનતમ અને મહત્તમ કિંમતો m અને M હોય, જ્યાં

$$m \leq f(x) \leq M \text{ તો } m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

$$\begin{aligned} \text{ઉકેલ : } f(x) &= \sin^6 x + \cos^6 x \\ &= (\sin^2 x)^3 + (\cos^2 x)^3 \\ &= (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x \\ &= 1 - \frac{3}{4} \sin^2 2x \end{aligned}$$

$$\text{હવે, } 0 \leq \sin^2 2x \leq 1$$

$$\therefore 0 \geq -\frac{3}{4} \sin^2 2x \geq -\frac{3}{4}$$

$$\therefore -\frac{3}{4} \leq -\frac{3}{4} \sin^2 2x \leq 0$$

$$\therefore \frac{1}{4} \leq 1 - \frac{3}{4} \sin^2 2x \leq 1$$

$$\therefore \frac{1}{4} \leq f(x) \leq 1$$

$$\therefore m = \frac{1}{4} \text{ અને } M = 1$$

$$\text{તેથી, } \left(\frac{\pi}{2} - 0\right) \frac{1}{4} < \int_0^{\frac{\pi}{2}} (\sin^6 x + \cos^6 x) dx < \left(\frac{\pi}{2} - 0\right) \cdot 1$$

$$\therefore \text{વિધાન 2 પ્રમાણે } \frac{\pi}{8} < g(x) < \frac{\pi}{2}.$$

$\therefore$ વિધાન 1 સત્ય છે તથા વિધાન 2 પણ સત્ય છે અને વિધાન 2 એ વિધાન 1ની યોગ્ય સમજૂતી માટે પર્યાપ્ત છે.

જવાબ : (A)

$$(89) \text{ વિધાન 1 : } \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\cot x}} = \frac{\pi}{12}$$

$$\text{વિધાન 2 : } \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(x) dx}{f(x) + f\left(\frac{\pi}{2} - x\right)} = \frac{\pi}{12}$$

$$\text{ઉકેલ : ધારો કે, } I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(x) dx}{f(x) + f\left(\frac{\pi}{2} - x\right)} \quad \dots(i)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right)}{f\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right) + f\left(\frac{\pi}{2} - \left(\frac{\pi}{6} + \frac{\pi}{3} - x\right)\right)} dx$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f\left(\frac{\pi}{2} - x\right)}{f\left(\frac{\pi}{2} - x\right) + f(x)} dx \quad \dots(ii)$$

સમીકરણ (i) અને (ii)નો સરવાળો કરતાં,

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore 2I = \frac{\pi}{6}. \quad \text{આથી, } I = \frac{\pi}{12}$$

$\therefore$ વિધાન 2 સત્ય છે.

$$\text{હવે ધારો કે, } f(x) = \frac{1}{1 + \sqrt{\cot x}} = \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}$$

$$\therefore f(x) = \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}$$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x) dx = \frac{\pi}{12}$$

$\therefore$ વિધાન 1 સત્ય છે તથા વિધાન 2 પણ સત્ય છે અને વિધાન 2 એ વિધાન 1ની યોગ્ય સમજૂતી માટે પર્યાપ્ત છે. જવાબ : (A)

(90) વિધાન 1 : $\int_0^{100} (x - [x]) dx = 50$, જ્યાં $[x]$ પૂર્ણાંક ભાગ વિધેય છે.

વિધાન 2 : વિધેય $f(x)$ એ આવર્તી વિધેય હોય તથા તેનું આવર્તમાન T હોય, તો

$$\int_0^{nT} f(x) dx = n \int_0^T f(x) dx.$$

ઉકેલ : વિધેય $f(x)$ એ આવર્તી વિધેય હોય તથા તેનું આવર્તમાન T હોય, તો

$$\int_0^{nT} f(x) dx = n \int_0^T f(x) dx.$$

$\therefore$ વિધાન 2 સત્ય છે.

(આવર્તી વિધેયનો ગુણધર્મ)

$$\text{હવે, } \int_0^{100} (x - [x]) dx = \int_0^{100(1)} (x - [x]) dx$$

$$= 100 \int_0^1 (x - [x]) dx \quad (x - [x] \text{ નું આવર્તમાન } 1 \text{ છે.})$$

$$= 100 \int_0^1 x dx \quad ([0, 1] \text{ માં } [x] = 0)$$

$$= 100 \left[\frac{x^2}{2} \right]_0^1$$

$$= 100 \times \frac{1}{2} = 50$$

જવાબ : (A)

$\therefore$ વિધાન 1 સત્ય છે તથા વિધાન 2 પણ સત્ય છે અને વિધાન 2 એ વિધાન 1ની યોગ્ય સમજૂતી માટે પર્યાપ્ત છે.

