2023

Sample Questions

Junior Research Fellowship (JRF) in Computer Science (CS)

Test Code: CSA (Forenoon Examination)

Note that all questions in the sample set are not at the same level of difficulty, and may not carry equal marks in an examination.

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1. Let us consider the following 2-person game: the players alternately choose a number. The first player starts with a number between 1 and 10, and the players then pick up a number within the next ten of the number that his opponent has chosen earlier. The player who is able to select the number 100 first, wins the game. Can the first player pick up a number between 1 and 10 such that whatever may be the strategy of his opponent, the first player will be able to reach 100 first?

2. Five men A, B, C, D, E are wearing caps of black or white color without each knowing the color of his cap. It is known that a man wearing a white cap will always speak the truth while a man wearing a black cap always lies. They make the following statements.

   A: I see three white and one black cap.

   B: I see four black caps.

   C: I see one white and three black caps.

   D: I see four white caps.

   Find the color of each person's cap.

3. Consider an array $X = [x_1, x_2, \ldots, x_n]$ of n real numbers sorted in ascending order as input. Another array $Y = [y_1, y_2, \ldots, y_n]$ is created such that $y_i = x_i^2 + x_i + 1$. Write an efficient algorithm to sort Y. (You will get full credit if your algorithm uses no more than kn comparisons, where k is some constant.)

4. Let A be an array of n integers containing the numbers $\{1, 2, \ldots, n\}$ in some arbitrary order. For integers i and j such that $1 \leq i < j \leq n$, let $\textsc{Reverse}(A, i, j)$ be a procedure that reverses the subarray $A[i], A[i+1], \ldots, A[j]$ of the array A while leaving the remaining elements of the array unaffected. For example, if $A = [3, 4, 1, 2, 5]$, then, after $\textsc{Reverse}(A, 2, 4)$, we will have $A = [3, 2, 1, 4, 5]$.

   Precisely what will happen to the array A if the following piece of code is executed? Justify your answer with an example enumerating different steps.

   for $i := 1$ to $n - 1$
       while $A[i] \neq i$ **do**
           $\textsc{Reverse}(A, i, A[i])$.

5. Consider a snail at the foot of a pole that is `h` metres high. The snail starts climbing up the pole at an initial speed of `c` metres / hour. After every `d` hours of climbing, the snail takes a nap for `z` hours. While it is asleep, it slips down the pole at `s` metres / hour. After waking up from the nap, it continues climbing, but its speed reduces by `p`% of the speed that it had just before it took this nap.

   In your answer booklet, write the expressions / conditions corresponding to the blanks in the pseudo-code below, so that the code determines the total number of hours it takes the snail to reach the top of the pole. If the snail can never reach the top of the pole, the code should print an appropriate message.

Step 1. Set `remainingHeight` to `h`, `speed` to `c`, and `numHours` to `0`.

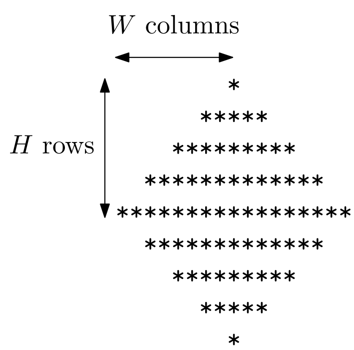
Step 2. If ____________________, print message:
"Snail reaches the top in ____________________ hours"
and STOP.

Step 3. Otherwise, increment `numHours` by ____________________;
decrement `remainingHeight` by ____________________.

Step 4. If ____________________, print message:
"Snail will never be able to reach the top"
and STOP.

Step 5. Otherwise, set `speed` to ____________________,
and repeat from Step 2.

6. Write pseudo-code to print a diamond formed using the character * as shown in the figure below. Let H and W denote the number of rows, and the number of columns occupied by the upper half and the left half of the diamond, respectively. In the example shown below, $H = 5$ and $W = 9$. Your code should take H and W as input parameters. You may assume that $W - 1$ is an integral multiple of $H - 1$.



7. Consider the pseudo-code given below.

   Input: Integers b and c.

   1. $a_0 \leftarrow \max(b, c)$, $a_1 \leftarrow \min(b, c)$.
   2. $i \leftarrow 1$.
   3. Divide a_{i-1} by a_i. Let q_i be the quotient and r_i be the remainder.
   4. If $r_i = 0$, then go to Step 8.
   5. $a_{i+1} \leftarrow a_{i-1} - q_i * a_i$.
   6. $i \leftarrow i + 1$.
   7. Go to Step 3.
   8. Print a_i.

   What is the output of the above algorithm when $b = 28$ and $c = 20$? What is the mathematical relation between the output a_i and the two inputs b and c?

8. The integers 1, 2, 3, 4 and 5 are to be inserted into an empty stack using the following sequence of PUSH() operations:

   PUSH(1) PUSH(2) PUSH(3) PUSH(4) PUSH(5)

Assume that POP() removes an element from the stack and outputs the same. Which of the following output sequences can be generated by inserting suitable POP() operations into the above sequence of PUSH() operations? Justify your answer.

(a) 5 4 3 2 1
(b) 1 2 3 4 5
(c) 3 2 1 4 5
(d) 5 4 1 2 3.

9. Given a function $f : A \to A$, an element $x \in A$ is said to be a fixed point of f if and only if $f(x) = x$. Let $f : \{1, 2, \ldots, 100\} \to \{1, 2, \ldots, 100\}$ be a function. For all $S \subseteq \{1, 2, \ldots, 100\}$, suppose a procedure `FIXED`(S) can determine whether the function f has at least one fixed point in S or not. Define a strategy to determine whether the function f has at least two fixed points by executing the procedure `FIXED` at most 15 times.

10. Give a strategy to sort four distinct integers a, b, c, d in increasing order that minimizes the number of pairwise comparisons needed to sort any permutation of a, b, c, d.

11. An $n \times n$ matrix is said to be *tridiagonal* if its entries a_{ij} are zero except when $|i - j| \leq 1$ for $1 \leq i, j \leq n$. Note that only $3n - 2$ entries of a tridiagonal matrix are non-zero. Thus, an array L of size $3n - 2$ is sufficient to store a tridiagonal matrix. Given i, j, write pseudo-code to

   (a) store a_{ij} in L, and
   (b) get the value of a_{ij} stored in L.

12. Given an array $A = \{a_1, a_2, \ldots, a_n\}$ of unsorted distinct integers, write a program in *pseudo-code* for the following problem: given an integer u, arrange the elements of the array A such that all the elements in A which are less than or equal to u are at the beginning of the array, and the elements which are greater than u are at the end of the array. You may use at most 5 extra variables apart from the array A.

13. Let n be a positive integer. Consider the set

$$S = \{-n, -n+1, -n+2, \ldots, -1, 0, 1, 2, \ldots, n-1, n\}.$$

Let $f : S \times S \to \{-1, 1\}$ be a function. If $f(-a, -b) = -f(a, b)$ $\forall a, b \in S$, f is said to be **odd**. If $f(-a, -b) = f(a, b)$ $\forall a, b \in S$, f is said to be **even**.

(a) What is the total number of different functions from $S \times S$ to $\{-1, 1\}$?

(b) How many of these are odd?

(c) How many of these are neither odd nor even?

14. How many functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there that are *monotone*; that is, for $i < j$ we have $f(i) \leq f(j)$?

15. How many 0's are there at the end of 50! ?

16. Let $b_q b_{q-1} \ldots b_1 b_0$ be the binary representation of an integer b, i.e.,

$$b = \sum_{j=0}^{q} 2^j b_j, \ \ b_j = 0 \text{ or } 1, \ \text{ for } j = 0, 1, \ldots, q$$

Show that b is divisible by 3 if $b_0 - b_1 + b_2 - \ldots (-1)^q b_q = 0$.

17. A set S contains integers 1 and 2. S also contains all integers of the form $3x+y$ where x and y are distinct elements of S, and every element of S other than 1 and 2 can be obtained as above. What is S? Justify your answer.

18. Show that if S is any subset of $\{1, 2, \ldots, 2n\}$ such that $|S| > n$, then S contains two numbers such that one of them divides the other.

19. Prove that every natural number $n \geq 1$ has a multiple whose decimal representation contains only 1s and 0s. (Hint: Consider the numbers 1, 11, 111, 1111, … and their remainders modulo n).

20. How many k-element subsets of $\{1, 2, \ldots, n\}$ exist that contain no two consecutive numbers?

21. Let T be an equilateral triangle of unit area. Let $P_1, \ldots, P_9$ be 9 points inside T. Show that, out of these 9 points, we can always find 3 points such that the area of the triangle formed by these three points will be at most $\frac{1}{4}$ unit.

22. Let $\mathbb{N}$ denote the set of natural numbers. For each of the following sets, state whether it is finite or infinite. Justify your answers.
    (a) $\{f : \mathbb{N} \to \{0,1\} \mid \forall n \in \mathbb{N},\ f(n) \leq f(n+1)\}$
    (b) $\{f : \mathbb{N} \to \{0,1\} \mid \forall n \in \mathbb{N},\ f(2n) \neq f(2n+1)\}$
    (c) $\{f : \mathbb{N} \to \{0,1\} \mid \forall n \in \mathbb{N},\ f(n) \neq f(n+1)\}$

23. Find the number of distinct positive integers that can be formed using 0, 1, 2 and 4, where each of these integers is used at most once.

24. Derive an expression for the maximum number of regions that can be formed within a circle by drawing n chords.

25. Given $A = \{1, 2, 3, \ldots, 70\}$, show that for any six elements a_1, a_2, a_3, a_4, a_5 and a_6 belonging to A, there exists one pair a_i and a_j for which $|a_i - a_j| \leq 14$ $(i \neq j)$.

26. Calculate how many integers in the set $\{1, 2, 3, \ldots, 1000\}$ are not divisible by 2, 5, or 11.

27. Consider all the permutations of the digits $1, 2, \ldots, 9$. Find the number of permutations each of which satisfies *all* of the following:
    - the sum of the digits lying between 1 and 2 (including 1 and 2) is 12,
    - the sum of the digits lying between 2 and 3 (including 2 and 3) is 23,
    - the sum of the digits lying between 3 and 4 (including 3 and 4) is 34, and
    - the sum of the digits lying between 4 and 5 (including 4 and 5) is 45.

28. State, with justification, which of the following expressions f, g and

h, define valid real-valued functions over the set of positive rational numbers. We denote a rational number by m/n, where m and n are positive integers.

(a) $f(m/n) = 2^m - 2^n$.
(b) $g(m/n) = \log m - \log n$.
(c) $h(m/n) = (m^2 - n^2)/(mn)$.

29. There are n students in a class. The students have formed k committees. Each committee consists of more than half of the students. Show that there is at least one student who is a member of more than half of the committees.

30. Consider an $m \times n$ integer grid. A *path* from the lower left corner at $(0, 0)$ to the grid point (m, n) can use three kinds of steps, namely (i) $(p, q) \to (p+1, q)$ (horizontal), (ii) $(p, q) \to (p, q+1)$ (vertical), or (iii) $(p, q) \to (p+1, q+1)$ (diagonal). Derive an expression in terms of m and n for $D_{m,n}$, the number of such distinct paths.

31. The numbers $1, 2, \ldots, 10$ are circularly arranged. Show that there are always three adjacent numbers whose sum is at least 17, irrespective of the arrangement.

32. Consider six distinct points in a plane. Let m and M denote respectively the minimum and the maximum distance between any pair of points. Show that $M/m \geq \sqrt{3}$.

33. A group of 15 boys plucked a total of 100 apples. Prove that two of those boys plucked the same number of apples.

34. Suppose X is a set such that for every function $f : X \to X$, f is one-to-one if and only if f is onto. Show that every one-to-one function $f : P(P(X)) \to P(P(X))$ is onto, where $P(A)$ denotes the set of all subsets of a set A.

35. Find the value of $\sum ij$, where the summation is over all integers i and j such that $1 \leq i < j \leq 10$.

36. Let $S = \{x \in \mathbb{R} : 1 \leq |x| \leq 100\}$. Find all subsets M of S such that for all x, y in M, their product xy is also in M.

37. A 3×3 magic square is an arrangement of the numbers from a set of odd integers $\{1, 3, 5, \ldots, 17\}$ in a 3×3 square grid, where the numbers in each row, in each column, and in the main and secondary diagonals, all add up to 27. Prove that the element at the center of the grid is 9.

38. How many triplets of real numbers (x, y, z) are simultaneous solutions of the equations $x + y = 2$ and $xy - z^2 = 1$?

39. The king's minter keeps mn coins in n boxes each containing m coins. Each box contains 2 false coins out of m coins. The king suspects the minter and randomly draws 1 coin from each of the n boxes and has these tested. What is the probability that the minter's dishonest actions go undetected?

40. What is the probability of getting a sum of 5 when throwing 3 fair dice?

41. A person throws a pair of fair dice. If the sum of the numbers on the dice is a perfect square, compute the probability that the number 3 appeared on at least one of the dice.

42. A fair die is rolled five times. What is the probability that the largest number rolled is 5?

43. In a 40-student class, what is the probability that at least two students have the same birthday? We assume that there are 365 days in each year to simplify our questions.

44. Suppose two positive numbers are chosen randomly from 1 to 50. What is the probability that their difference is divisible by 3?

45. The chance of a student getting admitted to colleges A and B are 60% and 40%, respectively. Assume that the colleges admit students

independently. If the student is told that he has been admitted to at least one of these colleges, compute the probability that he has got admitted to college A.

46. Let $A_1 = (0,0), A_2 = (1,0), A_3 = (1,1) and A_4 = (0,1)$ be the four vertices of a square. A particle starts from the point A_1 at time 0 and moves either to A_2 or to A_4 with equal probability. Similarly, in each of the subsequent steps, it randomly chooses one of its adjacent vertices and moves there. Let T be the minimum number of steps required to cover all four vertices. Compute $P(T = 4)$.

47. If G is a group such that $(a \cdot b)^2 = a^2 \cdot b^2$ for all $a, b \in G$, show that G must be Abelian.

48. Let G be a group and H a non-empty finite subset of G which is closed under the group operation of G. Prove that H is a subgroup of G.

49. Let G be a group which has only a finite number of subgroups. Prove that G must be a finite group.

50. Let G be a group. Let a and b be two elements of G such that both have finite orders. Is the order of ab always finite? Justify your answer.

51. In a group of n persons, each person is asked to write down the sum of the ages of all the other $(n-1)$ persons. Suppose the sums so obtained are $s_1, \ldots, s_n$. It is now desired to find the actual ages of the persons from these values.
    (a) Formulate the problem in the form of a system of linear equations.
    (b) Can the ages be always uniquely determined? Justify your answer.

52. Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ be linearly independent vectors in the vector space $V \subseteq \mathbb{R}^n$. Then, which of the following is/are true and why?
    (a) $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{z}, \mathbf{z}+\mathbf{u}\}$ spans V.
    (b) $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{z}, \mathbf{z}+\mathbf{u}\}$ is linearly independent.
    (c) $\text{span}\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{z}, \mathbf{z}+\mathbf{u}\}$ is contained in the span $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$.

(d) $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{z}, \mathbf{z}+\mathbf{u}\}$ is a basis of V.

53. Let $A = xy^T$ (matrix obtained by multiplying a column matrix (x) with row matrix (y^T)).
    (a) What is the rank of A?
    (b) Find an eigenvector of A and its corresponding eigenvalue.

54. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Prove or disprove that for every $x, y \in \mathbb{R}^2$ and for every θ, we have $x^T A^T A y = x^T y$.

55. For what value of α is the following true:

$$\det \begin{bmatrix} s+k & t+l & u+m \\ d+k & e+l & f+m \\ d+s & e+t & f+u \end{bmatrix} = \alpha \det \begin{bmatrix} d & e & f \\ s & t & u \\ k & l & m \end{bmatrix}.$$

56. Suppose P and Q are $n \times n$ matrices of real numbers such that
    - $P^2 = P$,
    - $Q^2 = Q$, and
    - $I - P - Q$ is invertible, where, I is a $n \times n$ identity matrix.

    Show that P and Q have the same rank.

57. If $1, a_1, a_2, \ldots, , a_{n-1}$ are the n roots of unity, find the value of

$$(1-a_1)(1-a_2)\ldots(1-a_{n-1}).$$

58. (a) Compute the number of real roots of the polynomial $x^3 - 2x + 7$.
    (b) Compute the number of positive real roots of the cubic equation $x^3 = 2x + 5$.

59. If α is a root of $x^2 - x + 1 = 0$, compute $\alpha^{2018} + \alpha^{-2018}$.

60. Compute the number of real solutions to the equation $x^7 + 2x^5 + 3x^3 + 4x = 2022$.

61. Let $a_1, a_2, \ldots, a_n$ be real numbers satisfying $\sum\limits_{k=1}^{n} \frac{a_k}{k} = 0$. Show that the polynomial $a_1 + a_2 x + \ldots + a_n x^{n-1}$ has a root in the interval $(0, 1)$.

62. Evaluate

$$\lim_{x \to 0} \left(x^2 \left(1 + 2 + 3 + \cdots + \left[\frac{1}{|x|} \right] \right) \right)$$

For any real number a, $[a]$ is the largest integer not greater than a.

63. Is $\sin(x|x|)$ differentiable for all real x? Justify your answer.

64. A sequence $\{x_n\}$ is defined by $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2 + x_n}$ for $n = 1, 2, \ldots$ Show that the sequence converges and find its limit.

65. Consider the sequence $a_n = a_{n-1}\, a_{n-2} + n$ for $n \geq 2$, with $a_0 = 1$ and $a_1 = 1$. Is a_{2011} odd? Justify your answer.