

Some Useful Results :

(1) Pascal's Triangle : The coefficients of x^r in the expansion of $(1 + x)^n$. ($r = 0, 1, 2, 3, \dots, n$)

$n = 1$				1		1								
$n = 2$				1		2		1						
$n = 3$				1		3		3		1				
$n = 4$				1		4		6		4		1		
$n = 5$				1		5		10		10		5		1
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

- Each row starts and ends in 1 and remaining elements of the row are sums of the two coefficients which lie on either side of each in the previous row.
- $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$ are $(n + 1)$ terms in n th row.
- The binomial coefficients of the terms which are equidistant from the beginning and from the end are always equal.
- The sum of the coefficients of n th row in Pascal's triangle is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$.
- In Pascal's triangle, the coefficient of $(r + 1)$ th term in n th row is $\binom{n}{r}$.

(2) Binomial theorem :

$$(a + b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n} a^0 b^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r \quad (n \in \mathbb{N})$$

(3) General term in the expansion of $(a + b)^n$:

$$T_{r+1} = (r + 1)\text{th term} = \binom{n}{r} a^{n-r} b^r \quad (0 \leq r \leq n)$$

(4) Middle term in the expansion of $(a + b)^n$:

$$(1) \text{ If } n \text{ is even, then the middle term} = \left(\frac{n}{2} + 1\right)\text{th term} = T_{\frac{n}{2}+1} = \binom{n}{\frac{n}{2}} a^{\frac{n}{2}} b^{\frac{n}{2}}$$

$$(2) \text{ If } n \text{ is odd, then the middle terms} = \left(\frac{n+1}{2}\right)\text{th term and } \left(\frac{n+3}{2}\right)\text{th term.}$$

$$= \frac{T_{n+1}}{2} = \binom{n}{\frac{n-1}{2}} a^{\frac{n+1}{2}} b^{\frac{n-1}{2}} \quad \text{and} \quad \frac{T_{n+3}}{2} = \binom{n}{\frac{n+1}{2}} a^{\frac{n-1}{2}} b^{\frac{n+1}{2}}$$

Here we get two middle terms.

(5) Characteristics of the expansion of $(a + b)^n$:

- There are $n + 1$ terms in the expansion.
- In each term, the sum of the exponents of a and b is n .
- The coefficients of the terms are $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$. They are called binomial coefficients and sum of the binomial coefficients is 2^n .
- The coefficient of $(r + 1)$ th term = coefficient of $a^{n-r} b^r = \binom{n}{r}$
- r th term from the end in the expansion of $(b + a)^n = \binom{n}{r-1} b^{n-r+1} a^{r-1} = (n + 2 - r)$ term from the beginning in the expansion of $(a + b)^n$.
- In each successive term the exponent of a goes on decreasing by 1 and that of b goes on increasing by 1.

(6) Results :

- $(a - b)^n = \binom{n}{0} a^n b^0 - \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 - \dots + (-1)^r \binom{n}{r} a^{n-r} b^r + \dots + (-1)^n \binom{n}{n} a^0 b^n$
- In the expansion of $(a - b)^n$, $(r + 1)$ th term = $T_{r+1} = (-1)^r \binom{n}{r} a^{n-r} b^r$
- $(a - b)^n = \sum_{r=0}^n (-1)^r \binom{n}{r} a^{n-r} b^r$
- $(1 + x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{n} x^n, n \in \mathbb{N} = \sum_{r=0}^n \binom{n}{r} x^r$
- In the expansion of $(1 + x)^n$, $(r + 1)$ th term = $T_{r+1} = \binom{n}{r} x^r$
- In the expansion of $(1 + x)^n$, the coefficient of $(r + 1)$ th term = coefficient of $x^r = \binom{n}{r}$
- $(1 - x)^n = \binom{n}{0} - \binom{n}{1} x + \binom{n}{2} x^2 - \binom{n}{3} x^3 + \dots + (-1)^n \binom{n}{n} x^n \quad (n \in \mathbb{N})$

- $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$
- $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = 2^{n-1}$
- The number of terms in the expansion of $(a + b + c)^n = \Sigma (n + 1) = \frac{(n+1)(n+2)}{2}$ and the sum of the coefficients $= 3^n$
- $\binom{n}{r} : \binom{n}{r-1} = \frac{n-r+1}{r}$
- $(x + y)^n + (x - y)^n = 2 \left[\binom{n}{0} x^n + \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 + \dots \right]$
- $(x + y)^n - (x - y)^n = 2 \left[\binom{n}{1} x^{n-1} y + \binom{n}{3} x^{n-3} y^3 + \binom{n}{5} x^{n-5} y^5 + \dots \right]$

(7) Binomial series for rational indices :

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots \text{infinite number of terms.}$$

where $n \in \mathbb{Q}$ and $|x| < 1$

For example,

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots \text{infinite number of terms.}$$

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots \text{infinite number of terms.}$$

$$(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots \text{infinite number of terms.}$$

$$(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \text{infinite number of terms.}$$

$$(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots \text{infinite number of terms.}$$

$$(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots \text{infinite number of terms.}$$

(8) General term :

(1) General term in the expansion of $(1 + x)^n$

$$T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r, n \in \mathbb{Q}$$

(2) General term in the expansion of $(1 - x)^n$

$$T_{r+1} = (-1)^r \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r, n \in \mathbb{Q}$$

(3) General term in the expansion of $(1 + x)^{-n}$

$$T_{r+1} = (-1)^r \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} x^r = x^r (-1)^r \binom{n+r-1}{r}, \text{ where } n \in \mathbb{N}$$

(4) General term in the expansion of $(1-x)^{-n}$

$$T_{r+1} = \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} x^r, n \in \mathbb{Q}$$

$$= \binom{n+r-1}{r} x^r, \text{ where } n \in \mathbb{N}$$

(9) $(1+x)^n = 1 + nx$, $|x| < 1$, $n \in \mathbb{Q}$, neglecting $x^2, x^3 \dots$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2, |x| < 1, n \in \mathbb{Q}, \text{ neglecting } x^3, x^4 \dots$$

(10) The total number of ways of distributing n identical objects among r persons so that each gets maximum n objects :

$$= \text{The coefficient of } x^n \text{ in } (1+x+x^2+\dots+x^n)^r$$

$$= \text{The coefficient of } x^n \text{ in } (1-x)^r (1+x+x^2+\dots+x^n)^r (1-x)^{-r}$$

$$= \text{The coefficient of } x^n \text{ in } (1-x^{n+1})^r (1-x)^{-r}$$

$$= \text{The coefficient of } x^n \text{ in } (1-x)^{-r}$$

$$= \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

(11) The total number of ways of distributing n identical objects among r persons so that every one of them receives atleast one object :

$$= \text{The coefficient of } x^n \text{ in } (x+x^2+x^3+\dots+x^n)^r$$

$$= \text{The coefficient of } x^{n-r} \text{ in } (1+x+x^2+\dots+x^{n-1})^r$$

$$= \text{The coefficient of } x^{n-r} \text{ in } (1-x)^r (1+x+x^2+\dots+x^{n-1})^r (1-x)^{-r}$$

$$= \text{The coefficient of } x^{n-r} \text{ in } (1-x^n)^r (1-x)^{-r}$$

$$= \text{The coefficient of } x^{n-r} \text{ in } (1-x)^{-r}$$

$$= \binom{r+n-r-1}{n-r} = \binom{n-1}{n-r} = \binom{n-1}{r-1}$$

(12) The number of ways in which n identical objects can be distributed among r persons when any one of them receives not less than m and not more than k items :

$$= \text{The coefficient of } x^n \text{ in } (x^m + x^{m+1} + x^{m+2} + \dots + x^k)^r.$$

The number of solutions of a linear equation :

The number of non-negative integral solutions of $x_1 + x_2 + x_3 + \dots + x_m = n$

$$= \text{The coefficient of } x^n \text{ in } (1+x+x^2+\dots+x^n)^r$$

$$= \text{The coefficient of } x^n \text{ in } (1-x)^r (1+x+x^2+\dots+x^n)^r (1-x)^{-r}$$

= The coefficient of x^n in $(1 - x^{n+1})^r (1 - x)^{-r}$

= The coefficient of x^n in $(1 - x)^{-r}$

$$= \binom{n+r-1}{n}$$

(13) The number of ways in which an ordered set of r whole numbers ($r \geq 2$) can be selected such that their sum is n :

= The coefficient of x^n in $(1+x+x^2+\dots+x^n)^r$

= The coefficient of x^n in $(1-x)^r (1+x+x^2+\dots+x^n)^r (1-x)^{-r}$

= The coefficient of x^n in $(1-x^{n+1})^r (1-x)^{-r}$

= The coefficient of x^n in $(1-x)^{-r}$

$$= \binom{n+r-1}{r-1} = \binom{n+r-1}{n}$$

(14) If there are m_1 items of one type, m_2 items of second type and so on m_r item of r th type, then the number of ways of choosing n items out of these :

= The coefficient of x^n in $(1+x+x^2+\dots+x^{m_1}) \times (1+x+x^2+\dots+x^{m_2}) \times \dots \times (1+x+x^2+\dots+x^{m_r})$

= The coefficient of x^n in $(1-x)^{-r} = \binom{n+r-1}{n}$

(15) If there are m_1 items of one type, m_2 items of second type, and so on m_r items of r th type, then the number of ways of choosing n items out of these, provided at least one item of each type is selected.

= The coefficient of x^n in $(x+x^2+\dots+x^{m_1})(x+x^2+\dots+x^{m_2}) \times \dots \times (x+x^2+\dots+x^{m_r})$

= The coefficient of x^n in $x^r (1+x+x^2+\dots+x^{m_1-1})(1+x+x^2+\dots+x^{m_2-1}) \times \dots \times (1+x+x^2+\dots+x^{m_r-1})$

= The coefficient of x^n in $(1+x+x^2+\dots+x^{m_1-1})(1+x+x^2+\dots+x^{m_2-1}) \times \dots \times (1+x+x^2+\dots+x^{m_r-1})$

= The coefficient of x^{n-r} in $(1-x)^{-r}$

$$= \binom{r+n-r-1}{n-r} = \binom{n-1}{n-r} = \binom{n-1}{r-1}$$

(16) Multinomial theorem :

$$(a_1 + a_2 + \dots + a_m)^n = \sum \frac{n!}{n_1! n_2! \dots n_m!} a_1^{n_1} a_2^{n_2} \dots a_m^{n_m} \text{ where } n_1 + n_2 + n_3 + \dots + n_m = n \text{ and}$$

$n_1, n_2, n_3, \dots, n_m$ are all non-negative integers.

$$\text{Here, the number of distinct terms} = \binom{n+m-1}{m-1} = \binom{n+m-1}{n}$$

$$\bullet \text{ The number of terms in } (a+b+c)^n = \binom{n+3-1}{3-1} = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2} = \sum (n+1)$$

(17) Properties of binomial coefficients :

- If binomial coefficient $\binom{n}{r-1}, \binom{n}{r}, \binom{n}{r+1}$ are in A.P., then $r = \frac{1}{2}(n + \sqrt{n+2})$
- The coefficient of x^n in $(1+x)^{2n} = 2 \times \{\text{the coefficient of } x^n \text{ in } (1+x)^{2n-1}\}$
- In the expansion of $(ax+by)^n$, if we take $x=y=1$, we get the sum of the coefficients of $(a+b)^n$.
- The greatest term in the expansion of $(x+y)^n, n \in \mathbb{N}, x > 0, y > 0$

$T_m = T_{m+1}$. If $m = \frac{(n+1)y}{x+y}$ is a positive integer.

and $T_{[m]+1}$, if $m = \frac{(n+1)y}{x+y}$ is not a positive integer.

- In the expansion of $(ax+by+cz)^n$ if we take $x=y=z=1$ we get the sum of the coefficients of $(a+b+c)^n$.

: Multiple Choice Questions :

- (1) The coefficient of x^r in the expansion of $(1+x)^{-n} (n \in \mathbb{N})$ is ($|x| < 1$)

(A) $\binom{n+r-1}{n+1}$ (B) $\binom{n+r-1}{r-1}$ (C) $\binom{n+r-1}{r}$ (D) $(-1)^r \binom{n+r-1}{r}$

Sol. : General term in the expansion of $(1+x)^{-n}$

$$\begin{aligned} T_{r+1} &= \frac{(-n)(-n-1)(-n-2)\dots(-n-r+1)}{r!} x^r \\ &= \frac{(-1)^r n(n+1)(n+2)\dots(n+r-1)}{r!} x^r \\ &= (-1)^r \binom{n+r-1}{r} x^r \end{aligned}$$

$\therefore$ The coefficient of x^r is $(-1)^r \binom{n+r-1}{r}$

Ans. : (D)

- (2) The coefficient of x^r in the expansion of $(1-x)^{-n} (n \in \mathbb{N})$ is ($|x| < 1$)

(A) $\binom{n+r-1}{r}$ (B) $\binom{n-r+1}{r}$ (C) $\binom{n+r-1}{n}$ (D) $(-1)^r \binom{n+r-1}{r}$

Sol. : General term in expansion of $(1-x)^{-n}$

$$\begin{aligned} T_{r+1} &= \frac{(-n)(-n-1)(-n-2)\dots(-n-r+1)}{r!} (-x)^r \\ &= \frac{(-1)^r n(n+1)(n+2)\dots(n+r-1)(-1)^r}{r!} x^r \end{aligned}$$

$$= \binom{n+r-1}{r} x^r \quad n \in \mathbb{N}$$

$$\therefore \text{The coefficient of } x^r = \binom{n+r-1}{r} \quad \text{Ans. : (A)}$$

(3) The coefficient of x^{10} in the expansion of $(1+2x)^{-7}$ is

$$(A) 2^{10} \binom{17}{10} \quad (B) 2^{10} \binom{16}{10} \quad (C) 2^{10} \binom{16}{9} \quad (D) \binom{16}{6}$$

$$\text{Sol. : Here, } T_{r+1} = (-1)^r \binom{n+r-1}{r} (2x)^r$$

$$T_{11} = (-1)^{10} \binom{7+10-1}{10} 2^{10} x^{10} \quad [n=7, r=10]$$

$$= 2^{10} \binom{16}{10} x^{10}$$

$$\therefore \text{The coefficient of } x^{10} = 2^{10} \binom{16}{10} \quad \text{Ans. : (B)}$$

(4) If $(1-x+x^2)^{-3} = a_0 + a_1x + a_2x^2 + \dots$, then $\sum_{r=0}^6 a_r = \dots$ ($|x| < 1$)

$$(A) 38 \quad (B) 10 \quad (C) -10 \quad (D) -7$$

$$\begin{aligned} \text{Sol. : } (1-x+x^2)^{-3} &= (1+x)^3 (1+x)^{-3} (1-x+x^2)^{-3} \\ &= (1+x)^3 (1+x^3)^{-3} \\ &= (1+3x+3x^2+x^3) (1-3x^3+6x^6-\dots) \end{aligned}$$

$$\therefore \sum_{r=0}^6 a_r = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

$$= 1 + 3 + 3 + (-3+1) + 3(-3) + 3(-3) + (-3+6) = -10 \quad \text{Ans. : (C)}$$

(5) The coefficient of x^{-2} in the expansion of $(1+x)^{40} \left(1+\frac{1}{x}\right)^{40}$ is

$$(A) \binom{107}{38} \quad (B) \binom{80}{37} \quad (C) \frac{80!}{38!} \quad (D) \binom{80}{38}$$

$$\text{Sol. : The coefficient of } x^{-2} \text{ in expansion of } (1+x)^{40} \left(1+\frac{1}{x}\right)^{40}$$

$$= \text{The coefficient of } x^{-2} \text{ in expansion of } x^{-40} (1+x)^{80}$$

$$= \text{The coefficient of } x^{38} \text{ in expansion of } (1+x)^{80} = \binom{80}{38} \quad \text{Ans. : (D)}$$

$$(6) \quad \sum_{r=0}^n r \binom{n}{r} \left(\frac{2}{7}\right)^r \left(\frac{5}{7}\right)^{n-r} = \dots$$

- (A) 1 (B) $\frac{5n}{7}$ (C) $\frac{2n}{7}$ (D) 0

Sol. : $\sum_{r=0}^n r \binom{n}{r} \left(\frac{2}{7}\right)^r \left(\frac{5}{7}\right)^{n-r}$

$$= \sum_{r=0}^n r \frac{n!}{(n-r)!r!} \left(\frac{2}{7}\right)^r \left(\frac{5}{7}\right)^{n-r}$$

$$= n \times \frac{2}{7} \sum_{r=0}^n \frac{(n-1)!}{(n-r)!(r-1)!} \left(\frac{2}{7}\right)^{r-1} \left(\frac{5}{7}\right)^{n-r}$$

$$= \frac{2n}{7} \sum_{r=0}^n \binom{n-1}{r-1} \left(\frac{2}{7}\right)^{r-1} \left(\frac{5}{7}\right)^{n-r}$$

$$= \frac{2n}{7} \left(\frac{5}{7} + \frac{2}{7}\right)^{n-1} = \frac{2n}{7}$$

Ans. : (C)

Note : In general if $a + b = 1$, then $\sum_{r=0}^n r \binom{n}{r} a^r b^{n-r} = na$

$$(7) \quad \text{If } x > 0, y > 0 \text{ then } \sum_{r=1}^{\infty} \left(\frac{x}{2x+3y}\right)^{r-1} = \dots$$

- (A) $\frac{2x+3y}{2x+y}$ (B) $\frac{2x+3y}{x+3y}$ (C) $\frac{2x}{x+3y}$ (D) $\frac{3y}{2x+3y}$

Sol. : Let $\frac{x}{2x+3y} = a$, then $0 < a < 1$, because $x > 0, y > 0$

$$\text{Now, } \sum_{r=1}^{\infty} \left(\frac{x}{2x+3y}\right)^{r-1} = \sum_{r=1}^{\infty} a^{r-1} = a^0 + a^1 + a^2 + \dots \infty$$

$$= \frac{1}{1-a}$$

$$0 < a < 1$$

$$= \frac{1}{1 - \left(\frac{x}{2x+3y}\right)}$$

$$= \frac{2x+3y}{2x+3y-x}$$

$$= \frac{2x+3y}{x+3y}$$

Ans. : (B)

(8) The coefficient of a^3b^4 in the expansion of $(a + 2b)^3 (2a - b)^4$ is

- (A) 15 (B) -15 (C) -16 (D) 593

Sol. : $(a + 2b)^3 (2a - b)^4 = (a^3 + 6a^2b + 12ab^2 + 8b^3) \times (16a^4 - 32a^3b + 24a^2b^2 - 8ab^3 + b^4)$

$\therefore$ The coefficient of $a^3b^4 = 1 \times 1 + 6 \times (-8) + 12 \times 24 + 8 \times (-32)$

$= 1 - 48 + 288 - 256 = -15$

Ans. : (B)

(9) The coefficient of x^9 in the expansion $\sum_{r=0}^{70} \binom{70}{r} (x - 5)^{70-r} 4^r$ is

- (A) $-\binom{70}{8}$ (B) $\binom{70}{9}$ (C) $-\binom{70}{61}$ (D) 0

Sol. : $\sum_{r=0}^{70} \binom{70}{r} (x - 5)^{70-r} \cdot 4^r$

$= (x - 5 + 4)^{70} = (x - 1)^{70} = (1 - x)^{70}$

Now, general term T_{r+1} in the expansion,

$\therefore T_{r+1} = \binom{70}{r} (-x)^r$

Let $r = 9$, $T_{10} = \binom{70}{9} (-x)^9 = -\binom{70}{61} x^9$

$\therefore$ The coefficient of $x^9 = -\binom{70}{61}$

Ans. : (C)

(10) $1 \cdot \frac{\binom{n}{1}}{\binom{n}{0}} + 2 \frac{\binom{n}{2}}{\binom{n}{1}} + 3 \frac{\binom{n}{3}}{\binom{n}{2}} + \dots + 31 \frac{\binom{n}{31}}{\binom{n}{30}} = \dots$

- (A) $30n - 465$ (B) $15(2n - 31)$ (C) $31n - 15$ (D) $31(n - 15)$

Sol. : $r \cdot \frac{\binom{n}{r}}{\binom{n}{r-1}} = r \cdot \frac{n!}{(n-r)!r!} \times \frac{(n-r+1)!(r-1)!}{n!} = n - r + 1$

Required sum $= \sum_{r=1}^{31} r \cdot \frac{\binom{n}{r}}{\binom{n}{r-1}} = \sum_{r=1}^{31} (n - r + 1)$

$= n + (n - 1) + (n - 2) + \dots + (n - 30) = \frac{1}{2}(31)(n - 30 + n)$

$= (31)(n - 15)$

Ans. : (D)

$$(11) \quad \frac{\binom{2n}{1}}{n} + \frac{\binom{2n}{2}}{n^2} + \frac{\binom{2n}{3}}{n^3} + \dots \quad 2n \text{ terms} = \dots$$

$$(A) \left(\frac{n+1}{n}\right)^{2n} - 1 \quad (B) \left(\frac{n+1}{n}\right)^{2n} \quad (C) \left(\frac{n}{n+1}\right)^{2n} - 1 \quad (D) \left(\frac{n+1}{n}\right)^{2n+1} - 1$$

$$\text{Sol. : } (1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$

$$\therefore \left(1+\frac{1}{n}\right)^{2n} = \binom{2n}{0} + \binom{2n}{1}\frac{1}{n} + \binom{2n}{2}\frac{1}{n^2} + \dots + \binom{2n}{2n}\frac{1}{n^{2n}}$$

$$\therefore \frac{\binom{2n}{1}}{n} + \frac{\binom{2n}{2}}{n^2} + \frac{\binom{2n}{3}}{n^3} + \dots \quad 2n \text{ terms} = \left(1+\frac{1}{n}\right)^{2n} - 1 = \left(\frac{n+1}{n}\right)^{2n} - 1 \quad \text{Ans. : (A)}$$

$$(12) \quad \text{If } x \text{ is so small that } \left(\frac{1+3x}{1-2x}\right)^2 = 1+kx, \text{ then } k = \dots$$

$$(A) 2 \quad (B) -10 \quad (C) 10 \quad (D) -2$$

$$\text{Sol. : } \left(\frac{1+3x}{1-2x}\right)^2 = (1+3x)^2(1-2x)^{-2} \\ = (1+6x+9x^2)(1+4x+\dots)$$

$$k = \text{the coefficient of } x \text{ in the expansion} = 1 \times 4 + 6 \times 1 = 10 \quad \text{Ans. : (C)}$$

$$(13) \quad \text{The expansion of } \frac{1}{\sqrt[5]{7-3x}} \text{ in exponents of } x \text{ is possible, if } \dots$$

$$(A) |x| < \frac{7}{3} \quad (B) |x| < \frac{11}{3} \quad (C) |x| > \frac{7}{3} \quad (D) |x| > \frac{3!}{7}$$

$$\text{Sol. : } \frac{1}{\sqrt[5]{7-3x}} = \frac{1}{7^{1/5}} \left(1-\frac{3x}{7}\right)^{-1/5}$$

$$\text{This expansion is valid if } \left|\frac{3x}{7}\right| < 1. \quad \text{So, } |x| < \frac{7}{3} \quad \text{Ans. : (A)}$$

$$(14) \quad \text{If } (1+x)^n = \sum_{r=0}^n c_r x^r, \text{ then } \sum_{r=0}^{200} \frac{c_r}{r+1} = \dots$$

$$(A) \frac{2^{201}+1}{201} \quad (B) \frac{2^{200}-1}{200} \quad (C) \frac{2^{201}}{201} \quad (D) \frac{2^{201}-1}{201}$$

$$\begin{aligned}\text{Sol. : } (1+x)^{200} &= \sum_{r=0}^{200} c_r x^r \\ &= c_0 + c_1 x + c_2 x^2 + \dots + c_{200} x^{200}\end{aligned}$$

$$\therefore \int_0^1 (1+x)^{200} dx = \int_0^1 (c_0 + c_1 x + c_2 x^2 + \dots + c_{200} x^{200}) dx$$

$$\therefore \left[\frac{(1+x)^{201}}{201} \right]_0^1 = \left[c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \dots + c_{200} \frac{x^{201}}{201} \right]_0^1$$

$$\therefore \frac{2^{201}}{201} - \frac{1}{201} = c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \dots + \frac{c_{200}}{201}$$

$$\therefore \sum_{r=0}^{200} \frac{c_r}{r+1} = \frac{2^{201}-1}{201}$$

Ans. : (D)

(15) The coefficient of x^{25} in $(1+x)^5 + (1+x)^6 + (1+x)^7 + \dots + (1+x)^{50}$ is

(A) $\binom{50}{25}$ (B) $\binom{51}{26}$ (C) $\binom{51}{25} - 1$ (D) $\binom{46}{25}$

Sol. : This is a sum of 46 terms of G.P. whose first term is $(1+x)^5$ and common ratio is $1+x$.

$$\therefore \text{Required coefficient} = \text{The coefficient of } x^{25} \text{ in } \frac{(1+x)^5 \{(1+x)^{46} - 1\}}{(1+x) - 1}$$

$$= \text{The coefficient of } x^{26} \text{ in } (1+x)^{51} - (1+x)^5$$

$$= \binom{51}{26} - 0 = \binom{51}{26}$$

Ans. : (B)

Note : In general the coefficient of x^m in $\sum_{i=r}^n (1+x)^i = \binom{n+1}{m+1}$ where $r \leq m \leq n$

(16) The coefficient of x^7 in the expansion of $(1-2x)^{-2}$ is

(A) 128 (B) 1024 (C) -1024 (D) 512

$$\text{Sol. : } T_{r+1} = \frac{(-2)(-3)(-4)\dots(-2-r+1)}{r!} (-2x)^r$$

$$= \frac{(-1)^r (2)(3)(4)\dots(r+1)(-1)^r 2^r x^r}{r!}$$

$$= \frac{(-1)^{2r} (r+1)r! 2^r}{r!} x^r$$

$$= (r+1) 2^r x^r$$

$$\text{Let } r = 7, T_8 = (7+1) 2^7 x^7 = 8 \times 128 x^7 = 1024 x^7$$

The coefficient of $x^7 = 1024$

Ans. : (B)

$$(17) \quad \frac{1}{\binom{101}{0}} - \frac{1}{\binom{101}{1}} + \frac{1}{\binom{101}{2}} - \frac{1}{\binom{101}{3}} + \dots + \frac{(-1)^{101}}{\binom{101}{101}} = \dots$$

(A) $\frac{100}{\binom{101}{50}}$ (B) -1 (C) 1 (D) 0

Sol. : Let $S = \left\{ \frac{(-1)^0}{\binom{101}{0}} + \frac{(-1)^{101}}{\binom{101}{101}} \right\} + \left\{ \frac{(-1)^1}{\binom{101}{1}} + \frac{(-1)^{100}}{\binom{101}{100}} \right\} + \dots + \left\{ \frac{(-1)^{50}}{\binom{101}{50}} + \frac{(-1)^{51}}{\binom{101}{51}} \right\}$

$$= \left\{ \frac{1-1}{\binom{101}{0}} \right\} + \left\{ \frac{-1+1}{\binom{101}{1}} \right\} + \dots + \left\{ \frac{1-1}{\binom{101}{50}} \right\} = 0 + 0 + \dots + 0 = 0 \quad \text{Ans. : (D)}$$

Note : In general $\sum_{r=0}^n \frac{(-1)^r}{\binom{n}{r}} = 0$ (If n is odd)

$$(18) \quad 2 \binom{n}{2} + 4 \binom{n}{4} + 6 \binom{n}{6} + \dots = \dots$$

(A) $n \times 2^{n+1}$ (B) $n \times 2^{n-1}$ (C) $n \times 2^{n-2}$ (D) $n \times 2^{n+2}$

Sol. : $2 \binom{n}{2} + 4 \binom{n}{4} + 6 \binom{n}{6} + \dots$

$$= 2 \frac{n(n-1)}{2} + 4 \frac{n(n-1)(n-2)(n-3)}{24} + \dots$$

$$= n \left[(n-1) + \frac{(n-1)(n-2)(n-3)}{6} + \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{120} + \dots \right]$$

$$= n \left[\binom{n-1}{1} + \binom{n-1}{3} + \binom{n-1}{5} + \dots \right] = n \times 2^{n-2}$$

Note : $\sum_{r=1}^{\left[\frac{n}{2}\right]} 2r \binom{n}{2r} = n \cdot 2^{n-2} \Rightarrow \sum_{r=1}^{\left[\frac{n}{2}\right]} r \binom{n}{2r} = n \cdot 2^{n-3}$ **Ans. : (C)**

$$(19) \quad \binom{n}{0} \binom{n}{r} + \binom{n}{1} \binom{n}{r+1} + \dots + \binom{n}{n-r} \binom{n}{n} = \dots$$

(A) $\binom{2n}{r-1}$ (B) $\binom{2n}{r}$ (C) $\binom{2n}{n-r}$ (D) $\binom{2n}{n-r+1}$

Sol. : $(1+x)^n \times (x+1)^n = (1+x)^{2n}$

$$\therefore \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \right] \times$$

$$\left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \dots + \binom{n}{n-1}x + \binom{n}{n}x^0 \right]$$

$$= \binom{2n}{0} + \binom{2n}{1}x + \dots + \binom{2n}{r}x^r + \dots + \binom{2n}{2n}x^{2n}$$

Comparing the coefficients of x^{n-r} on both the sides.

$$\binom{n}{0} \binom{n}{r} + \binom{n}{1} \binom{n}{r+1} + \dots + \binom{n}{n-r} \binom{n}{n} = \binom{2n}{n-r}$$

Ans. : (C)

$$(20) \quad \binom{n}{1}^2 + 2 \binom{n}{2}^2 + 3 \binom{n}{3}^2 + \dots + n \binom{n}{n}^2 = \dots$$

(A) $\frac{(2n-1)!}{[(n-1)!]^2}$

(B) $n \binom{2n}{n}$

(C) $\frac{(2n-1)!}{(n-1)!}$

(D) $\frac{(2n-1)!}{n!}$

Sol. : Let $S = 0 \binom{n}{0}^2 + 1 \binom{n}{1}^2 + 2 \binom{n}{2}^2 + \dots + n \binom{n}{n}^2$ (1)

$$\therefore S = n \binom{n}{n}^2 + (n-1) \binom{n}{n-1}^2 + (n-2) \binom{n}{n-2}^2 + \dots + 0 \binom{n}{0}^2$$

$$= n \binom{n}{0}^2 + (n-1) \binom{n}{1}^2 + (n-2) \binom{n}{2}^2 + \dots + 0 \binom{n}{n}^2$$
 (2)

$$\therefore 2S = n \left[\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 \right]$$

$$= n \times [\text{the coefficient of } x^n \text{ in the expansion of } (1+x)^n (x+1)^n]$$

$$= n \times [\text{the coefficient of } x^n \text{ in the expansion of } (1+x)^{2n}]$$

$$= n \times \binom{2n}{n}$$

$$= n \times \frac{(2n)!}{n!n!}$$

$$\therefore S = \frac{n \times 2n(2n-1)!}{2 \times n(n-1)!n(n-1)!}$$

$$= \frac{(2n-1)!}{[(n-1)!]^2}$$

Ans. : (A)

(21) If $(1+x)^{100} = c_0 + c_1x + c_2x^2 + \dots + c_{100}x^{100}$, then $2^1 \frac{c_0}{1} + 2^2 \frac{c_1}{2} + 2^3 \frac{c_2}{3} + \dots + 2^{101} \frac{c_{100}}{101} = \dots$

(A) $\frac{2^{101}-1}{101}$ (B) $\frac{3^{101}-1}{101}$ (C) $\frac{3^{101}}{101}$ (D) $\frac{3^{101}+1}{101}$

Sol. : $\int_0^2 [c_0 + c_1x + c_2x^2 + \dots + c_{100}x^{100}] dx = \int_0^2 (1+x)^{100} dx$

$$\therefore \left[c_0x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \dots + c_{100} \frac{x^{101}}{101} \right]_0^2 = \left[\frac{(1+x)^{101}}{101} \right]_0^2$$

$$\therefore 2c_0 + \frac{2^2}{2}c_1 + \frac{2^3}{3}c_2 + \dots + \frac{2^{101}}{101}c_{100} = \frac{3^{101}-1}{101} \quad \text{Ans. : (B)}$$

(22) $\sum_{r=0}^n (-1)^r (a-r) \binom{n}{r} = \dots$

(A) a (B) na (C) 0 (D) $a+1$

Sol. : $\sum_{r=0}^n (-1)^r (a-r) \binom{n}{r} = (a-0) \binom{n}{0} - (a-1) \binom{n}{1} + (a-2) \binom{n}{2} + \dots + (-1)^n (a-n) \binom{n}{n}$

$$= a \left[\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n} \right] + \left[\binom{n}{1} - 2 \binom{n}{2} + 3 \binom{n}{3} - \dots - (-1)^n n \binom{n}{n} \right] \quad \text{(i)}$$

Now, $(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \dots + (-1)^n \binom{n}{n} x^n \quad \text{(ii)}$

Let $x=1$. So, $0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n} \quad \text{(iii)}$

Differentiating both the sides of (ii) with respect to x ,

$$-n(1-x)^{n-1} = -\binom{n}{1} + 2\binom{n}{2}x - 3x^2 \binom{n}{3} + \dots + (-1)^n nx^{n-1} \binom{n}{n}$$

Taking $x=1$, $0 = -\binom{n}{1} + 2\binom{n}{2} - 3\binom{n}{3} + \dots + (-1)^n n \binom{n}{n}$

$$\therefore 0 = \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \dots - (-1)^n n \binom{n}{n} \quad \text{(iv)}$$

From (i), (iii) and (iv),

$$\sum_{r=0}^n (-1)^r (a-r) \binom{n}{r} = a(0) + 0 = 0 \quad \text{Ans. : (C)}$$

$$(23) \quad \binom{n+1}{1} \binom{n+1}{2} \binom{n+1}{3} \dots \binom{n+1}{n} = \dots$$

$$(A) (n+1)^n \binom{n}{1} \binom{n}{2} \dots \binom{n}{n}$$

$$(B) n^n \binom{n}{1} \binom{n}{2} \dots \binom{n}{n}$$

$$(C) \frac{(n+1)^n}{n!} \binom{n}{1} \binom{n}{2} \dots \binom{n}{n}$$

$$(D) \frac{(n-1)^n}{n!} \binom{n}{1} \binom{n}{2} \dots \binom{n}{n}$$

$$\begin{aligned} \text{Sol. : } \binom{n+1}{r} &= \frac{(n+1)!}{r!(n+1-r)!} = \frac{(n+1) n!}{(n+1-r)(n-r)!r!} \\ &= \frac{n+1}{n+1-r} \times \binom{n}{r} \end{aligned}$$

$$\therefore \binom{n+1}{1} \binom{n+1}{2} \binom{n+1}{3} \dots \binom{n+1}{n}$$

$$= \frac{n+1}{n} \binom{n}{1} \times \frac{n+1}{n-1} \binom{n}{2} \times \frac{n+1}{n-2} \binom{n}{3} \times \dots \times \frac{n+1}{1} \binom{n}{n}$$

$$= \frac{(n+1)^n}{n!} \times \binom{n}{1} \binom{n}{2} \dots \binom{n}{n}$$

Ans. : (C)

$$(24) \quad \text{If } (1+x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n, \text{ then } c_0 - \frac{1}{2}c_1 + \frac{1}{3}c_2 - \dots + (-1)^n \frac{1}{n+1}c_n = \dots$$

$$(A) \frac{2}{n+1}$$

$$(B) \frac{2^{n+1}+1}{n+1}$$

$$(C) \frac{2^{n+1}-1}{n+1}$$

$$(D) \frac{1}{n+1}$$

$$\text{Sol. : } c_0 - \frac{1}{2}c_1 + \frac{1}{3}c_2 - \dots + (-1)^n \frac{1}{n+1}c_n$$

$$= \binom{n}{0} - \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} - \dots + (-1)^n \frac{1}{n+1} \binom{n}{n}$$

$$= 1 - \frac{1}{2}n + \frac{n(n-1)}{6} - \dots + (-1)^n \frac{1}{n+1}$$

$$= \frac{1}{n+1} \left[(n+1) - \frac{n(n+1)}{2} + \frac{(n+1)n(n-1)}{6} - \dots + (-1)^n \right]$$

$$= \frac{1}{n+1} \left[\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \dots + (-1)^n \binom{n+1}{n+1} \right]$$

$$\begin{aligned}
&= \frac{1}{n+1} \left[1 - \left\{ \binom{n+1}{0} - \binom{n+1}{1} + \binom{n+1}{2} - \dots + (-1)^{n+1} \binom{n+1}{n+1} \right\} \right] \\
&= \frac{1}{n+1} [1 - (1 - 1)^{n+1}] \\
&= \frac{1}{n+1} [1 - 0] \\
&= \frac{1}{n+1}
\end{aligned}$$

Ans. : (D)

→ **Second method :**

$$\begin{aligned}
(1-x)^n &= \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \dots + (-1)^n \binom{n}{n}x^n \\
&= c_0 - c_1x + c_2x^2 - \dots + (-1)^n c_n x^n \\
\therefore \int_0^1 (1-x)^n dx &= \int_0^1 (c_0 - c_1x + c_2x^2 - \dots + (-1)^n c_n x^n) dx \\
\therefore \left[\frac{-(1-x)^{n+1}}{n+1} \right]_0^1 &= \left[c_0 x - c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} - \dots + (-1)^n \frac{c_n x^{n+1}}{n+1} \right]_0^1 \\
\therefore 0 + \frac{1}{n+1} &= c_0 - \frac{c_1}{2} + \frac{c_2}{3} - \dots + \frac{(-1)^n}{n+1} \\
\therefore c_0 - \frac{1}{2}c_1 + \frac{1}{3}c_2 - \dots + \frac{(-1)^n}{n+1} &= \frac{1}{n+1}
\end{aligned}$$

Ans. : (D)

- (25) An examiner is supposed to give 25 marks from 8 equations. If he has to give at least 2 marks in a question, in how many ways can he do so ?

(A) $\binom{16}{8}$

(B) ${}_{16}P_7$

(C) $\binom{17}{9}$

(D) $\binom{16}{7}$

Sol. : The number of ways

$$\begin{aligned}
&= \text{The number of ways in which eight integers, greater than 1 can be found such that their sum is 25 subject to condition.} \\
&= \text{The coefficient of } x^{25} \text{ in } (x^2 + x^3 + x^4 + \dots)^8 \\
&= \text{The coefficient of } x^{25} \text{ in } x^{16} (1 + x + x^2 + \dots)^8 \\
&= \text{The coefficient of } x^9 \text{ in } (1 + x + x^2 + \dots)^8 \\
&= \text{The coefficient of } x^9 \text{ in } (1 - x)^{-8} \qquad (1 + x + x^2 + \dots = (1 - x)^{-1}) \\
&= \binom{9+8-1}{9} = \binom{16}{9} = \binom{16}{7}
\end{aligned}$$

Ans. : (D)

(26) In how many ways 12 identical pens can be distributed among 5 students ?

- (A) $\binom{11}{4}$ (B) $\binom{16}{4}$ (C) $\binom{16}{5}$ (D) ${}_{16}P_4$

Sol. : Required sum

= The number of ways of finding 5 non-negative integer values whose sum is 12.

= The coefficient of x^{12} in $(1 + x + x^2 + \dots + x^{12})^5$

= The coefficient of x^{12} in $(1 - x)^5 (1 + x + x^2 + \dots + x^{12})^5 (1 - x)^{-5}$

= The coefficient of x^{12} in $(1 - x^{13})^5 (1 - x)^{-5}$

= The coefficient of x^{12} in $(1 - x)^{-5}$ $[(1 - x^{13})^5 = 1 - 5x^{13} + 10x^{26} \dots]$

= $\binom{12+5-1}{12}$ [The coefficient of x^r in $(1 - x)^{-r} = \binom{n+r-1}{n}, r \in \mathbb{N}$]

= $\binom{16}{12} = \binom{16}{4}$ **Ans. : (B)**

(27) In how many ways 4 coins can be chosen out of 20 coins of ₹ 1, 15 coins of ₹ 2 and 10 coins of ₹ 5 ?

- (A) 21 (B) 20 (C) 30 (D) 15

Sol. : Required sum

= The coefficient of x^4 in $(1 + x + x^2 + \dots + x^{20}) \times (1 + x + x^2 + \dots + x^{15}) \times (1 + x + x^2 + \dots + x^{10})$

= The coefficient of x^4 in $\frac{(1-x^{21})}{1-x} \times \frac{(1-x^{16})}{1-x} \times \frac{(1-x^{11})}{1-x}$

= The coefficient of x^4 in $(1 - x^{21}) (1 - x^{16}) (1 - x^{11}) (1 - x)^{-3}$

= The coefficient of x^4 in $(1 - x)^{-3}$ (in expansion of $(1 - x^{21})(1 - x^{16})(1 - x^{11})$ the first term is 1 and all other terms have exponent greater than 4.)

= $\binom{4+3-1}{4} = \binom{6}{4} = \binom{6}{2} = \frac{6 \times 5}{2} = 15$ **Ans. : (D)**

(28) There are at least 10 books of each of the subjects Mathematics, Physics and Chemistry. In how many ways 5 books can be chosen from these books ?

- (A) 21 (B) 6 (C) 42 (D) 24

Sol. : The number of ways in which 5 books can be chosen.

= The coefficient of x^5 in $(1 + x + x^2 + \dots)^3$

= The coefficient of x^5 in $(1 - x)^{-3}$ $(1 + x + x^2 + \dots = (1 - x)^{-1})$

$$= \binom{5+3-1}{5}$$

$$= \binom{7}{5} = \binom{7}{2} = 21$$

Ans. : (A)

- (29) There are numerous balls of red, white, black and green colour. In how many ways 11 balls can be chosen from these ?

(A) 165

(B) 120

(C) 330

(D) unlimited

Sol. : Required number of ways

$$= \text{The coefficient of } x^{11} \text{ in } (x + x^2 + x^3 + \dots)^4$$

$$= \text{The coefficient of } x^7 \text{ in } (1 + x + x^2 + \dots)^4$$

$$= \text{The coefficient of } x^7 \text{ in } (1 - x)^{-4}$$

$$= \binom{7+4-1}{7} = \binom{10}{7} = \binom{10}{3} = \frac{10 \times 9 \times 8}{6} = 120$$

Ans. : (B)

- (30) A student is taking examination of three subjects and each paper has maximum of n marks. In how many ways, can he score $2n$ marks ?

(A) $\frac{(n+1)(n+2)}{2}$

(B) $\frac{n(n+1)}{2}$

(C) $\binom{3n}{2n}$

(D) $n(n+1)(n+2)$

Sol. : Required number of ways

$$= \text{The coefficient of } x^{2n} \text{ in } (1 + x + x^2 + \dots + x^n)^3$$

$$= \text{The coefficient of } x^{2n} \text{ in } (1 - x)^3 (1 + x + x^2 + \dots + x^n)^3 (1 - x)^{-3}$$

$$= \text{The coefficient of } x^{2n} \text{ in } (1 - x^{n+1})^3 (1 - x)^{-3}$$

$$= \text{The coefficient of } x^{2n} \text{ in } (1 - 3x^{n+1} + \dots) (1 - x)^{-3}$$

$$= \text{The coefficient of } x^{2n} \text{ in } (1 - x)^{-3} - 3 \times \text{The coefficient of } x^{2n-n-1} \text{ in } (1 - x)^{-3}$$

$$= \binom{2n+3-1}{2n} - 3 \binom{n-1+3-1}{n-1}$$

$$= \binom{2n+2}{2n} - 3 \binom{n+1}{n-1} = \binom{2n+2}{2} - 3 \binom{n+1}{2} = \frac{(2n+2)(2n+1)}{2} - 3 \frac{(n+1)n}{2}$$

$$= \frac{(n+1)}{2} (4n+2-3n) = \frac{(n+1)(n+2)}{2}$$

Ans. : (A)

(31) If a and b are positive integers, then $\binom{a}{r} + \binom{a}{r-1}\binom{b}{1} + \binom{a}{r-2}\binom{b}{2} + \dots + \binom{a}{0}\binom{b}{r} = \dots$

- (A) $\binom{a+b}{r+1}$ (B) $\binom{a+b}{r}$ (C) $\frac{(a+b)!}{r!}$ (D) $\frac{a!b!}{r!}$

Sol. : $(1+x)^a (1+x)^b = (1+x)^{a+b}$.

Expanding both the sides, $\left[\binom{a}{0} + \binom{a}{1}x + \binom{a}{2}x^2 + \dots + \binom{a}{a}x^a \right] \left[\binom{b}{0} + \binom{b}{1}x + \binom{b}{2}x^2 + \dots + \binom{b}{b}x^b \right]$

$$= \binom{a+b}{0} + \binom{a+b}{1}x + \binom{a+b}{2}x^2 + \dots + \binom{a+b}{a+b}x^{a+b}$$

Comparing coefficient of x^r on both the sides,

$$\therefore \binom{a}{r}\binom{b}{0} + \binom{a}{r-1}\binom{b}{1} + \binom{a}{r-2}\binom{b}{2} + \dots + \binom{a}{0}\binom{b}{r} = \binom{a+b}{r}$$

$$\therefore \text{The required sum} = \binom{a+b}{r}$$

Ans. : (B)

(32) There are numerous balls of red, yellow and blue colours. In how many ways 12 balls can be chosen from them ?

- (A) 105 (B) 55 (C) 91 (D) unlimited

Sol. : The required sum

= The coefficient of x^{12} in $(1+x+x^2+\dots)^3$

= The coefficient of x^{12} in $(1-x)^{-3}$ $(1+x+x^2+\dots \infty = (1-x)^{-1})$

$$= \binom{12+3-1}{12} = \binom{14}{12} = \binom{14}{2} = \frac{14 \times 13}{2} = 91$$

Ans. : (C)

(33) In how many ways 4 alphabets can be chosen from the word ASSASSINATION ?

- (A) 60 (B) 72 (C) 80 (D) 71

Sol. : The number of ways of choosing 4 different alphabets from S, A, N, I, T, O = $\binom{6}{4} = \binom{6}{2} = 15$

(For example SANI, AITO etc.)

The number of ways of selecting two identical alphabets and two different alphabets

$$= \binom{4}{1} \binom{5}{2} = 40 \quad (\text{For example SSAN, AATO, NIIO etc.})$$

The number of ways of selecting two pairs of two identical alphabets = $\binom{4}{2} = 6$

(For example SSAA, NNII, AANN etc.)

The number of ways of selecting three identical alphabets and one different alphabet

$$= \binom{2}{1} \binom{5}{1} = 10 \quad (\text{For example SSSA, AAAI, AAAN etc.})$$

The total number of ways = $15 + 40 + 6 + 10 + 1 = 72$ (1 for all same alphabets) **Ans. : (B)**

$$(34) \quad \frac{3}{4} + \left(\frac{3}{4} \times \frac{5}{8}\right) + \left(\frac{3}{4} \times \frac{5}{8} \times \frac{7}{12}\right) + \dots \infty = \dots$$

$$(A) \sqrt{2} - 1 \quad (B) 2\sqrt{2} + 1 \quad (C) 2\sqrt{2} - 1 \quad (D) 2\sqrt{2}$$

$$\text{Sol. :} \quad (1+x)^n - 1 = nx + \frac{n(n-1)}{2} x^2 + \dots \infty$$

$$= \frac{3}{4} + \left(\frac{3}{4} \times \frac{5}{8}\right) + \left(\frac{3}{4} \times \frac{5}{8} \times \frac{7}{12}\right) \dots \infty$$

$$\therefore nx = \frac{3}{4} \dots\dots (i) \quad \text{and} \quad \frac{n(n-1)}{2} x^2 = \frac{15}{32} \dots(ii)$$

$$(ii) \div \text{square of (i)} \quad \frac{n(n-1)x^2}{2n^2x^2} = \frac{15}{32} \times \frac{4}{3} \times \frac{4}{3}$$

$$\therefore \frac{n-1}{n} = \frac{5}{3}$$

$$\therefore 3n - 3 = 5n. \quad \text{So } 2n = -3. \quad \text{Hence } n = -\frac{3}{2}$$

$$\text{Substituting, } n = -\frac{3}{2} \text{ in (i) we get, } \left(-\frac{3}{2}\right)x = \frac{3}{4}. \quad \text{So } x = \frac{-1}{2}$$

$$\therefore (1+x)^n - 1 = \left(1 - \frac{1}{2}\right)^{-\frac{3}{2}} - 1 = \left(\frac{1}{2}\right)^{-\frac{3}{2}} - 1 = 2^{\frac{3}{2}} - 1 = 2\sqrt{2} - 1 \quad \text{Ans. : (C)}$$

(35) The coefficient of x^5 in the expansion of $(1+x+x^2+x^3+x^4)^3$ is

$$(A) 15 \quad (B) 18 \quad (C) 21 \quad (D) 24$$

$$\text{Sol. :} \quad (1+x+x^2+x^3+x^4)^3 = \frac{(1-x)^3 (1+x+x^2+x^3+x^4)^3}{(1-x)^3}$$

$$= (1-x^5)^3 (1-x)^{-3}$$

$$\therefore \text{The required coefficient of } x^5 = [\text{The coefficient of } x^5 \text{ in } (1-x^5)^3] \times$$

$$[\text{The coefficient of } x^0 \text{ in } (1-x)^{-3}] +$$

$$[\text{The coefficient of } x^0 \text{ in } (1-x^5)^3] \times$$

$$[\text{The coefficient of } x^5 \text{ in } (1-x)^{-3}]$$

$$= -\binom{3}{1} \times (1) + (1) \times \binom{5+3-1}{5}$$

$$[\text{The coefficient of } x^r \text{ in } (1-x)^{-k}, k \in \mathbb{N} \text{ is } \binom{k+r-1}{r}]$$

$$= -3 + \binom{7}{5} = -3 + \binom{7}{2} = -3 + 21 = 18 \quad \text{Ans. : (B)}$$

(36) The number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 31$, $x_1, x_2, x_3, x_4 \in \mathbb{W}$ is

- (A) $\binom{34}{4}$ (B) $\binom{34}{3}$ (C) $\binom{35}{4}$ (D) $\binom{33}{2}$

Sol. : The number of solutions = The coefficient of x^{31} in $(1 + x + x^2 + \dots + x^{31})^4$

$$= \text{The coefficient of } x^{31} \text{ in } \frac{(1-x)^4 (1+x+x^2+\dots+x^{31})^4}{(1-x)^4}$$

$$= \text{The coefficient of } x^{31} \text{ in } (1-x^{32})^4 (1-x)^{-4}$$

$$= \text{The coefficient of } x^{31} \text{ in } (1-x)^{-4}$$

$$= \binom{31+4-1}{31} \quad \left(\text{The coefficient of } x^r \text{ in } (1-x)^{-n} = \binom{n+r-1}{r} \right)$$

$$= \binom{34}{31} = \binom{34}{3} \quad \text{Ans. : (B)}$$

(37) The number of solutions of the equation $x + y + z = 20$, $x \geq 1$, $y \geq 2$, $z \geq 3$, $x, y, z \in \mathbb{W}$ is

- (A) 210 (B) 136 (C) 120 (D) 124

Sol. : The number of solutions,

$$= \text{The coefficient of } x^{20} \text{ in } (x^1 + x^2 + \dots + x^{20}) (x^2 + x^3 + \dots + x^{20}) (x^3 + x^4 + \dots + x^{20})$$

$$= \text{The coefficient of } x^{20} \text{ in } x^6 (1 + x + \dots + x^{19}) (1 + x + \dots + x^{18}) (1 + x + \dots + x^{17})$$

$$= \text{The coefficient of } x^{14} \text{ in } (1 + x + \dots + x^{19}) (1 + x + \dots + x^{18}) (1 + x + \dots + x^{17})$$

$$= \text{The coefficient of } x^{14} \text{ in } \left(\frac{1-x^{20}}{1-x} \right) \left(\frac{1-x^{19}}{1-x} \right) \left(\frac{1-x^{18}}{1-x} \right)$$

$$= \text{The coefficient of } x^{14} \text{ in } (1-x^{20}) (1-x^{19}) (1-x^{18}) (1-x)^{-3}$$

$$= \text{The coefficient of } x^{14} \text{ in } (1-x)^{-3}$$

$$= \binom{3+14-1}{14} = \binom{16}{14} = \binom{16}{2} = \frac{16 \times 15}{2} = 120 \quad \text{Ans. : (C)}$$

(38) The number of non-negative integral solutions of inequality $x + y + z \leq 30$ is

- (A) 5465 (B) 5456 (C) 5654 (D) 5646

Sol. : The number of non-negative integral solutions of $x + y + z \leq 30$ = the number of non-negative integer solutions of $x + y + z + a = 30$.

The number of solutions = the coefficient of x^{30} in $(1 + x^1 + x^2 + \dots + x^{30})^4$ = the coefficient of x^{30} in $(1-x)^4 (1 + x^1 + x^2 + \dots + x^{30})^4 (1-x)^{-4}$.

= The coefficient of x^{30} in $(1 - x^{31})^4 (1 - x)^{-4}$

= The coefficient of x^{30} in $(1 - x)^{-4}$

$$= \binom{4+30-1}{30} = \binom{33}{30} = \binom{33}{3} = \frac{33 \times 32 \times 31}{6} = 5456 \quad \text{Ans. : (B)}$$

(39) In how many ways can 15 objects be placed in 3 boxes ? (Each box can hold atmost 15 objects).

- (A) 91 (B) ${}_{15}P_3$ (C) 3^{15} (D) 136

Sol. : Here in each box, we can put 0 or 1 or 2.... or 15 objects.

The required number of types,

= The coefficient of x^{15} in $(1 + x + x^2 + \dots + x^{15})^3$

= The coefficient of x^{15} in $(1 - x)^3 (1 + x + x^2 + \dots + x^{15})^3 (1 - x)^{-3}$

= The coefficient of x^{15} in $(1 - x^{16})^3 (1 - x)^{-3}$

= The coefficient of x^{15} in $(1 - x)^{-3}$

$$= \binom{15+3-1}{15} = \binom{17}{15} = \binom{17}{2} = \frac{17 \times 16}{2} = 136 \quad \text{Ans. : (D)}$$

(40) If $c_r = \binom{30}{r}$, then $\left(\frac{c_1}{c_0}\right)^2 + 8 \left(\frac{c_2}{c_1}\right)^2 + 27 \left(\frac{c_3}{c_2}\right)^2 + \dots + 30^3 \left(\frac{c_{30}}{c_{29}}\right)^2 = \dots$

- (A) 76808 (B) 76080 (C) 75880 (D) 76880

Sol. : $\frac{c_r}{c_{r-1}} = \frac{\binom{30}{r}}{\binom{30}{r-1}} = \frac{30!}{(30-r)!r!} \times \frac{(30-r+1)!(r-1)!}{30!} = \frac{31-r}{r}$

$$\begin{aligned} \text{The required sum} &= \sum_{r=1}^{30} r^3 \left(\frac{c_r}{c_{r-1}} \right)^2 \\ &= \sum_{r=1}^{30} r^3 \left(\frac{31-r}{r} \right)^2 \\ &= \sum_{r=1}^{30} r (961 - 62r + r^2) \\ &= \sum_{r=1}^{30} (961r - 62r^2 + r^3) \\ &= \left[961 \frac{n(n+1)}{2} - 62 \frac{n(n+1)(2n+1)}{6} + \frac{n^2(n+1)^2}{4} \right]_{n=30} \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{n(n+1)}{2} \left(961 - \frac{124n}{3} - \frac{62}{3} + \frac{n^2+n}{2} \right) \right]_{n=30} \\
&= \left[\frac{n(n+1)}{2} \left(\frac{5766 - 248n - 124 + 3n^2 + 3n}{6} \right) \right]_{n=30} \\
&= \left[\frac{n(n+1)}{12} (3n^2 - 245n + 5642) \right]_{n=30} \\
&= \frac{155}{2} (8342 - 7350) = \frac{155}{2} (992) = 155 \times 496 = 76880 \quad \text{Ans. : (D)}
\end{aligned}$$

$$(41) \quad \sum_{m=1}^n \left(\sum_{r=0}^{m-1} \binom{n}{r} \right) = \dots$$

- (A) $n \cdot 2^{n-1}$ (B) $n \cdot 2^n$ (C) $2^{n-1} + n$ (D) $(n+1) 2^{n-1}$

Sol. : $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$

Differentiating both the sides with respect to x ,

$$n(1+x)^{n-1} = \binom{n}{1} + 2x \binom{n}{2} + \dots + nx^{n-1} \binom{n}{n}. \text{ Let } x = 1$$

$$\begin{aligned}
n \cdot 2^{n-1} &= \binom{n}{1} + 2 \binom{n}{2} + \dots + n \binom{n}{n} \\
&= \binom{n}{n-1} + 2 \binom{n}{n-2} + \dots + (n-1) \binom{n}{1} + n \binom{n}{0} \\
&= n \binom{n}{0} + (n-1) \binom{n}{1} + \dots + 2 \binom{n}{n-2} + \binom{n}{n-1} \\
&= \binom{n}{0} + \left(\binom{n}{0} + \binom{n}{1} \right) + \left(\binom{n}{0} + \binom{n}{1} + \binom{n}{2} \right) + \dots + \left(\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n-1} \right) \\
&= \sum_{m=1}^n \left(\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{m-1} \right) \\
&= \sum_{m=1}^n \left(\sum_{r=0}^{m-1} \binom{n}{r} \right)
\end{aligned}$$

Ans. : (A)

(42) The coefficient of x^5 in the expansion of $\sum_{r=0}^9 \{(x+4)^{9-r} \cdot (x+3)^r\}$ is

- (A) $781 \cdot \binom{10}{5}$ (B) $780 \binom{9}{5}$ (C) 781 (D) $\binom{10}{5}$

Sol. : Here, $k =$ The coefficient of x^5 in $\sum_{r=0}^9 \{(x+4)^{9-r} \cdot (x+3)^r\}$

$=$ The coefficient of x^5 in $(x+4)^9 + (x+4)^8(x+3) + (x+4)^7(x+3)^2 + \dots + (x+3)^9$

$=$ The coefficient of x^5 in $(x+4)^9 \frac{\left[1 - \left(\frac{x+3}{x+4}\right)^{10}\right]}{1 - \left(\frac{x+3}{x+4}\right)}, \quad \left[a = (x+4)^9, r = \frac{x+3}{x+4}, n=10\right]$

$=$ The coefficient of x^5 in $(x+4)^{10} \left[1 - \left(\frac{x+3}{x+4}\right)^{10}\right]$

$=$ The coefficient of x^5 in $(x+4)^{10} - (x+3)^{10} = \binom{10}{5} 4^5 - \binom{10}{5} 3^5 = \binom{10}{5} \cdot 781$

Ans. : (A)

(43) If $n \in \mathbb{N} - \{1\}$, then $\sum_{r=0}^n \left\{ (-1)^r (100-r) \binom{n}{r} \right\} = \dots$

- (A) 100 (B) $100(2^n)$ (C) $2^n - 1$ (D) 0

Sol. : $\sum_{r=0}^n \left\{ (-1)^r (100-r) \binom{n}{r} \right\} = \sum_{r=0}^n (-1)^r 100 \binom{n}{r} - \sum_{r=0}^n (-1)^r r \binom{n}{r}$

$= 100 \left\{ \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n} \right\} - \left\{ -\binom{n}{1} + 2\binom{n}{2} - 3\binom{n}{3} + \dots + (-1)^n n \binom{n}{n} \right\}$

$= 100 \{0\} + \left\{ \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \dots + (-1)^{1+n} n \binom{n}{n} \right\} \dots \dots \dots (1)$

$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \dots + (-1)^n x^n$

Differentiating both the sides with respect to x ,

$n(1-x)^{n-1}(-1) = -\binom{n}{1} + 2x\binom{n}{2} - 3x^2\binom{n}{3} + \dots + (-1)^n nx^{n-1}$

Taking $x = 1, 0 = -\binom{n}{1} + 2\binom{n}{2} - 3\binom{n}{3} + \dots + (-1)^n n$

$\therefore 0 = \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \dots + (-1)^{n+1} n \dots \dots \dots (2)$

From equations (1) and (2), the required sum = 0

Ans. : (D)

(44) If $C_r = \binom{n}{r}$, then $\sum_{m=0}^n \left(\sum_{r=0}^m C_r \right) = \dots$

- (A) $(n+2) \cdot 2^n$ (B) $(n+1) \cdot 2^{n-1}$ (C) $n \cdot 2^{n-1}$ (D) $(n+2) \cdot 2^{n-1}$

Sol. : The required sum $= \sum_{m=0}^n (C_0 + C_1 + C_2 + \dots + C_m)$

$$\begin{aligned}
 &= C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + \dots + (C_0 + C_1 + C_2 + \dots + C_n) \\
 &= (n+1) \cdot C_0 + n \cdot C_1 + (n-1) \cdot C_2 + (n-2) \cdot C_3 + \dots + C_n \\
 &= (n+1) \cdot C_n + n \cdot C_{n-1} + (n-1) \cdot C_{n-2} + \dots + 3 \cdot C_2 + 2 \cdot C_1 + C_0 \\
 &= (C_0 + C_1 + C_2 + \dots + C_n) + (C_1 + 2 \cdot C_2 + \dots + n \cdot C_n) \\
 &= 2^n + n \cdot 2^{n-1} = (n+2) \cdot 2^{n-1}
 \end{aligned}$$

[Expand $(1+x)^n$, differentiate w.r.t. x and then substitute $x=1$, we get,

$$C_1 + 2 \cdot C_2 + 3 \cdot C_3 + \dots + n \cdot C_n = n \cdot 2^{n-1}] \quad \text{Ans. : (D)}$$

(45) If $S_k = \sum_{r=1}^n r^k$, then $\sum_{r=1}^{k+1} \binom{k+1}{r} S_{k+1-r} = \dots$

- (A) $(n+1)^{k+1} - 1$ (B) $(n+1)^{k+1}$ (C) $n^{k+1} - 1$ (D) $(n+1)^k - 1$

Sol. : The required sum

$$\begin{aligned}
 &= \sum_{r=0}^{k+1} \left[\binom{k+1}{r} S_{k+1-r} \right] - \binom{k+1}{0} S_{k+1} \\
 &= \sum_{r=0}^{k+1} \left[\binom{k+1}{r} \{1^{k+1-r} + 2^{k+1-r} + 3^{k+1-r} + \dots + n^{k+1-r}\} \right] - S_{k+1} \\
 &= \sum_{r=0}^{k+1} \binom{k+1}{r} 1^{k+1-r} + \sum_{r=0}^{k+1} \binom{k+1}{r} 2^{k+1-r} + \sum_{r=0}^{k+1} \binom{k+1}{r} 3^{k+1-r} + \dots + \sum_{r=0}^{k+1} \binom{k+1}{r} n^{k+1-r} - S_{k+1} \\
 &= 2^{k+1} + 3^{k+1} + 4^{k+1} + \dots + (n+1)^{k+1} - (1^{k+1} + 2^{k+1} + 3^{k+1} + \dots + n^{k+1}) \\
 &= (n+1)^{k+1} - 1 \quad \text{Ans. : (A)}
 \end{aligned}$$

(46) If n is a multiple of 3, then $\sum_{r=0}^{\frac{n}{3}} \binom{n}{3r} = \dots$

- (A) $\frac{1}{3} (2^n + (-1)^n 2)$ (B) 0 (C) $2^n - 2 (-1)^n$ (D) $2^n + 2 (-1)^n$

Sol. : $(1+x)^n = \sum_{r=0}^n \binom{n}{r} x^r$. Let $x = 1$, $x = w$ and $x = w^2$ respectively, (where $1 + w + w^2 = 0$)

$$(1+1)^n = \sum_{r=0}^n \binom{n}{r}$$

$$(1+w)^n = \sum_{r=0}^n \binom{n}{r} w^r$$

$$(1+w^2)^n = \sum_{r=0}^n \binom{n}{r} w^{2r}$$

$$\therefore 2^n + (1+w)^n + (1+w^2)^n = \sum_{r=0}^n \binom{n}{r} (1 + w^r + w^{2r})$$

$$\therefore 2^n + (-w^2)^n + (-w)^n = \binom{n}{0} (1+1+1) + \binom{n}{1} (1+w+w^2) + \dots + \binom{n}{n} (1+w^n+w^{2n})$$

$$\therefore 2^n + (-1)^n w^{2n} + (-1)^n (w)^n = 3 \binom{n}{0} + \binom{n}{1} (0) + \binom{n}{2} (0) + \binom{n}{3} (3) + \dots + \binom{n}{n} (3)$$

$$\therefore 2^n + (-1)^n (1) + (-1)^n (1) = 3 \left[\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \dots + \binom{n}{n} \right] \quad (n = 3k, \text{ so, } w^n = 1)$$

$$\therefore \frac{2^n + 2(-1)^n}{3} = \sum_{r=0}^{\frac{n}{3}} \binom{n}{3r}. \quad \text{So, } \sum_{r=0}^{\frac{n}{3}} \binom{n}{3r} = \frac{1}{3} (2^n + (-1)^n 2) \quad \text{Ans. : (A)}$$

(47) The coefficient of x^5 in $\sum_{r=0}^{40} [(r+1) x^r (1+x)^{40-r}]$ is

(A) $\binom{40}{5}$

(B) $\binom{42}{5}$

(C) $\binom{41}{5}$

(D) $\binom{39}{5}$

Sol. : $S = \sum_{r=0}^{40} [(r+1) x^r (1+x)^{40-r}]$

$$= (1+x)^{40} \sum_{r=0}^{40} \left[(r+1) \left(\frac{x}{1+x} \right)^r \right]$$

$$= (1+x)^{40} \sum_{r=0}^{40} (r+1) p^r \quad \left(\frac{x}{1+x} = p \right)$$

$$\therefore S = (1+x)^{40} [1 + 2p + 3p^2 + \dots + 41 p^{40}] \quad (1)$$

$$p S = (1+x)^{40} [p + 2p^2 + \dots + 40 p^{40} + 41 p^{41}] \quad (2)$$

(1) – (2) gives

$$(1-p) S = (1+x)^{40} [1 + p + p^2 + \dots + p^{40} - 41 p^{41}]$$

$$= (1+x)^{40} \left[1 \frac{(1-p^{41})}{1-p} - 41 p^{41} \right]$$

$$\left(1 - \frac{x}{1+x}\right) S = (1+x)^{40} \left[\frac{1 - \left(\frac{x}{1+x}\right)^{41}}{1 - \frac{x}{1+x}} - 41 \left(\frac{x}{1+x}\right)^{41} \right]$$

$$S = (1+x)^{41} \left[\frac{(1+x)^{42} - x^{41} (1+x)}{(1+x)^{41}} - 41 \frac{x^{41}}{(1+x)^{41}} \right]$$

$$= (1+x)^{42} - x^{41} (1+x) - 41 \cdot x^{41} = (1+x)^{42} - 42 \cdot x^{41} - x^{42}$$

$$\therefore \text{The coefficient of } x^5 \text{ in } S = \text{The coefficient of } x^5 \text{ in } (1+x)^{42} = \binom{42}{5} \quad \text{Ans. : (B)}$$

(48) If $|x| < 1$, then the coefficient of x^n in the expansion of $(1 + x + x^2 + x^3 + \dots)^2$ is

(AIEEE : 2002)

- (A) n (B) $n+2$ (C) $n-1$ (D) $n+1$

$$\text{Sol. : As } |x| < 1, 1 + x + x^2 + x^3 + \dots \infty = \frac{1}{1-x}$$

$\therefore$ The coefficient of x^n in $(1 + x + x^2 + x^3 + \dots)^2$

$$= \text{The coefficient of } x^n \text{ in } (1-x)^{-2} = \binom{n+2-1}{n} = \binom{n+1}{1} = n+1 \quad \text{Ans. : (D)}$$

(49) The smallest positive integer greater than $(1+0.0001)^{10^3}$ is (AIEEE : 2002)

- (A) 4 (B) 5 (C) 2 (D) 3

$$\text{Sol. : } m = (1+0.0001)^{1000} = (1+10^{-4})^{1000}$$

$$= 1 + 1000 (10^{-4})^1 + \frac{1000 \times 999}{2} (10^{-4})^2 + \dots \text{ 1001 terms}$$

$$< 1 + \frac{1}{10} + 1000 \times 1000 \times 10^{-8} + \dots \infty$$

$$\leq 1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \dots \infty$$

$$= \frac{1}{1 - \frac{1}{10}} \quad (a = 1, r = \frac{1}{10})$$

$$= \frac{10}{9}$$

$\therefore m < \frac{10}{9}$. Also, $m > 1$ is true.

$\therefore$ The smallest positive integer greater than m is 2.

Ans. : (C)

(50) The coefficient of x^5 in $(1+2x+3x^2+\dots)^{\frac{-3}{2}}$ is (AIEEE : 2002)

(A) 21 (B) 25 (C) 26 (D) None of these

$$\begin{aligned} \text{Sol. : } (1+2x+3x^2+\dots)^{\frac{-3}{2}} &= \{(1-x)^{-2}\}^{\frac{-3}{2}} \\ &= (1-x)^3 \\ &= \binom{3}{0} - \binom{3}{1}x + \binom{3}{2}x^2 + \binom{3}{3}x^3 \end{aligned}$$

$\therefore$ The coefficient of $x^5 = 0$

$\therefore$ Out of given options (A), (B), (C), none is correct.

Ans. : (D)

(51) If $x > 0$, then the first negative term in the expansion of $(1+x)^{\frac{27}{5}}$ is (AIEEE : 2003)

(A) 7th term (B) 5th term (C) 8th term (D) 6th term

$$\begin{aligned} \text{Sol. : } T_{r+1} &= \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} \times x^r \\ &= \frac{\frac{27}{5}\left(\frac{27}{5}-1\right)\left(\frac{27}{5}-2\right)\dots\left(\frac{27}{5}-r+1\right)}{r!} \times x^r \end{aligned}$$

Now, if the last factor in the numerator is negative, T_{r+1} will be first negative term.

$\therefore \frac{27}{5} - r + 1 < 0$ is necessary for this to happen.

$\therefore \frac{32}{5} < r$ is necessary.

$\therefore$ The smallest value of $r = 7$ will make T_{r+1} negative.

$\therefore$ First negative term = T_{r+1} th term = T_8 th term

Note : For $r = 6$, $\frac{27}{5} \left(\frac{27}{5} - 1 \right) \dots \left(\frac{27}{5} - 6 + 1 \right) > 0$

Ans. : (C)

- (52) The number of integer terms in the expansion of $(\sqrt{3} + \sqrt[8]{5})^{256}$ is (AIEEE : 2003)
- (A) 35 (B) 33 (C) 34 (D) 32

Sol. : Here, $T_{r+1} = \binom{256}{r} (\sqrt{3})^{256-r} (\sqrt[8]{5})^r = \binom{256}{r} (3)^{\frac{256-r}{2}} (5)^{\frac{r}{8}}$

$\frac{256-r}{2}$ and $\frac{r}{8}$ must be both positive integers for integer term T_{r+1} .

Hence, r is a multiple of 8. Again for this $\frac{256-r}{2}$ is also an integer.

Here, $0 \leq r \leq 256$, and so maximum value of r is 256.

$\therefore$ Values of $r = 0, 8, 16, \dots, 256$.

$\therefore T_1, T_9, T_{17}, \dots, T_{257}$ are integers.

$\therefore 257 = 1 + 8(n-1)$

$\therefore n = 33$

The total number of integer terms = 33.

Ans. : (B)

- (53) The coefficient of x^n in the expansion of $(1+x)(1-x)^n$ (AIEEE : 2003)
- (A) $n-1$ (B) $(-1)^n (n+1)$ (C) $(-1)^{n-1} n$ (D) $(-1)^n (1-n)$

Sol. : $(1+x)(1-x)^n$

$$= (1+x) \left\{ 1 - nx + \binom{n}{2} x^2 - \dots + \binom{n}{n-1} (-x)^{n-1} + \binom{n}{n} (-x)^n \right\}$$

$\therefore$ The coefficient of x^n in the expansion,

$$= (1) \times \binom{n}{n} (-1)^n + (1) \times \binom{n}{n-1} (-1)^{n-1} = (-1)^n + n(-1)^{n-1} = (-1)^{n-1} (-1+n) = (-1)^n (1-n)$$

Ans. : (D)

- (54) If $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} + \frac{b}{x^2} \right)^{2x} = e^2$, then a and b satisfy (AIEEE : 2004)
- (A) $b = 2, a \in \mathbb{R}$ (B) $a = 1, b \in \mathbb{R}$ (C) $a, b \in \mathbb{R}$ (D) $a = 1, b = 2$

Sol. : $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} + \frac{b}{x^2} \right)^{2x} = e^2$

$\therefore \lim_{x \rightarrow \infty} 2x \log \left(1 + \frac{ax+b}{x^2} \right) = 2 \log_e e$

$\therefore \lim_{x \rightarrow \infty} 2x \left\{ \frac{ax+b}{x^2} - \frac{1}{2} \left(\frac{ax+b}{x^2} \right)^2 + \frac{1}{3} \left(\frac{ax+b}{x^2} \right)^3 + \dots \right\} = 2$

$$[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty]$$

$$\therefore \lim_{x \rightarrow \infty} \frac{2x}{x} \left\{ \frac{ax+b}{x} - \frac{1}{2} \frac{(ax+b)^2}{x^3} + \frac{1}{3} \frac{(ax+b)^3}{x^5} - \dots \infty \right\} = 2$$

$$\therefore \lim_{x \rightarrow \infty} 2 \left\{ a + \frac{b}{x} - \frac{1}{2x} \left(a + \frac{b}{x} \right)^2 + \frac{1}{3x^2} \left(a + \frac{b}{x} \right)^3 - \dots \infty \right\} = 2$$

$$\therefore 2 \{ a + 0 - 0 (a+0)^2 + 0 (a+0)^3 - \dots \infty \} = 2$$

$$\therefore 2a = 2$$

$$\therefore a = 1, b \in \mathbb{R}$$

Ans. : (B)

- (55) If the coefficients of the middle term in the expansions of $(1 + \alpha x)^4$ and $(1 - \alpha x)^6$ are same, then the value of non-zero α is (AIEEE : 2004)

(A) $\frac{-3}{10}$

(B) $\frac{-5}{3}$

(C) $\frac{3}{5}$

(D) $\frac{10}{3}$

Sol. : The coefficient of the middle term in $(1 + \alpha x)^4$ = The coefficient of the middle term in $(1 - \alpha x)^6$

$$\therefore \binom{4}{2} \alpha^2 = \binom{6}{3} (-\alpha)^3$$

$$\therefore 6 \alpha^2 = -20 \alpha^3$$

$$\therefore \alpha = \frac{-3}{10}$$

$$(\alpha \neq 0)$$

Ans. : (A)

- (56) If x is so small that x^3 and higher power of x may be neglected, then $\frac{(1+x)^{\frac{3}{2}} - \left(1 + \frac{x}{2}\right)^3}{(1-x)^{\frac{1}{2}}}$ may be approximated as (AIEEE : 2005)

(A) $3x + \frac{3}{8} x^2$

(B) $1 - \frac{3}{8} x^2$

(C) $\frac{x}{2} - \frac{3}{8} x^2$

(D) $-\frac{3}{8} x^2$

Sol. : Here, $\left\{ (1+x)^{\frac{3}{2}} - \left(1 + \frac{x}{2}\right)^3 \right\} (1-x)^{-\frac{1}{2}}$

$$= \left\{ \left(1 + \frac{3}{2}x + \frac{1}{2!} \left(\frac{3}{2} \right) \left(\frac{1}{2} \right) x^2 \right) - \left(1 + \frac{3x}{2} + (3) \frac{x^2}{4} \right) \right\} (1-x)^{-\frac{1}{2}} \quad [\text{neglecting terms containing } x^3, x^4, x^5 \dots]$$

$$= \left(\frac{3}{8} x^2 - \frac{3}{4} x^2 \right) \left(1 + \frac{1}{2}x + \frac{\frac{1}{2} \times \frac{3}{2}}{2} x^2 \right) \quad (\text{Since, higher powers of } x \text{ can be neglected})$$

$$= \left(\frac{-3x^2}{8} \right) \left(1 + \frac{x}{2} + \frac{3x^2}{8} \right)$$

$$= \frac{-3x^2}{8} - \frac{3x^3}{16} - \frac{9x^4}{64} = \frac{-3x^2}{8}$$

[After neglecting terms containing x^3, x^4]

Ans. : (D)

- (57) If the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$, then a and b satisfy the relation (AIEEE : 2005)
- (A) $a - b = 1$ (B) $a + b = 1$ (C) $ab = 1$ (D) $a = b$

Sol. : $(r + 1)$ st term in $\left(ax^2 + \frac{1}{bx}\right)^{11} = \binom{11}{r} (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r$

$$= \binom{11}{r} \frac{a^{11-r}}{b^r} \times x^{22-2r-r}.$$

So, to find coefficient of x^7 , $22 - 3r = 7$. So $r = 5$

and $(r + 1)$ st term in $\left(ax - \frac{1}{bx^2}\right)^{11} = \binom{11}{r} (ax)^{11-r} \left(\frac{-1}{bx^2}\right)^r$

$$= \binom{11}{r} \frac{a^{11-r}}{b^r} \times (-1)^r x^{11-3r}$$

So, comparing with x^{-7} , $r = 6$

Taking $r = 5$ and $r = 6$ respectively,

$$\therefore \binom{11}{5} \frac{a^6}{b^5} = \binom{11}{6} \frac{a^5}{b^6} (-1)^6$$

$$\therefore a^6 b^6 = a^5 b^5$$

$$\therefore ab = 1$$

Ans. : (C)

- (58) For natural numbers m and n , if $(1 - y)^m (1 + y)^n = 1 + a_1 y + a_2 y^2 + \dots$ and $a_1 = a_2 = 10$, then $(m, n) = \dots$ (AIEEE : 2006)
- (A) (20, 45) (B) (35, 20) (C) (45, 35) (D) (35, 45)

Sol. : $(1 - y)^m (1 + y)^n = \left(1 - my + \binom{m}{2} y^2 - \binom{m}{3} y^3 + \dots (m+1) \text{ terms}\right) \times$

$$\left(1 + ny + \binom{n}{2} y^2 + \binom{n}{3} y^3 + \dots (n+1) \text{ terms}\right)$$

$$= 1 + (n - m) y + \left(\binom{m}{2} + \binom{n}{2} - mn\right) y^2 + \dots$$

$$\therefore n - m = a_1 = 10, \binom{m}{2} + \binom{n}{2} - mn = a_2 = 10$$

$$\therefore n - m = 10, \frac{m(m-1)}{2} + \frac{n(n-1)}{2} - mn = 10$$

$$\therefore n - m = 10, m^2 - m + n^2 - n - 2mn = 20$$

$$\therefore n - m = 10, (m - n)^2 - (m + n) = 20$$

$$\therefore n - m = 10, 100 - (m + n) = 20$$

$$\therefore n - m = 10, m + n = 80$$

$$\therefore n = 45, m = 35$$

$$\therefore (m, n) = (35, 45)$$

Ans. : (D)

(59) If the sum of the 5th and 6th terms in the expansion of $(a - b)^n$, $n \geq 5$ is zero, then $\frac{a}{b} = \dots$

(AIEEE : 2007)

- (A) $\frac{6}{n-5}$ (B) $\frac{5}{n-4}$ (C) $\frac{n-5}{6}$ (D) $\frac{n-4}{5}$

Sol. : $T_5 + T_6 = 0$

$$\therefore \binom{n}{4} a^{n-4} (-b)^4 = -\binom{n}{5} a^{n-5} (-b)^5$$

$$\therefore \frac{n(n-1)(n-2)(n-3)}{24} a = \frac{n(n-1)(n-2)(n-3)(n-4)}{120} b$$

$$\therefore \frac{a}{b} = \frac{n-4}{5}$$

Ans. : (D)

(60) If the expansion of the function $\frac{1}{(1-ax)(1-bx)}$ in powers of x is $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$, then $a_n = \dots$ (AIEEE : 2006)

- (A) $\frac{a^n - b^n}{b-a}$ (B) $\frac{a^{n+1} - b^{n+1}}{b-a}$ (C) $\frac{b^{n+1} - a^{n+1}}{b-a}$ (D) $\frac{b^n - a^n}{b-a}$

Sol. : $(1 - ax)^{-1} (1 - bx)^{-1} = (1 + ax + a^2x^2 + \dots) \times (1 + bx + b^2x^2 + \dots)$

$a_n =$ The coefficient of x^n

$$= b^n + ab^{n-1} + a^2b^{n-2} + \dots + a^n$$

$$= b^n \left(1 + \frac{a}{b} + \frac{a^2}{b^2} + \dots + \frac{a^n}{b^n} \right)$$

$$= b^n \left(\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1} \right) = b^n \left(\frac{a^{n+1} - b^{n+1}}{a - b} \right) \frac{b}{b^{n+1}} = \frac{b^{n+1} - a^{n+1}}{b - a}$$

Ans. : (C)

(61) $\binom{20}{0} - \binom{20}{1} + \binom{20}{2} - \binom{20}{3} + \dots + \binom{20}{10} = \dots$ (AIEEE : 2007)

- (A) $-\binom{20}{10}$ (B) $\frac{1}{2} \binom{20}{10}$ (C) 0 (D) $\binom{20}{10}$

Sol. : $(1 + x)^{20} = \binom{20}{0} + \binom{20}{1}x + \binom{20}{2}x^2 + \binom{20}{3}x^3 + \dots + \binom{20}{10}x^{10} + \dots + \binom{20}{20}x^{20}$

$$\text{Let } x = -1. \text{ So } 0 = \binom{20}{0} - \binom{20}{1} + \binom{20}{2} - \dots - \binom{20}{9} + \binom{20}{10} - \binom{20}{11} + \dots + \binom{20}{20}$$

$$= \binom{20}{0} - \binom{20}{1} + \dots - \binom{20}{9} + \binom{20}{10} - \binom{20}{9} + \binom{20}{8} - \dots + \binom{20}{0}$$

$$\binom{20}{10} = 2 \binom{20}{0} - 2 \binom{20}{1} + \dots - 2 \binom{20}{9} + \binom{20}{10} + \binom{20}{10} \quad \text{(Adding } \binom{20}{10} \text{ on both the sides)}$$

$$\binom{20}{10} = 2 \left\{ \binom{20}{0} - \binom{20}{1} + \dots - \binom{20}{9} + \binom{20}{10} \right\}$$

$$\therefore \binom{20}{0} - \binom{20}{1} + \dots - \binom{20}{9} + \binom{20}{10} = \frac{1}{2} \binom{20}{10} \quad \text{Ans. : (B)}$$

(62) On dividing $8^{2n} - (62)^{2n+1}$ by 9, the remainder obtained is (AIEEE : 2009)

- (A) 0 (B) 2 (C) 7 (D) 8

Sol. : $8^{2n} - (62)^{2n+1} = (1 + 63)^n - (63 - 1)^{2n+1}$

$$= (1 + 63)^n + (1 - 63)^{2n+1}$$

$$= \left[1 + 63n + \binom{n}{2}(63)^2 + \dots + (63)^n \right] +$$

$$\left[1 - (2n+1)63 + \binom{2n+1}{2}(63)^2 - \dots - (63)^{2n+1} \right]$$

$$= 2 + 63 \left[n + \binom{n}{2}(63) + \dots + (63)^{n-1} - (2n+1) + \binom{2n+1}{2}(63) - \dots - (63)^{2n} \right]$$

So, dividing $8^{2n} - (62)^{2n+1}$ by 9, the remainder obtained is 2. **Ans. : (B)**

Second method : $8^{2n} - (62)^{2n+1} \equiv (1)^n - (-1)^{2n+1} \pmod{9} \equiv 1 + 1 = 2 \pmod{9}$

(63) The degree of polynomial $\left\{ x + (x^3 - 1)^{\frac{1}{2}} \right\}^5 + \left\{ x - (x^3 - 1)^{\frac{1}{2}} \right\}^5$ in x is (IIT-JEE : 1992)

- (A) 5 (B) 6 (C) 7 (D) 8

Sol. : We know that, $(x+a)^n + (x-a)^n = 2 \left[x^n + \binom{n}{2}x^{n-2}a^2 + \binom{n}{4}x^{n-4}a^4 + \dots \right]$

Taking $a = (x^3 - 1)^{\frac{1}{2}}$ and $n = 5$, $\left\{ x + (x^3 - 1)^{\frac{1}{2}} \right\}^5 + \left\{ x - (x^3 - 1)^{\frac{1}{2}} \right\}^5$

$$= 2 \left[x^5 + \binom{5}{2}x^3(x^3 - 1) + \binom{5}{4}x(x^3 - 1)^2 \right]$$

The index of the first term is 5, that of the second term is 6 and that of the third term is 7.

The given polynomial is of degree 7. **Ans. : (C)**

(64) If n is a positive integer, then $(\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n}$ is (AIEEE-2012)

- (A) an irrational number (B) an odd positive integer
(C) an even positive integer (D) a non-integer rational number

$$\begin{aligned}\text{Sol. : } (\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n} &= (4 + 2\sqrt{3})^n - (4 - 2\sqrt{3})^n \\ &= 2^n [(2 + \sqrt{3})^n - (2 - \sqrt{3})^n] \\ &= 2^n \cdot 2 \left[\binom{n}{1} 2^{n-1} \sqrt{3} + \binom{n}{3} 2^{n-3} (\sqrt{3})^3 + \dots \right] \\ &= 2^{n+1} \sqrt{3} \left[\binom{n}{1} 2^{n-1} + \binom{n}{3} 2^{n-3} \cdot 3 + \dots \right] \text{ is an irrational number.}\end{aligned}$$

Ans. : (A)

(65) Given that positive integers $r > 1$, $n > 2$ and the coefficient of $(3r)$ th and $(r + 2)$ nd terms in the expansion of $(1 + x)^{2n}$ are equal, then $n = \dots\dots$. (IIT-JEE : 1983)

- (A) $n = 2r$ (B) $n = 2r + 1$ (C) $n = 3r$ (D) None of these

Sol. : In the expansion of $(1 + x)^{2n}$, coefficients of T_{3r} and T_{r+2} are equal.

$$\begin{aligned}\therefore \binom{2n}{3r-1} &= \binom{2n}{r+1} \\ \therefore 3r - 1 &= r + 1 \text{ or } (3r - 1) + (r + 1) = 2n \\ \therefore 2r &= 2 \text{ or } 4r = 2n \\ \therefore r &= 1 \text{ or } 2r = n\end{aligned}$$

But $r > 1$ is given.

So, $n = 2r$

Ans. : (A)

: Questions Asked in JEE Main and Advanced :

(66) The term independent of x in the expansion of $\left(\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \frac{x-1}{x - x^{\frac{1}{2}}} \right)^{10}$ is

(JEE-MAIN : 2013)

- (A) 4 (B) 120 (C) 210 (D) 310

$$\begin{aligned}\text{Sol. : } &\left[\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \frac{x-1}{x - x^{\frac{1}{2}}} \right]^{10} \\ &= \left[\frac{1}{(x^{\frac{1}{3}} + 1)} - \frac{(x^{\frac{1}{2}} - 1)(x^{\frac{1}{2}} + 1)}{x^{\frac{1}{2}}(x^{\frac{1}{2}} - 1)} \right]^{10} \quad \left(x + 1 = (x^{\frac{1}{3}} + 1)(x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1) \right)\end{aligned}$$

$$= \left[\left(x^{\frac{1}{3}} + 1 \right) - \left(1 + x^{\frac{-1}{2}} \right) \right]^{10} = \left(x^{\frac{1}{3}} - x^{\frac{-1}{2}} \right)^{10}$$

$$\begin{aligned} \text{Now, } T_{r+1} &= \binom{10}{r} \left(x^{\frac{1}{3}} \right)^{10-r} \left(-x^{\frac{-1}{2}} \right)^r = \binom{10}{r} x^{\frac{10-r}{3}} x^{\frac{-r}{2}} (-1)^r \\ &= \binom{10}{r} (-1)^r x^{\frac{20-5r}{6}} \end{aligned}$$

For constant term, $\frac{20-5r}{6} = 0$. So, $r = 4$

$$\therefore T_5 = \binom{10}{4} (-1)^4 = \frac{10 \times 9 \times 8 \times 7}{24} = 210$$

Ans. : (C)

(67) The greatest term in the expansion of $(10 + 3x)^{12}$, when $x = 4$ is and its value is

$$(A) T_8, \binom{12}{5} (120)^5 \times 144$$

$$(B) T_7, \binom{12}{7} 10^5 \times 12^8$$

$$(C) T_7 = T_8, \binom{12}{8} 10^5 \cdot 12^7$$

$$(D) T_7, \binom{12}{5} 10^5 \times 12^7$$

Sol. : Comparing $(10 + 3x)^{12}$ with $(x + y)^n$, $x = 10$, $y = 3x$, $n = 12$

$$\text{Now, } m = \frac{(n+1)y}{x+y} = \frac{(12+1)3x}{10+3x} = \frac{13 \times 3 \times 4}{10+12} = \frac{156}{22} = \frac{78}{11} \text{ is not an integer.}$$

$$\text{So greatest term, } T_{[m] + 1} = T_{7+1} = T_8$$

$$\text{and } T_8 = \binom{12}{7} (10)^5 \times (3x)^7 = \binom{12}{5} 10^5 \times 12^7 = \binom{12}{5} 120^5 \times 144$$

Ans. : (A)

$$\text{Second method : } T_{r+1} \geq T_r \Rightarrow \binom{12}{r} 10^{12-r} (3x)^r \geq \binom{12}{r-1} 10^{12-r+1} (3x)^{r-1}$$

$$\Rightarrow \frac{3x}{r} \geq \frac{10}{13-r}$$

$$\Rightarrow r \leq \frac{78}{11}$$

($x = 4$)

$\therefore$ The greatest value of $r = 7$

$$\therefore \text{ The greatest term } = T_{r+1} = T_8 = \binom{12}{5} 10^5 \times 12^7$$

$$= \binom{12}{5} (120)^5 \times 144$$

Ans. : (A)

- (68) The sum of the coefficients of integral powers of x in the binomial expansion of $(1 - 2\sqrt{x})^{50}$
(JEE-MAIN : 2015)

(A) $\frac{1}{2} (3^{50} + 1)$ (B) $\frac{1}{2} (3^{50})$ (C) $\frac{1}{2} (3^{50} - 1)$ (D) $\frac{1}{2} (2^{50} + 1)$

Sol. : $(1 - 2\sqrt{x})^{50} = \binom{50}{0} - \binom{50}{1} (2\sqrt{x})^1 + \binom{50}{2} (2\sqrt{x})^2 - \binom{50}{3} (2\sqrt{x})^3 + \dots$

Let k = The coefficient of integral powers of x .

$$= \binom{50}{0} + \binom{50}{2} (2)^2 + \binom{50}{4} (2)^4 + \dots + \binom{50}{50} 2^{50}$$

Now, $(1 + x)^{50} = \binom{50}{0} + \binom{50}{1}x + \binom{50}{2}x^2 + \binom{50}{3}x^3 + \binom{50}{4}x^4 + \dots + \binom{50}{50}x^{50}$ (1)

Put $x = 2$ in (1), $3^{50} = \binom{50}{0} + \binom{50}{1}2 + \binom{50}{2}2^2 + \binom{50}{3}2^3 + \binom{50}{4}2^4 + \dots + \binom{50}{50}2^{50}$ (2)

Put $x = -2$ in (1),

$$1^{50} = \binom{50}{0} - \binom{50}{1}2 + \binom{50}{2}2^2 - \binom{50}{3}2^3 + \binom{50}{4}2^4 - \dots + \binom{50}{50}2^{50}$$
 (3)

Adding (2) and (3),

$$3^{50} + 1 = 2 \left[\binom{50}{0} + \binom{50}{2}2^2 + \binom{50}{4}2^4 + \dots + \binom{50}{50}2^{50} \right]$$

$$\frac{3^{50} + 1}{2} = \left[\binom{50}{0} + \binom{50}{2}2^2 + \binom{50}{4}2^4 + \dots + \binom{50}{50}2^{50} \right] = k$$

$$\therefore k = \frac{3^{50} + 1}{2}$$

Ans. : (A)

- (69) If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in power of x are both zero, then $(a, b) = \dots$
(JEE-MAIN : 2014)

(A) $\left(16, \frac{251}{3}\right)$ (B) $\left(14, \frac{251}{3}\right)$ (C) $\left(14, \frac{272}{3}\right)$ (D) $\left(16, \frac{272}{3}\right)$

Sol. : $(1 + ax + bx^2)(1 - 2x)^{18}$

$$= (1 + ax + bx^2) \left(1 - \binom{18}{1}2x + \binom{18}{2}(2x)^2 - \binom{18}{3}(2x)^3 + \binom{18}{4}(2x)^4 - \dots \right)$$

$$\text{Now, the coefficient of } x^3 = -\binom{18}{3}2^3 + a\binom{18}{2}2^2 - b\binom{18}{1}2 = 0 \quad (1)$$

$$\text{The coefficient of } x^4 = \binom{18}{4}2^4 + a\left(-\binom{18}{3}2^3\right) + b\binom{18}{2}2^2 = 0 \quad (2)$$

$$\text{From (1), } -153a + 9b + 1632 = 0 \text{ giving } -51a + 3b + 544 = 0 \quad (3)$$

$$\text{From (2), } 32a - 3b - 240 = 0. \quad (4)$$

$$\text{So } -19a + 304 = 0. \quad (\text{Add (3) and (4)})$$

$$\therefore a = 16. \quad \text{So, } b = \frac{272}{3}$$

$$\therefore (a, b) = \left(16, \frac{272}{3}\right) \quad \text{Ans. : (D)}$$

$$(70) \quad \text{Prove that } 7^{2n} + 2^{3n-3} \cdot 3^{n-1} \text{ is divisible by 25.} \quad (\text{IIT : 1992})$$

$$\text{Sol. : } 7^{2n} + 2^{3n-3} \cdot 3^{n-1} = 49^n + 2^{3(n-1)} \cdot 3^{n-1} = 49^n + 24^{(n-1)}$$

$$= (50 - 1)^n + (25 - 1)^{n-1}$$

$$= (-1)^n (1 - 50)^n + (-1)^{n-1} (1 - 25)^{n-1}$$

$$= (-1)^n \left\{ [1 - 50n + \binom{n}{2}(50)^2 - \binom{n}{3}(50)^3 + \binom{n}{4}(50)^4 - \dots + (-50)^n] \right.$$

$$\left. - [1 - (n-1)25 + \binom{n-1}{2}25^2 - \binom{n-1}{3}25^3 + \dots + (-1)^{n-1}(25)^{n-1}] \right\}$$

$$= (-1)^n \left\{ -50n + \binom{n}{2}(50)^2 - \binom{n}{3}(50)^3 + \binom{n}{4}(50)^4 - \dots + (-1)^n 50^n + (n-1)25 \right.$$

$$\left. - \binom{n-1}{2}25^2 + \binom{n-1}{3}25^3 - \dots - (-1)^{n-1}(25)^{n-1} \right\}$$

Each term is divisible by 25.

$$\therefore 7^{2n} + 2^{3n-3} \cdot 3^{n-1} \text{ is divisible by 25.}$$

$$\text{Second method : } 7^{2n} + 2^{3n-3} \cdot 3^{n-1} = 49^n + 24^{(n-1)}$$

$$49 \equiv (-1) \pmod{25} \text{ and } 24 \equiv (-1) \pmod{25}$$

$$\therefore 49^n + 24^{(n-1)} \equiv (-1)^n + (-1)^{n-1} \pmod{25} \equiv (0) \pmod{25}$$

$$\therefore 49^n + 24^{n-1} \text{ is divisible by 25.}$$

$$(71) \quad \text{Let } n \text{ be a positive integer and } (1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_{2n}x^{2n}. \text{ Then show that } a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots - a_{2n-1}^2 + a_{2n}^2 = a_n \quad (\text{IIT : 1994})$$

$$\text{Sol. : } (1 + x + x^2)(1 - x + x^2) = (1 + x^2)^2 - x^2 = 1 + x^2 + x^4$$

$$\therefore (1 + x + x^2)^n (1 - x + x^2)^n = (1 + x^2 + x^4)^n$$

$$\text{Now, } (1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_{2n}x^{2n} \quad (1)$$

Replace x by $\frac{1}{x}$ in (1),

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \dots + \frac{a_{2n}}{x^{2n}}$$

Multiplying both the sides by x^{2n} ,

$$(x^2 + x + 1)^n = a_0x^{2n} + a_1x^{2n-1} + a_2x^{2n-2} + \dots + a_{2n}$$

Replace x by $-x$,

$$(x^2 - x + 1)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} - \dots + a_{2n} \quad (2)$$

Replace x by x^2 in (1),

$$(1 + x^2 + x^4)^n = a_0 + a_1x^2 + a_2x^4 + \dots + a_nx^{2n} + \dots + a_{2n}x^{4n} \quad (3)$$

Multiplying (1) and (2), we get

$$(1 + x + x^2)^n \times (1 - x + x^2)^n = (a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}) \times (a_0x^{2n} - a_1x^{2n-1} + \dots + a_{2n}) \quad (4)$$

Equating (3) and (4),

$$a_0 + a_1x^2 + a_2x^4 + \dots + a_nx^{2n} + \dots + a_{2n}x^{4n} = (a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}) \times (a_0x^{2n} - a_1x^{2n-1} + \dots + a_{2n})$$

Comparing coefficients of x^{2n} on both the sides,

$$a_n = a_0^2 - a_1^2 + a_2^2 - \dots + a_{2n}^2$$

$$\therefore a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + a_{2n}^2 = a_n$$

$$(72) \quad \text{Let } R = (5\sqrt{5} + 11)^{2n+1} \text{ and } f = R - [R] \text{ where } [] \text{ denotes the greatest integer function,} \\ \text{prove that } R \cdot f = 4^{2n+1}. \quad (\text{IIT : 1988})$$

Sol. : For integer I , let $I < R < I + 1$. So, $[R] = I$ (R is not an integer.)

$$0 < R - [R] < 1. \text{ Hence, } 0 < f < 1 \text{ and } R = f + I$$

$$\text{Now, } 121 < 125 < 144 \Rightarrow 11 < 5\sqrt{5} < 12 \Rightarrow 0 < 5\sqrt{5} - 11 < 1$$

$$\Rightarrow 0 < (5\sqrt{5} - 11)^{2n+1} < 1$$

$$\therefore 0 < f' < 1 \text{ where, } f' = (5\sqrt{5} - 11)^{2n+1}$$

$$\therefore I + f - f' = (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$$

$$= 2 \left\{ \binom{2n+1}{1} (5\sqrt{5})^{2n} 11 + \binom{2n+1}{3} (5\sqrt{5})^{2n-2} \cdot 11^3 + \dots \right\}$$

$$= \text{an even integer}$$

$$\therefore f - f' = \text{an integer} \quad (\text{I is an integer.})$$

$$\text{But } 0 < f < 1, 0 < f' < 1 \Rightarrow -1 < f - f' < 1$$

$$\therefore f - f' = 0 \Rightarrow f = f'$$

$$\begin{aligned} \therefore R \cdot f &= R \cdot f' = (5\sqrt{5} + 11)^{2n+1} \cdot (5\sqrt{5} - 11)^{2n+1} \\ &= \{(5\sqrt{5} + 11)(5\sqrt{5} - 11)\}^{2n+1} = (125 - 121)^{2n+1} = 4^{2n+1} \end{aligned}$$

$$(73) \quad \text{If } \sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r \text{ and } a_k = 1, \text{ for all } k \geq n, \text{ then show that } b_n = \binom{2n+1}{n+1}$$

(IIT : 92)

Sol. : Substitute $x - 3 = y$

$$\sum_{r=0}^{2n} a_r (1+y)^r = \sum_{r=0}^{2n} b_r y^r$$

$$\begin{aligned} \therefore a_0 + a_1(1+y) + a_2(1+y)^2 + \dots + a_{n-1}(1+y)^{n+1} + 1(1+y)^n + 1(1+y)^{n+1} + 1(1+y)^{n+2} + \dots + \\ 1(1+y)^{2n} = b_0 + b_1 y + \dots + b_n y^n + \dots + b_{2n} y^{2n} \quad (a_n, a_{n+1}, \dots = 1) \end{aligned}$$

Comparing coefficients of y^n on both the sides,

$$1 + \binom{n+1}{1} + \binom{n+2}{2} + \dots + \binom{n+n}{n} = b_n$$

$$\begin{aligned} \therefore b_n &= \binom{n+1}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \dots + \binom{2n}{n} \\ &= \binom{n+2}{1} + \binom{n+2}{2} + \binom{n+3}{3} + \dots + \binom{2n}{n} \\ &= \binom{n+3}{2} + \binom{n+3}{3} + \dots + \binom{2n-1}{n-1} + \binom{2n}{n} \\ &= \binom{n+4}{3} + \dots + \binom{2n-1}{n-1} + \binom{2n}{n} = \binom{2n-1}{n-2} + \binom{2n-1}{n-1} + \binom{2n}{n} \\ &= \binom{2n}{n-1} + \binom{2n}{n} = \binom{2n+1}{n} = \binom{2n+1}{n+1} \end{aligned}$$

$$(74) \quad \text{Prove that, } \frac{3!}{2(n+3)} = \sum_{r=0}^n (-1)^r \frac{{}_n C_r}{{}_{r+3} C_r} \quad (\text{I.I.T. : 1997})$$

$$\text{Sol. : } \frac{{}_n C_r}{{}_{r+3} C_r} = \frac{n!}{r!(n-r)!} \times \frac{r!3!}{(r+3)!} = \frac{3!(n+3)!}{(n+1)(n+2)(n+3)(n-r)!(r+3)!}$$

$$= \frac{3!}{n+3P_3} \times \binom{n+3}{r+3}$$

$$\begin{aligned} \text{R.H.S.} &= \sum_{r=0}^n \left\{ \frac{(-1)^r 3!}{n+3P_3} \times \binom{n+3}{r+3} \right\} = \frac{-3!}{n+3P_3} \sum_{r=0}^n \left\{ (-1)^{r+3} \binom{n+3}{r+3} \right\} \\ &= \frac{-3!}{n+3P_3} \left\{ (-1)^3 \binom{n+3}{3} + (-1)^4 \binom{n+3}{4} + (-1)^5 \binom{n+3}{5} + \dots + (-1)^{n+3} \binom{n+3}{n+3} \right\} \\ &= \frac{-3!}{n+3P_3} \left\{ (-1)^0 \binom{n+3}{0} + (-1)^1 \binom{n+3}{1} + (-1)^2 \binom{n+3}{2} + (-1)^3 \binom{n+3}{3} + \dots + (-1)^{n+3} \binom{n+3}{n+3} \right\} \\ &= \frac{-3!}{n+3P_3} \left\{ (1-1)^{n+3} - \binom{n+3}{0} + \binom{n+3}{1} - \binom{n+3}{2} \right\} \\ &= \frac{-3!}{n+3P_3} \left\{ 0 - 1 + n + 3 - \frac{(n+3)(n+2)}{2} \right\} \\ &= \frac{-3!}{(n+3)(n+2)(n+1)} \times \left\{ \frac{-2 + 2n + 6 - n^2 - 5n - 6}{2} \right\} \\ &= \frac{-3!}{(n+3)(n+2)(n+1)} \times \left(\frac{-n^2 - 3n - 2}{2} \right) = \frac{3!}{2(n+3)} = \text{L.H.S.} \end{aligned}$$

(75) If the sum $\sum_{i=0}^m \binom{10}{i} \binom{20}{m-i}$ is maximum, then $m = \dots$ (where $\binom{p}{q} = 0$ if $p < q$)

(IIT-JEE : 2002)

(A) 15

(B) 5

(C) 20

(D) 10

$$\begin{aligned} \text{Sol. : } \sum_{i=0}^m \binom{10}{i} \binom{20}{m-i} &= \binom{10}{0} \binom{20}{m} + \binom{10}{1} \binom{20}{m-1} + \dots + \binom{10}{m} \binom{20}{0} \\ &= \text{The coefficient of } x^m \text{ in the expansion of } (1+x)^{10} (1+x)^{20} \\ &= \text{The coefficient of } x^m \text{ in the expansion of } (1+x)^{30} \\ &= \binom{30}{m} \text{ which is maximum when } m = 15. \end{aligned}$$

Ans. : (A)

(76) The coefficient of t^{24} in $(1+t^2)^{12} (1+t^{12}) (1+t^{24}) \dots$

(IIT-JEE : 2003)

(A) $\binom{12}{6}$

(B) $\binom{12}{6} + 3$

(C) $\binom{12}{6} + 2$

(D) $\binom{12}{6} + 1$

$$\begin{aligned} \text{Sol. : } (1+t^2)^{12} (1+t^{12}) (1+t^{24}) \\ = (1+t^{12}+t^{24}+t^{36}) (1+t^{12}) \end{aligned}$$

$$= (1 + t^{12} + t^{24} + t^{36}) \left[\binom{12}{0} + \binom{12}{1} t^2 + \binom{12}{2} t^4 + \dots + \binom{12}{12} t^{24} \right]$$

$$\text{The coefficient of } t^{24} = 1 \times \binom{12}{12} + 1 \times \binom{12}{6} + 1 \times \binom{12}{0}$$

$$= 1 + \binom{12}{6} + 1$$

$$= \binom{12}{6} + 2$$

Ans. : (C)

(77) The coefficient of x^4 in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is (IIT-JEE : 2003)

(A) $\frac{450}{263}$

(B) $\frac{405}{256}$

(C) $\frac{504}{259}$

(D) None of these

Sol. : Here, $T_{r+1} = \binom{10}{r} \left(\frac{x}{2}\right)^{10-r} \left(\frac{-3}{x^2}\right)^r = \binom{10}{r} x^{10-3r} \frac{(-1)^r 3^r}{2^{10-r}}$

To find the coefficient of x^4 , we should have $10 - 3r = 4$ i.e. $r = 2$

$$\text{The coefficient of } x^4 = \binom{10}{2} \frac{(-1)^2 3^2}{2^8} = \frac{45 \times 9}{256}$$

$$= \frac{405}{256}$$

Ans. : (B)

(78) If the coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio of 5:10:14, then $n = \dots\dots$.

Sol. : Let the coefficients of three consecutive terms in $(1+x)^{n+5}$ namely T_{r-1} , T_r , T_{r+1} be in the ratio 5:10:14. (IIT-JEE : 2013)

$$\binom{n+5}{r-2} : \binom{n+5}{r-1} : \binom{n+5}{r} = 5 : 10 : 14$$

$$\therefore \frac{\binom{n+5}{r-2}}{5} = \frac{\binom{n+5}{r-1}}{10} \Rightarrow \frac{2(n+5)!}{(n-r+7)!(r-2)!} = \frac{(n+5)!}{(n-r+6)!(r-1)!}$$

$$\therefore 2(r-1) = n-r+7$$

$$\therefore n-3r+9=0 \quad (1)$$

$$\text{and } \frac{\binom{n+5}{r-1}}{10} = \frac{\binom{n+5}{r}}{14} \Rightarrow \frac{7(n+5)!}{(n-r+6)!(r-1)!} = \frac{5(n+5)!}{(n-r+5)! r!}$$

$$\Rightarrow 7r = 5(n-r+6)$$

$$\Rightarrow 5n-12r+30=0. \text{ Also, } 4n-12r+36=0 \quad (\text{From (1)})$$

$$\therefore n-6=0.$$

$$\text{So, } n=6$$

Ans. : $n=6$

(79) The coefficient of x^{11} in the expansion of $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ is

(JEE- Advanced : 2014)

(A) 1051

(B) 1106

(C) 1113

(D) 1120

Sol. : Here, $(x^2)^a \cdot (x^3)^b \cdot (x^4)^c = x^{11}$

$$\therefore x^{2a+3b+4c} = x^{11}$$

$$\therefore 2a + 3b + 4c = 11$$

$$\therefore (a, b, c) = (0, 1, 2), (1, 3, 0), (2, 1, 1), (4, 1, 0)$$

$$\begin{aligned} \therefore \text{The coefficient of } x^{11} &= \binom{4}{0} \binom{7}{1} \binom{12}{2} + \binom{4}{1} \binom{7}{3} \binom{12}{0} + \binom{4}{2} \binom{7}{1} \binom{12}{1} + \binom{4}{4} \binom{7}{1} \binom{12}{0} \\ &= (1 \times 7 \times 66) + (4 \times 35 \times 1) + (6 \times 7 \times 12) + (1 \times 7 \times 1) \\ &= 462 + 140 + 504 + 7 = 1113 \end{aligned}$$

Ans. : (C)

: Questions with More Than One Correct Answers :

(80) The coefficient of $a^3 b^4 c^5$ in the expansion of $(a + b + c)^{12}$ is

(A) $\frac{(12)!}{3!4!5!}$

(B) $\binom{12}{5} \times \binom{7}{3}$

(C) $\binom{12}{3} \binom{12}{4} \binom{12}{5}$

(D) $\binom{12}{4} \times \binom{8}{3}$

Sol. : The general term in the expansion of $(a + b + c)^{12} = \frac{(12)!}{p!q!r!} a^p b^q c^r = \frac{(12)!}{3!4!5!} a^3 b^4 c^5$

$$\therefore \text{The required coefficient} = \frac{(12)!}{3!4!5!} = \frac{(12)!}{7!5!} \times \frac{7!}{3!4!} \text{ (required)}$$

$$= \binom{12}{5} \times \binom{7}{3}$$

$$\text{Also, } \frac{(12)!}{3!4!5!} = \frac{(12)!}{8!4!} \times \frac{8!}{3!5!}$$

$$= \binom{12}{4} \times \binom{8}{3}$$

Ans. : (A), (B), (D)

(81) If in the expansion of $\left\{ (9^{x-1} + 7)^{\frac{1}{2}} + (3^{x-1} + 1)^{\frac{-1}{5}} \right\}^7$, $T_6 = 84$, then $x = \dots$

(A) 3

(B) 1

(C) 2

(D) 1 or 2

Sol. : $T_6 = \binom{7}{5} (9^{x-1} + 7)^{\frac{2}{2}} (3^{x-1} + 1)^{\frac{-5}{5}} = 84$

$$\therefore 21 (9^{x-1} + 7) = 84 (3^{x-1} + 1)$$

$$\therefore \frac{3^{2x}}{9} + 7 = 4 \left(\frac{3^x}{3} + 1 \right)$$

$$\therefore 3^{2x} + 63 = 12 \cdot 3^x + 36$$

$$\therefore (3^x)^2 - 12 \cdot 3^x + 27 = 0$$

$$\therefore (3^x - 3) (3^x - 9) = 0$$

$$\therefore 3^x = 3 \text{ or } 3^x = 9$$

$$\therefore x = 1 \text{ or } x = 2$$

Ans. : (B), (C), (D)

(82) The number of terms in the expansion of $(5x + 6y - 7z)^n$

- (A) $\binom{n+2}{3}$ (B) $\Sigma (n+2)$ (C) $\binom{n+2}{2}$ (D) $\Sigma (n+1)$

Sol. : The number of terms $= \binom{n+k-1}{n} = \binom{n+3-1}{n} = \binom{n+2}{n} = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$

$\therefore$ The number of terms $= \Sigma (n+1)$

Ans. : (C), (D)

(83) If the sums of the coefficients in the expansion of $(1 - 3x^2 + 10x^5)^n$ and $(1 + x^7)^n$ are p and q respectively, then ($n > 1$)

- (A) $p : q = 4$ (B) $p + q = pq$ (C) $p = q^3$ (D) $pq = 4^{2n}$

Sol. : Substitute $x = 1$ in $(1 - 3x^2 + 10x^5)^n$ and $(1 + x^7)^n$

$p = (1 - 3 + 10)^n = 8^n$ and $q = (1 + 1)^n = 2^n$

Here, $q^3 = (2^n)^3 = 2^{3n} = 8^n = p$, So, $p = q^3$. (C)

$pq = 8^n \cdot 2^n = 16^n = 4^{2n}$ (D)

$p : q = 8^n : 2^n = 2^{3n} : 2^n = 4^n \neq 4$, ($n \neq 1$)

$p + q = 8^n + 2^n \neq 4^{2n}$

$\therefore p + q \neq pq$

$\therefore p = q^3$ and $pq = 4^{2n}$

Ans. : (C), (D)

(84) If $p = \dots\dots$, then x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^4$.

- (A) 6 (B) 2 (C) 5 (D) 8, 2, 5, 3

Sol. : Here, $T_{r+1} = \binom{4}{r}(x^2)^{4-r} \left(\frac{1}{x}\right)^r = \binom{4}{r}(x)^{8-2r} \cdot x^{-r} = \binom{4}{r}x^{8-3r}$

$\therefore p = 8 - 3r$

$\therefore$ If $r = \frac{8-p}{3}$ is an integer, then x^p exists. Also, $0 \leq r \leq 4$

$\therefore p = 8$ or $p = 2$ or $p = 5$. $r = 3, 4$ correspond to $p = -1, -4$ not given. **Ans. :** (B), (C)

(85) The largest coefficient in $(1 + x)^{2n+12}$ is

- (A) $\binom{2n+12}{n+6}$ (B) $\frac{(2n+12)!}{[(n+6)!]^2}$ (C) $\frac{2!}{(n+6)!}$ (D) $\binom{2n+12}{n+7}$

Sol. : If n is even, then largest coefficient in the expansion of $(1 + x)^n = \binom{n}{\frac{n}{2}}$

$\therefore$ The required coefficient $= \binom{2n+12}{n+6} = \frac{(2n+12)!}{(n+6)!(n+6)!} = \frac{(2n+12)!}{[(n+6)!]^2}$ **Ans. :** (A), (B)

(86) If $T_1 = 729$, then in the expansion of $(x + 5)^6$

(A) $T_2 = 7290, x = -3$

(B) $T_3 = 30375, x = \pm 3$

(C) $x = 3, T_2 = -7290$

(D) $T_4 = \pm 67500, x = \pm 3$

Sol. : $T_{r+1} = \binom{6}{r} x^{6-r} 5^r$

$$\therefore T_1 = \binom{6}{0} x^6 5^0 = 729 \Rightarrow x^6 = 3^6 \Rightarrow x = \pm 3$$

$$\therefore T_2 = \binom{6}{1} x^5 5^1 = 6 (\pm 3)^5 5 = \pm 7290$$

$$\therefore T_3 = \binom{6}{2} x^4 5^2 = 15 \times (\pm 3)^4 \times 25 = 30375$$

$$\therefore T_4 = \binom{6}{3} x^3 5^3 = 20 (\pm 3)^3 (125) = \pm 67500$$

Ans. : (B), (D)

(87) If $n = \dots$, then the coefficients of T_5, T_6 and T_7 in the expansion of $(1 + x)^n$ are in A.P.

(A) 13

(B) 7

(C) 14

(D) 7, 14

Sol. : The coefficients of T_5, T_6 and $T_7, \binom{n}{4}, \binom{n}{5}$ and $\binom{n}{6}$ are in A.P. $\Leftrightarrow \binom{n}{4} + \binom{n}{6} = 2 \binom{n}{5}$

$$\Leftrightarrow \frac{n!}{4!(n-4)!} + \frac{n!}{6!(n-6)!} = \frac{2n!}{5!(n-5)!}$$

$$\Leftrightarrow \frac{1}{(n-4)(n-5)} + \frac{1}{6 \times 5} = \frac{2}{5(n-5)}$$

$$\Leftrightarrow 30 + (n-4)(n-5) = 12(n-4)$$

$$\Leftrightarrow 30 + n^2 - 9n + 20 = 12n - 48$$

$$\Leftrightarrow n^2 - 21n + 98 = 0$$

$$\Leftrightarrow (n-7)(n-14) = 0$$

$$\Leftrightarrow n = 7 \text{ or } n = 14$$

Ans. : (B), (C), (D)

: Assertion and Reason Type Questions :

Explanation : Each of these questions contains statement 1 (Assertion) and statement 2 (Reason). Each question has four choices, option (A), (B), (C) and (D), which are as follows. You have to select the correct choice.

(A) Statement 1 is true, statement 2 is true. Statement 2 is the correct explanation for statement 1.

(B) Statement 1 is true, statement 2 is true. Statement 2 is not a correct explanation for statement 1.

(C) Statement 1 is true, statement 2 is false.

(D) Statement 1 is false, statement 2 is true.

(88) **Statement 1** : $2^{3n} - 7n$ leaves remainder 1 when divided by 49.

$$\text{Statement 2 : } (1+x)^n = \sum_{r=0}^n \binom{n}{r} x^r, \forall n \in \mathbb{N}$$

$$\text{Sol. : } 2^{3n} = 8^n = (1+7)^n$$

$$= 1 + 7n + \binom{n}{2} 7^2 + \binom{n}{3} 7^3 + \dots + \binom{n}{n} 7^n$$

$$\therefore 2^{3n} - 7n = 1 + 49 \left[\binom{n}{2} + \binom{n}{3} 7 + \dots + 7^{n-2} \right]$$

$$= 1 + 49m \text{ where } m \text{ is a positive integer.}$$

$$\therefore 2^{3n} - 7n \text{ leaves remainder 1 when divided by 49.}$$

$\therefore$ Statement (1) and (2) are true and statement (2) is the correct explanation of statement (1).

Ans. : (A)

(89) **Statement 1** : The number of terms in the expansion $(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)^{50}$ are 251.

$$\text{Statement 2 : } (1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$

$$\text{Sol. : } (1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)^{50}$$

$$= [(1+x)^5]^{50} = (1+x)^{250} \text{ and the number of terms in its expansion} = 250 + 1 = 251$$

$\therefore$ Statement 1 is true.

$\therefore$ Statement (2) is also true and statement (2) is the correct explanation of statement (1).

Ans. : (A)

(90) **Statement 1** : The number of terms in power of x in the expansion of $\left(1+x+\frac{1}{x}\right)^{50}$ is 101.

$$\text{Statement 2 : The number of terms in the expansion of } (x_1 + x_2 + x_3 + \dots + x_r)^n \text{ is } \binom{n+r-1}{n}.$$

$$\text{Sol. : } \left(1+x+\frac{1}{x}\right)^{50} = \frac{1}{x^{50}} (1+x+x^2)^{50}$$

$$= \frac{1}{x^{50}} \left[1 + \binom{50}{1}(x+x^2) + \binom{50}{2}(x+x^2)^2 + \dots + (x+x^2)^{50} \right]$$

$$= \frac{1}{x^{50}} [a_1 + a_2x + a_3x^2 + \dots + a_{101}x^{100}]$$

$\therefore$ The total number of terms = 100 + 1 = 101

$\therefore$ Statement 1 is true.

The number of terms in the expansion of $(x_1 + x_2 + x_3 + \dots + x_r)^n$

= The number of non-negative solutions of the equation $a_1 + a_2 + a_3 + \dots + a_r = n$

= The coefficient of t^n in $(1 + t + t^2 + \dots + t^n)^r$

= The coefficient of t^n in $(1 - t^{n+1}) (1 - t)^{-r}$

= The coefficient of t^n in $(1 - t)^{-r} = \binom{n+r-1}{n}$

So the number of terms in the expansion of $(x_1 + x_2 + x_3 + \dots + x_r)^n = \binom{n+r-1}{n}$

∴ Statement (2) is true, but statement (2) is not the correct explanation of statement (1)

Ans. : (B)

(91) **Statement 1 :** $\sum_{r=0}^n (r+1) \cdot {}_nC_r = (n+2) 2^{n-1}$

Statement 2 : $\sum_{r=0}^n (r+1) \cdot {}_nC_r \cdot x^r = (1+x)^n + nx(1+x)^{n-1}$ (AIEEE : 2008)

Sol. : $(1+x)^n = \sum_{r=0}^n {}_nC_r \cdot x^r$. Multiply by x on both the sides.

$x(1+x)^n = \sum_{r=0}^n {}_nC_r \cdot x^{r+1}$. Now, differentiating w.r.t. x ,

$(1+x)^n + nx(1+x)^{n-1} = \sum_{r=0}^n (r+1) {}_nC_r \cdot x^r$ (1)

∴ Statement 2 is true.

Substitute $x = 1$, in the equation 1,

$2^n + n 2^{n-1} = \sum_{r=0}^n (r+1) \cdot {}_nC_r$

∴ $\sum_{r=0}^n (r+1) {}_nC_r = (n+2) 2^{n-1}$

∴ Statement 1 is also true and statement 2 is the correct explanation of statement 1. **Ans. :** (A)

(92) **Statement 1 :** $32^{32} - 4$ is divisible by 28.

Statement 2 : $32^{32} - 1$ is divisible by 3.

Sol. : $32^{32} = (2^5)^{32} = 2^{160} = (1-3)^{160}$

$= 1 - \binom{160}{1}3 + \binom{160}{2}3^2 + \dots + 3^{160}$

$= 1 + 3k \quad k \in \mathbb{Z}$

∴ $32^{32} - 1 = 3k$ is divisible by 3.

(Note : $32^{32} - 1 \equiv (-1)^{32} - 1 \equiv 0 \pmod{3}$)

$\therefore 32^{32} - 1$ is divisible by 3.

$\therefore$ Statement 2 is true.

$$\begin{aligned}\text{Now, } 32^{32^{32}} &= 32^{3k+1} = 2^{5(3k+1)} \\ &= 2^{15k+5} \\ &= 4 \cdot 2^{3(5k+1)} \\ &= 4 \cdot 8^{5k+1} \\ &= 4 \cdot (1+7)^{5k+1} \\ &= 4 \left[1 + \binom{5k+1}{1} 7 + \binom{5k+1}{2} 7^2 + \dots + 7^{5k+1} \right] \\ &= 4(1+7n) \quad n \in \mathbb{Z} \\ &= 4 + 28n\end{aligned}$$

$\therefore 32^{32^{32}} - 4 = 28n$ is divisible by 28.

$\therefore$ Thus, statement 1 and statement 2 both are true but statement 2 is not the correct explanation of statement 1.

Note : $32^{32^{32}}$ and 4 both are divisible by 4.

$\therefore 32^{32^{32}} - 4$ is divisible by 4.

$$\begin{aligned}32^{32^{32}} &\equiv 4^{4^4} \pmod{7} \\ &\equiv 4^{16 \cdot 16} \pmod{7} \\ &\equiv 4^{2 \cdot 2} \pmod{7} \\ &\equiv 4^4 \equiv 256 \equiv 4 \pmod{7}\end{aligned}$$

$\therefore 32^{32^{32}} - 4$ is divisible by 7.

$\therefore 32^{32^{32}} - 4$ is divisible by 28.

Ans. : (B)

