## CS Group (Computer Science)

**Note: Answer all questions from Part-A and any eight questions from Part-B.**

### Part-A

CS1. Suppose a crime has been committed and there are three suspects: Professor Plum, Mrs. Peacock, and Mr. Green. Given that:

- At least one of the above three suspects is guilty.
- Not all of them are guilty.
- If Mrs. Peacock is guilty, then so is Professor Plum.
- If Mr. Green is innocent, then so is Professor Plum.

Prove or disprove each of the following statements:

(i) Mr. Green is guilty.

(ii) Mrs. Peacock is innocent.

[3+3 = 6]

CS2. A room has four walls. In how many different ways can you color the four walls, using colors from the set {Red, Green, Blue, Yellow}, such that no two adjacent walls have the same color?

[7]

CS3. Consider a round-robin football tournament among $n \geq 3$ teams, where every team plays a match with every other team exactly once. For any team, each match results in a *win, draw or loss.* In the tournament, each team loses at least one match. Prove or disprove the following statement:

*There must be two teams with the same number of wins at the end of the tournament.*

[7]

**Part-B**

CS4. Prove or disprove the following statements:

(i) $L_1 = \{0^i 1^j \mid i, j \in \mathbb{N}\}$ is a regular language.

(ii) $L_2 = \{0^n 1^n \mid n \in \mathbb{N}\}$ is a regular language.

(iii) For $x \in \{0,1\}^*$,

- $\mathcal{R}(x)$ denotes the string obtained by reversing the string $x$.
- $\mathcal{C}(x)$ denotes the string obtained by flipping 0 to 1 and 1 to 0 in the string $x$.

$L_3 = \{x \in \{0,1\}^* \mid x = \mathcal{R}(\mathcal{C}(x))\}$ is a regular language.

[2+3+5 = 10]

CS5. Let $\mathcal{B}$ be a *binary search tree* (BST) on eight nodes filled with the following set of eight integer keys $A = \{10, 2, 5, 3, 20, 15, 9, 22\}$. The order in which these keys were inserted to create $\mathcal{B}$ is not known. However, it is known that for the given BST, $(8 \times 9)/2 = 36$ comparisons are needed to check the presence of all the eight keys in $A$. Using this information as a clue or otherwise, construct and depict pictorially four possible BSTs each of which requires 36 comparisons to check the presence of all the eight keys in $A$. Justify your answer. [10]

CS6. Let $A[0 \ \ldots \ n-1]$ and $B[0 \ \ldots \ n-1]$ be two arrays containing $n$ real numbers such that $A[k] \leq A[k+1]$ and $B[k] \leq B[k+1]$ for all $k \in \{0, 1, \ldots, n-2\}$. Design an efficient algorithm to find whether there exists any $k \in \{0, 1, \ldots, n-1\}$ such that $A[k] + iB[k]$, where $i = \sqrt{-1}$, forms a complex root of the equation $x^2 + 2cx + c^2 + d^2 = 0$ and $c$ and $d$ are real numbers. State and justify the time complexity of your algorithm. [10]

CS7. Consider an operating system consisting of a pair of processes $P_0$ and $P_1$. The structure of each process $P_{\mathtt{i}}$ is shown below.

```
while(TRUE){
    j = 1-i;
    flag[i] = TRUE;
    turn = j;
    while(TRUE){
        if(flag[j] == FALSE || turn == i)
            break;
    }
    /* CRITICAL SECTION */
    flag[i] = FALSE;
    /* REMAINDER SECTION */
}
```

Suppose we initialize the system as follows.

```
turn = 0; // Taken as a shared integer variable
flag[0] = FALSE; // Taken as a shared Boolean variable
flag[1] = FALSE; // Taken as a shared Boolean variable
```

Explain whether the above implementation satisfies mutual exclusion and progress.

[4+6 = 10]

CS8. Let $G := (V, E)$ be a simple, connected, undirected, weighted graph with vertex set $V$, $|V| \geq 4$, and edge set $E$, $|E| \geq 4$. The edges have *distinct* positive weights on them. Prove that a minimum spanning tree of $G$ either contains the edge with the third smallest weight or the edge with the fourth smallest weight. [10]

CS9. Consider an $n \times n$ grid as follows:

$$S = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq i, j \leq n\}.$$

A triangle with its vertices on the grid points of $S$ is *nice* if

- the area of the triangle is 3, and
- at least one of its edges is parallel to either $x$-axis or $y$-axis.

Find the number of nice triangles. [10]

CS10. A sequential circuit has two $D$ flip-flops $D_A$ and $D_B$ with outputs $A$ and $B$, respectively. The circuit has two inputs $x$ and $y$, and one output $z$. The input equations for the flip-flops and the circuit output are as follows:

$$D_A = \overline{x}\, y + A\, \overline{x}$$

$$D_B = x\, \overline{y} + B\, x$$

$$z = Ax + B\overline{x}$$

(i) Draw the logic diagram of the sequential circuit.

(ii) Draw the corresponding state transition diagram.

[4+6 = 10]

CS11. Consider a computer with two-level cache system $L_1$ and $L_2$ along with a main memory $M$. The $L_1$ cache is divided into two parts: $I$-cache (Instruction cache) and $D$-cache (Data cache). The probability that the processor reads a memory for an instruction is $1/4$ and a data is $3/4$. The miss ratio of $D$-cache is two times of the miss ratio of the $L_2$ cache, but the miss ratios of $I$-cache and $L_2$ cache are the same. Let the access times of $I$, $D$, $L_2$ and $M$ be 4 ns, 4 ns, 40 ns and 400 ns, respectively. Let the average memory access time be 11 ns. Compute the hit ratio of the $D$-cache. [10]

CS12. Consider a pair of relations $R_1$ and $R_2$ having $t_1$ and $t_2$ number of tuples, respectively. Let us define the *semijoin* ($R_1 \ltimes R_2$) and *antijoin* ($R_1 \rhd R_2$) operations on $R_1$ and $R_2$ in the form of the following relational algebra expressions:

$$R_1 \ltimes R_2 = \pi_{A(R_1)}(R_1 \bowtie R_2)$$

$$R_1 \rhd R_2 = R_1 - (R_1 \ltimes R_2) = R_1 - \pi_{A(R_1)}(R_1 \bowtie R_2)$$

Recall that $\bowtie$ denotes the natural join operation.

A *dangling tuple* in a relation does not have a matching attribute value in another relation while performing a join operation. If the ratio between the number of tuples returned by the semijoin and antijoin is 1:2, then what is the number of dangling tuples in $R_1$ with respect to $R_2$? [10]

CS13. Suppose node $A$ generates data frames and sends to node $C$ through node $B$. Assume that the link between $A$ and $B$ as well as the link between $B$ and $C$ are full-duplex and error free. All data frames are $10^3$ bits long and ACK frames are separate frames of negligible length. The data rates of the links $A \to B$ and $B \to C$ are $10^5$ bps and $R$ bps, respectively. Propagation delays of both the links $A \to B$ and $B \to A$ are 0.02 seconds, and that of the links $B \to C$ and $C \to B$ are both 0.004 seconds. The *sliding window* protocol with window size $W$ is used between $A$ and $B$. The *stop and wait* protocol is used between $B$ and $C$.

(i) Determine the window size $W$ such that the utilization of the link $A \to B$ is maximum.

(ii) Determine the minimum value of $R$ such that the utilization of the link $A \to B$ is maximum and the buffer at $B$ is not *flooded*. (**Note:** In order not to flood the buffer at $B$, the time taken by $B$ to send $W$ frames to $C$ must be at most the time taken by $A$ to send $W$ frames to $B$.)

[5+5 = 10]

**Non-CS Group (Mathematics)**

**Note: Answer all questions from Part-A and any eight questions from Part-B.**

**Part-A**

NC1. Suppose a crime has been committed and there are three suspects: Professor Plum, Mrs. Peacock, and Mr. Green. Given that:

- At least one of the above three suspects is guilty.
- Not all of them are guilty.
- If Mrs. Peacock is guilty, then so is Professor Plum.
- If Mr. Green is innocent, then so is Professor Plum.

Prove or disprove each of the following statements:

(i) Mr. Green is guilty.

(ii) Mrs. Peacock is innocent.

[3+3 = 6]

NC2. A room has four walls. In how many different ways can you color the four walls, using colors from the set {Red, Green, Blue, Yellow}, such that no two adjacent walls have the same color?

[7]

NC3. Consider a round-robin football tournament among $n \geq 3$ teams, where every team plays a match with every other team exactly once. For any team, each match results in a *win, draw or loss.* In the tournament, each team loses at least one match. Prove or disprove the following statement:

*There must be two teams with the same number of wins at the end of the tournament.*

[7]

**Part-B**

NC4. Consider an $n \times n$ grid as follows:

$$S = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \ \mid 1 \leq i, j \leq n\}.$$

A triangle with its vertices on the grid points of $S$ is *nice* if

- the area of the triangle is 3, and
- at least one of its edges is parallel to either $x$-axis or $y$-axis.

Find the number of nice triangles. [10]

NC5. (i) Let $f : [0, 1] \to [0, 1]$ be a strictly decreasing continuous function with $f(0) = 1$ and $f(1) = 0$. Show that

$$\int_0^1 f(x)dx = \int_0^1 f^{-1}(y)dy.$$

(ii) Let $f$ be a differentiable function on $[-2, 2]$ such that $f(-2) = 1$, $f(2) = 5$ and $|f'(x)| \leq 1$ for all $x \in [-2, 2]$. Find the value of $f(0)$.

[6+4 = 10]

NC6. (i) Consider a simple, undirected, connected graph $G = (V, E)$ whose every vertex has degree at least 2. Show that $G$ contains a cycle.

(ii) A simple, undirected, connected graph is 2-*edge connected* if at least 2 edges need to be removed to make $G$ disconnected.

(a) Show an example of a tree with 11 vertices that can be converted to a 2-*edge connected* graph by adding an edge.

(b) Show an example of a tree with 11 vertices that can be converted to a 2-*edge connected* graph by adding five edges.

[5+(2+3) = 10]

NC7. Let $a_0, a_1, \ldots, a_n$ be real numbers with property that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

Prove that the equation

$$a_0 + a_1 x + \cdots + a_n x^n = 0$$

has a solution in $(0, 1)$. [10]

NC8. Find the largest positive real number $\delta$ such that, for real numbers $x$ and $y$

$$|x - y| < \delta \ \text{ implies } \ |\cos x - \cos y| < \sqrt{2}.$$

[10]

NC9. Let $f(x)$ be a nonconstant polynomial with real coefficients. If there exists a complex root $\alpha$ of $f(x)$ with multiplicity greater than 1, show that the polynomial $f(f(x))$ also has a complex root with multiplicity greater than 1. [10]

NC10. A particular Covid-19 vaccine produces a side effect (fever) with probability $p \in [0, 1]$. A clinic vaccinates 500 people each day and records the number of people having side effects on 5 randomly chosen days.

(i) What is the probability that the number of people who developed side effects on the 5 observed days are $10, 15, 0, 20, 25$?

(ii) Find the value of $p$ that maximizes the probability obtained in part (i).

(iii) Based on the value of $p$ obtained in part (ii), what is the expected number of people who will develop side effects on each day?

[4+4+2 = 10]

NC11. Let $A, A_1, A_2, \ldots, A_m$ be $n \times n$ matrices with real entries such that

$$A = \sum_{t=1}^{m} A_t.$$

(i) Show that

$$\operatorname{rank}(A) \leq \sum_{t=1}^{m} \operatorname{rank}(A_t).$$

(ii) An $n \times n$ matix $B$ with real entries is defined as a *two-block matrix* if there exist two disjoint submatrices $\mathcal{C}$ and $\mathcal{D}$ in $B$ such that

- All entries of $\mathcal{C}$ are the same, and all entries of $\mathcal{D}$ are the same.
- All other entries of $B$ not in $\mathcal{C}$ and $\mathcal{D}$ are 0.

If the matrices $A_t$ are *two-block matrices* for all $t \in \{1, \ldots, m\}$ then show that $m \geq \operatorname{rank}(A)/2$.

[3+7 = 10]

NC12. Let $\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$ denote the following set of matrices

$$\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \, a_{ij} \in \mathbb{Z}/m\mathbb{Z} \text{ and } a_{11}a_{22} - a_{12}a_{21} \in (\mathbb{Z}/m\mathbb{Z})^* \right\}$$

with matrix multiplication as group operation. Note that $(\mathbb{Z}/m\mathbb{Z})^*$ denotes the multiplicative group of units in $\mathbb{Z}/m\mathbb{Z}$.

(i) Show that the map $\psi : \mathbb{Z}/m\mathbb{Z} \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$ given by $\psi(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is an injective group homomorphism.

(ii) Show that there exists a $2 \times 2$ matrix $M$ such that $M$ is of order 2 in $\mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z})$ and $M$ is of order 5 in $\mathrm{GL}_2(\mathbb{Z}/15\mathbb{Z})$.

[3+7 = 10]

NC13. Find the perimeter of the region bounded by $x^2 + y^2 - 144 \leq 0$ and $x^2 + y^2 - (24\sqrt{2})x + 144 \leq 0$. [10]