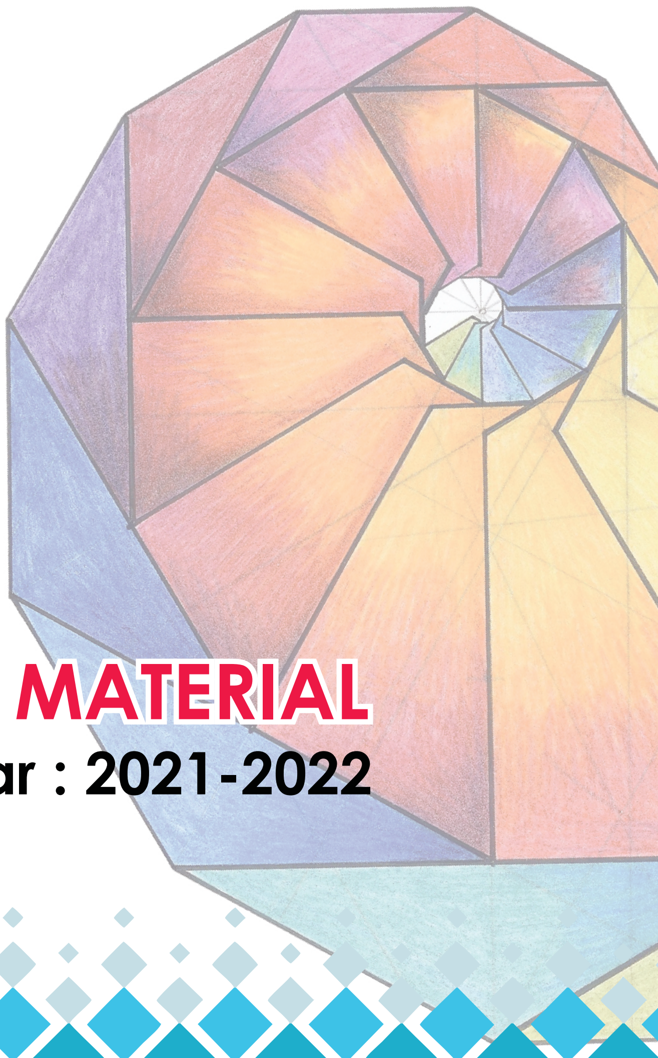
Telangana State Board of
INTERMEDIATE Education
FIRST YEAR

MATHEMATICS
I B



BASIC LEARNING MATERIAL
For The Academic Year : 2021-2022

TELANGANA STATE BOARD OF INTERMEDIATE EDUCATION

MATHEMATICS - IB

FIRST YEAR

(English Medium)

BASIC LEARNING MATERIAL

ACADEMIC YEAR 2021-2022

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PREFACE

The ongoing Global Pandemic Covid-19 that has engulfed the entire world has changed every sphere of our life. Education, of course is not an exception. In the absence of Physical Classroom Teaching, Department of Intermediate Education Telangana has successfully engaged the students and imparted education through TV lessons. In the back drop of the unprecedented situation due to the pandemic TSBIE has reduced the burden of curriculum load by considering only 70% syllabus for class room instruction as well as for the forthcoming Intermediate Examinations. It has also increased the choice of questions in the examination pattern for the convenience of the students.

To cope up with exam fear and stress and to prepare the students for annual exams in such a short span of time , TSBIE has prepared "Basic Learning Material" that serves as a primer for the students to face the examinations confidently. It must be noted here that, the Learning Material is not comprehensive and can never substitute the Textbook. At most it gives guidance as to how the students should include the essential steps in their answers and build upon them. I wish you to utilize the Basic Learning Material after you have thoroughly gone through the Text Book so that it may enable you to reinforce the concepts that you have learnt from the Textbook and Teachers. I appreciate ERTW Team, Subject Experts, who have involved day in and out to come out with the Basic Learning Material in such a short span of time.

I would appreciate the feedback from all the stake holders for enriching the learning material and making it cent percent error free in all aspects.

The material can also be accessed through our website**www.tsbie.cgg.gov.in.**

Commissioner & Secretary
Intermediate Education, Telangana.

CONTENTS

Unit **1**

Locus

Def : The path of a point moving subject to the given condition is called locus.

The equation representing the locus is called equation of locus.

Steps to find locus

Step 1 : Let the point P be P(h, k)

Step 2 : Use the formula related to given condition

Step 3 : Simplify

Step 4 : The simplified algebraic equation represents locus

Short Answer Questions (4 Marks)

1. Find the equation of locus of a point which is ata distance 5 from A(4, –3).

Sol: Let P(h, k) be any point

Given point A(4, –3)

|PA| = 5 given condition

$\Rightarrow |PA|^2 = 25$

$(h-4)^2 + (k+3)^2 = 25$ (distance between two points $= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$)

equation of locus $(h-4)^2+(k+3)^2 = 25$

$h^2+k^2-8h+6k+25=25 \Rightarrow x^2+y^2-8x+6y=0$

2. Find the equation of locus of apoint which is equidistant from the points A(–3, 2), and B(0, 4)

Sol: Let P(h, k) be any point

|PA| = |PB| given condition

$\Rightarrow |PA|^2 = |PB|^2$

$(h+3)^2 + (k-2)^2 = h^2 + (k-4)^2$

$h^2+6h+9+k^2-4k+4 = h^2+k^2-8k+16$

$6h - 4k + 13 = -8x + 16$

$6h + 4k = 3$

$\therefore$ equation of locus $6x + 4y - 3 = 0$

3. Find the equation of locus of a point P such that the distance of P from the origin is twice the distance of P and A(1,2).

Sol: Let P(h, k) be any point

O(0, 0) be the origin

A(1, 2) given point

|PO| = 2|PA| given condition

$\Rightarrow |PO|^2 = 4\ |PA|^2$

$h^2 + k^2 = 4[(h-1)^2 + (k-2)^2]$

simplifying $3h^2 + 3k^2 - 8h - 16k + 20 = 0$

equation of locus $= 3x^2 + 3y^2 - 8x - 16y + 20 = 0$

4. Find the equation of locus of a point which is equidistant from the coordinate axes.

Sol: Let P(h, k) be any point

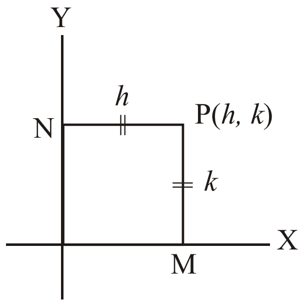
distance from X-axis = |PM|

distance from Y-axis = |PN|

$|PM| = |PN| \Rightarrow |PM|^2 = |PN|^2$

$k^2 = h^2$.

equation of locus $y^2 = x^2 \Rightarrow x^2 = y^2$



5. Find the equation of locus of a point equidistant from A(2,0) and the Y-axis.

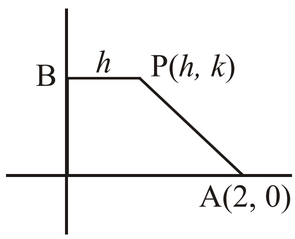
Sol: Let P(h, k) be any point

given point = A(2, 0)

given condition $|PA| = h$

$|PA|^2 = h^2 \Rightarrow (h-2)^2 + k^2 = h^2$

equation of locus $y^2 - 4x + 4 = 0$.



6. Find the equation of locus of a point P such that $|PA|^2 + |PB|^2 = 2c^2$ where A(a, 0), B($-a$, 0) and $0 < |a| < |c|$

Sol: Let P(h, k) be any point

given point = A(a, 0), B($-a$, 0)

given condition $|PA|^2 + |PB|^2 = 2c^2$

$(h-a)^2 + k^2 + (h+a)^2 + k^2 = 2c^2$

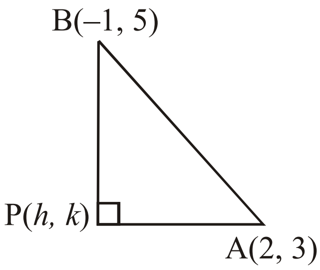
simplifying $h^2 + k^2 = c^2 - a^2$

$\therefore$ Equation of locus $x^2 + y^2 = c^2 - a^2$.

7. Find the equation of locus of P, if the line segment joining (2, 3) and (–1, 5) subtends a right angle at P.

Sol: Let P(h, k) be any point

given point = A(2, 3), B(–1, 5)



slope of PA $m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{k-3}{h-2}$

slope of PB $m_2 = \frac{k-5}{h+1}$

PA $\perp$ PB

$m_1 \times m_2 = -1$

$\Rightarrow \frac{k-3}{h-2} \times \frac{k-5}{h+1} = -1$

simplifying $h^2 + k^2 - h - 8k + 13 =$

$\therefore$ equation of locus $x^2 + y^2 - x - 8y + 13 = 0$.

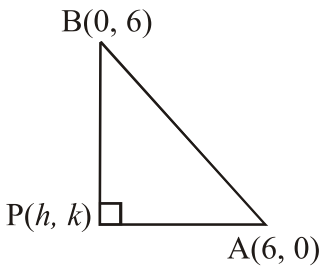
8. The ends of the hypotenuse of a right angle triangle are (0,6) and (6,0). Find the equation of the locus of its third vertex.

Sol: Let P(h, k) be any point

given point = (6, 0), (0, 6)

slope of PA $m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{k-0}{h-0}$

slope of PB $m_2 = \frac{k-6}{h-0}$



given condition PA $\perp$ PB

$m_1 \times m_2 = 1$

$\Rightarrow \frac{k}{h-6} \times \frac{k-6}{h} = -1$

$\Rightarrow k^2 - 6k = [h^2 - 6h]$

simplifying $h^2 + k^2 - 6h - 6k = 0$.

$\therefore$ equation of locus $x^2 + y^2 - 6x - 6y = 0$.

9. Find the equation of the locus of a point, the difference of whose distance from (−5, 0) and(3, 0) is 8.

Sol: Let P(h, k) be any point

given point = A(–5, 0), B(3, 0)

given condition |PA – PB| = 8

|PA| = 8 + |PB|

squaring both sides

$|PA|^2 = [8 + |PB|^2$

simplifying $|PA|^2 - |PB|^2 - 64 = 16$ PB

$[(h+5)^2 + k^2] - [(h-3)^2 + k^2] - 64 = 16\,|\,PB\,|$

$\Rightarrow 20h - 64 = 16\,|PB|$

$\Rightarrow 5h - 16 = 4\,|PB|$

$\Rightarrow (5h - 16)^2 = 16\,|PB|^2$

$\Rightarrow (5h - 16)^2 = 16\,[(h - 3)^2 + k^2]$

simplifying $\frac{h^2}{16} - \frac{k^2}{9} = 1$

$\therefore$ equation of locus $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

10. Find the equation of the locus of P, if A(4, 0), B(–4, 0) and |PA – PB| = 4

Sol: Let P(h, k) be any point

given points = A(4, 0), B(–4, 0)

given condition |PA – PB| = 4

$\Rightarrow |PA| = 4 + |PB|$

$[|PA|^2 - |PB|^2] - 16 = 8\,|PB|$

$[(h-4)^2 + k^2] - [(h+4)^2 + k^2] - 16 = 8|PB|$

$-16h - 16 = 8\,|PB|$

$4(h + 1)^2 = (h + 4)^2 + k^2$

simplifying $3h^2 - k^2 = 12$

$\therefore$ equation of locus $\frac{x^2}{4} - \frac{y^2}{12} = 1$.

11. Find the equation of the locus of a point, the sum of whose distances from (0,2) and (0,–2) is 6.

Sol: Let P(h, k) be any point

given points = A(2, 3), B(2, –3)

given condition |PA| + |PB| = 8

$\Rightarrow |PA| = 8 - |PB|$

$[|PA|^2 - |PB|^2] - 64 = -16\,|PB|$

$[(h-2)^2 + (k-3)^2] - [(h-2)^2 + (k+3)^2] - 64 = -16|PB|$

$-12k - 64 = -16\,|PB|$

$(3k+16)^2 = 16[(h-2)^2 + (k+3)^2]$

simplifying $16h^2 + 7k^2 - 64h - 48 = 0$

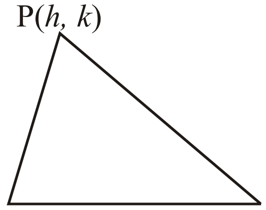
$\therefore$ equation of locus $16x^2 + 7y^2 - 64x - 48 = 0$.

12. A(5, 3), B(3, –2) are two fixed points. Find the equation of the locus of P, so that the area of triangle PAB is 9.

Sol: Let 3rd vertex be P(h, k)

given points = A(5, 3), B(3, –2)

given condition |PA| + |PB| = 8



$$\text{area of } \Delta \text{ PAB} = \frac{1}{2}\begin{vmatrix} h & k \\ 5 & 3 \\ 3 & -2 \\ h & k \end{vmatrix} = 9$$

$|3h - 5k - 10 - 9 + 3k + 2h| = 18$

$|5h - 2k - 19| = 18$

$(5h - 2k - 1)(5h - 2k - 37) = 0$

$\therefore$ equation of locus $(5h - 2k - 1)(5h - 2k - 37) = 0$.

$\therefore (5x - 2y - 1)(5x - 2y - 37) = 0$

Practice Problem

A(2,3) and B(–3,4) are two gien points. Find the equation of locus of P so that the area of the triangle PAB is 8.5

13. If the distance from P to the points (2, 3), (2, –3) are in the ratio 2 : 3, then find the equation of the locus of P.

Sol: Let (2, 3), (2, –3) (h, k) are the vertices of the triangle

Let P(h, k) be any point

Given points = A(2, 3), B(2, –3)

given condition $\dfrac{|\text{PA}|}{|\text{PB}|} = \dfrac{2}{3}$

$\Rightarrow 3|\text{PA}| = 2|\text{PB}|$

$9|\text{PA}|^2 = 4\,|\text{PB}|^2$

$9[(h-2)^2 + (k-3)^2] = 4[(h-2)^2 + (k+3)^2]$

simplifying $5h^2 + 5k^2 - 20h - 78k + 65 = 0$

$\therefore$ equation of locus $5x^2 + 5y^2 - 20x - 78y + 65 = 0$.

14. A(1, 2), B(2, –3) and C(–2, 3) are three points. A point P moves such that $|PA|^2 + |PB|^2 = 2|PC|^2$. Show that the equation to the locus of P is $7x - 7y + 4 = 0$.

Sol: Let $P(h, k)$ be any point

$|PA|^2 + |PB|^2 = (h-1)^2 + (k-2)^2 + (h-2)^2 + (k+3)^2$

$2\,|PC|^2 = 2[(h+2)^2 + (k-3)^2]$

substituting in $|PA|^2 + |PB|^2 = 2\,|PC|^2$

simplifying $7h - 7k + 4 = 0$

$\therefore$ equation of locus $7x - 7y + 4 = 0$.

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Unit **2**

Transformation of Axis

Def : Without changing the direction of co-ordinate axes of the origin is shifted to a given point then the change occured is called translation of axes.

Let P(x, y) original coordinates

and P(X, Y) transformed coordinates

$x = PQ = ON = OL + LN = OL + O'M = h + X = X + h$

$y = PN = PM + MN = Y + O'L = Y + k$

$\therefore\ x = X + h,\ \ y = Y + k \Rightarrow X = x - h,\ \ Y = y - k$

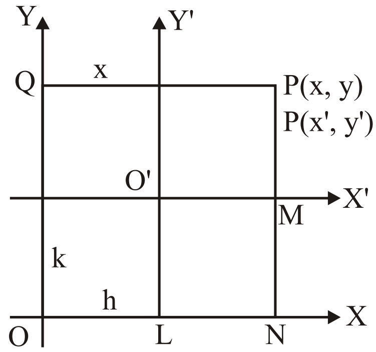
$P(x, y) = (X + h, Y + k)$

$P(X, Y) = (x - h, y - k)$

$(h, k) = (x - X, y - Y)$

original equation of the curve $f(x, y)$

transfored equation of the curve $f(X, Y)$



translation of axis

Rotation of axes : without the changing the position of origin the axes are rotated through an angle then it is called rotation of axes.

$x = OL = OQ - LQ = X\cos\theta - NM$

$= X\cos\theta - Y\sin\theta.$

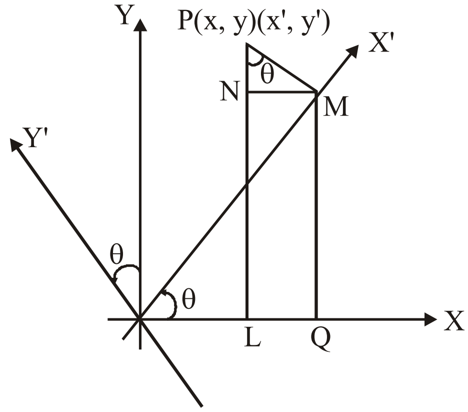
$y = PL = PN + NL = PN + MQ$

$= PM\cos\theta + OM\sin\theta$

$= Y\cos\theta + X\sin\theta = X\sin\theta + Y\cos\theta.$

$P(x, y) = (X\cos\theta - Y\sin\theta, X\sin\theta + Y\cos\theta)$

θ	X	Y
x	cosθ	–sinθ
y	sinθ	cosθ



Rotation of axis

Short Answer Questions (4 Marks)

1. When the origin is shifted to (–1, 2) by the translation of axes, find the transformed equations of the following.

Sol : (1) $x^2 + y^2 + 2x - 4y + 1 = 0$

$(h, k) = (-1, 2)$

$x = X + h = X - 1$ $y = Y + k = Y + 2$

substituting $(X-1)^2 + (Y+2)^2 + 2(X-1) - 4(Y+2) + 1 = 0$

simplifying $X^2 + Y^2 - 4 = 0$ required equation

(2) $2x^2 + y^2 - 4x + 4y = 0$

$x = X + h = X - 1$ $y = Y + k = Y + 2$

substituting $2(X-1)^2 + (Y+2)^2 - 4(X-1) + 4(Y+2) = 0$

simplifying $2[X^2 - 2X + 1] + [Y^2 + 4Y + 4] - 4X + 4 + 4Y + 8 = 0$

$2X^2 + Y^2 - 8X + 8Y + 18 = 0$ required equation

2. **The point to which the origin is shifted and the transformed equation are given below. Find the original equation.**

Sol : (1) (3, –4), $x^2 + y^2 = 4$

Let the transformed equation indicated by $X^2 + Y^2 = 4$ ...... (1)

simplifying (h, k) = (3, –4)

$X = x - h$ $Y = y - k$

$X = x - 3$ $Y = y + 4$

original equation after substitution and simplification

$x^2 + y^2 - 6x + 8y + 21 = 0$

(2) (–1, 2), $x^2 + y^2 + 16 = 0$

changed origin is (h, k) = (–1, 2)

given transformed equation $x^2 + 2y^2 + 16 = 0$

Let this denoted by $X^2 + 2Y^2 + 16 = 0$ ....(1)

$X = x - h = x + 1$ $Y = y - k = y - 2$

original equation after substitution and simplification

$x^2 + 2y^2 + 2x - 8y + 25 = 0$

3. **Fiond the point to which the origin is to be shifted so as to remove the first degree terms from the equation.**

Sol : given equation $f(x, y) = 4x^2 + 9y^2 - 8x + 36y + 4 = 0$

transformed equation

$f(X+h, Y+k) = f(X, Y) = 4(X+h)^2 + 9(Y+k)^2 - 8(X+h) + 36(Y+k) + 4 = 0$

To eliminate X, Y terms equate coefficient of X and coefficient of Y to 0

$8h - 8 = 0 \Rightarrow h = 1$

$18k + 36 = 0 \Rightarrow k = -2$

the point to which the origin to be shifted is $(h, k) = (1, -2)$ or

use the formula $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

4. When the origin is shifted to (2, 3) then the transformed equation of a curve is $\mathbf{x^2 + 3xy - 2y^2 + 17x - 7y - 11 = 0}$, then find the original equation of the curve.

Sol : transformed equation $x^2 + 3xy - 2y^2 + 17x - 7y - 11 = 0$

for convenient let the equation $X^2 + 3XY - 2Y^2 + 17X - 7Y - 11 = 0$ ...... (1)

$(h, k) = (2, 3)$

$X = x - h = x - 2$ $\qquad$ $Y = y - k = y - 3$

substituting in (1)

$(x-2)^2 + 3(x-2)(y-3) - 2(y-3)^2 + 17(x-2) - 7(y-3) - 11 = 0$

original equation $x^2 + 3xy - 2y^2 + 4x - y - 20 = 0$

1. When the axes are rotated through an angle $\frac{\pi}{6}$, find the transformed equation of $\mathbf{x^2 + 2\sqrt{3}xy - y^2 = 2a^2}$

Sol : $\theta = \pi/6$

$f(x, y) = x^2 + 2\sqrt{3}xy - y^2 = 2a^2$

$x = X\cos\theta - Y\sin\theta = X\cos\frac{\pi}{6} - Y\sin\frac{\pi}{6} = \frac{X\sqrt{3}}{2} - \frac{Y}{2}$

$y = X\sin\theta + Y\cos\theta = X\sin\frac{\pi}{6} + Y\cos\frac{\pi}{6} = \frac{X}{2} + \frac{Y\sqrt{3}}{2}$

transformed equation $= f(X, Y)$

$$= \left(\frac{\sqrt{3}X - Y}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}X - Y}{2}\right)\left(\frac{X + Y\sqrt{3}}{2}\right) - \left(\frac{X + Y\sqrt{3}}{2}\right)^2 = 2a^2$$

simplifying $X^2 - Y^2 = a^2$.

2. When the axes are rotated through an angle $\pi/4$, find the transformed equation of $3x^2 + 10xy + 3y^2 = 9$

Sol : $\theta = \pi/4$

$f(x, y) = 3x^2 + 10xy + 3y^2 = 9$

$x = X\cos\theta - Y\sin\theta = \dfrac{X - Y}{\sqrt{2}}$

$y = X\sin\theta + Y\cos\theta = \dfrac{X + Y}{\sqrt{2}}$

by substituting and simplifying the transformed equation $8X^2 - 2Y^2 = 9$

3. When the axes are rotated through an angle α, find the transformed equation of $x\cos\alpha + y\sin\alpha = p$

Sol : $x = X\cos\alpha - Y\sin\alpha$

$y = X\sin\alpha + Y\cos\alpha$

by substituting

$(X\cos\alpha - Y\sin\alpha)\cos\alpha + (X\sin\alpha + Y\cos\alpha)\sin\alpha = p$

$X(\cos^2\alpha + \sin^2\alpha) = p$

$X = p$.

4. When the axes are rotated through an angle 45^0 , the transformed equation of a curve is $17x^2 - 16xy + 17y^2 = 225$. Find the original equation of the curve.

Sol : angle of rotation $= \theta = 45^o$

let the transformed equation be

$17X^2 - 16XY + 17Y^2 = 225$

$X = x\cos\theta + y\sin\theta$, $\quad Y = -x\sin\theta + y\cos\theta$

$= \dfrac{x + y}{\sqrt{2}} \qquad\qquad = \dfrac{-x + y}{\sqrt{2}}$

simplifying

$$17\left(\frac{x+y}{\sqrt{2}}\right)^2 - 16\left(\frac{x+y}{\sqrt{2}}\right)\left(\frac{-x+y}{\sqrt{2}}\right) + 17\left(\frac{-x+y}{\sqrt{2}}\right)^2 = 225$$

required equation $25x^2 + 9y^2 = 225$

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Unit **3**

The Straight Line

Chapter 3(a)

Note :

1 If a non - vertical straight line makes an angle 'θ' with the X- axis measured counter - clock wise from the positive direction of X - axis, then tanθ is called the slope of the line L.

$m = \tan\theta$

2. Slope of X- axis & its parallel line is zero

3. Slope of Y- axis & ils parallal line is not define

4. A line which is passing through $A(x_1, y_1)$, $B(x_2, y_2)$ then its slope $(m) = \frac{y_2 - y_1}{x_2 - x_1}$.

5. The equation of the S.L. which cut off non - zero intercepts '*a*' and '*b*' on the X-axis and the Y-axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$

Intercept Form

6. The equation of the S.L. with slope 'm' and cutting off y - intercept 'C' is $y = mx + c$ (slope - intercept form)

7. If it passes through origin then the equation is $y = mx$

8. **Point - slope form :**

The equation of the straight line with slope 'm' and passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

9. The equation of the S.L. passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$ (Two point form).

10. $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ are collinear $\Leftrightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ (OR)

Slope of AC = Slope of AB (OR) Aera of $\Delta ABC = 0$.

Questions

1. Find the equation of the S.L. passing through the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$

Sol: $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$

$$\boxed{y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)}$$

$$y - 2at_1 = \left(\frac{2at_2 - 2at_1}{at_2^2 - at_1^2}\right)(x - at_1^2)$$

$$y - 2at_1 = \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)}(x - at_1^2)$$

$$y - 2at_1 = \frac{2(t_2 - t_1)}{(t_2 - t_1)(t_2 + t_1)}(x - at_1^2)$$

$$(t_2 + t_1)(y - 2at_1) = 2(x - at_1^2)$$

$$(t_1 + t_2)y - (t_1 + t_2)2at_1 = 2x - 2at_1^2$$

$$2x - 2at_1^2 - (t_1 + t_2)y + (t_1 + t_2)2at_1 = 0$$

$$2x - 2at_1^2 - (t_1 + t_2)y + 2at_1^2 + 2at_1t_2 = 0$$

$$2x - (t_1 + t_2)y + 2at_1t_2 = 0$$

2. Find the value of x, if the slope of the line passing through (2, 5) and (x, 3) is 2.

Sol: A(2, 5), B(x, 3), m = 2

$$\boxed{\text{Slope of AB} = \frac{y_2 - y_1}{x_2 - x_1}}$$

$$= \frac{3 - 5}{x - 2} = 2$$

$-2 = 2(x - 2)$

$2x - 4 = -2$

$2x = -2 + 4 \Rightarrow 2x = 2 \Rightarrow x = 1$.

3. Find the value of *y* if the line joining points (3, y) and (2, 7) is parallel to the line the points (–1, 4) and (0, 6)

Sol: A(3, y), B(2, 7), C(–1, 4), D(0, 6)

Slope of AB = Slope of CD

$$\frac{7-y}{2-3}=\frac{6-4}{0-(-1)}\Rightarrow\frac{7-y}{-1}=\frac{2}{1}$$

$7 - y = -2 \Rightarrow 7 + 2 = y \Rightarrow y = 9.$

4. Find the condition for the points (a, 0) and (0, b), where ab ≠ 0, to be collinear

Sol: A(a, 0), B(h, k), C(0, b)

points A, B & C are collinear ⇔ Slope of AC = Slope of AB

$$\frac{b-0}{0-a}=\frac{k-0}{h-a}\Rightarrow\frac{-b}{a}=\frac{k}{h-a}$$

$$\Rightarrow -b(h-a)=ak$$

$$\Rightarrow -bh+ab=ak\Rightarrow ak+bh=ab$$

$$\frac{ak}{ab}+\frac{bh}{ab}=\frac{ab}{ab}\Rightarrow\frac{h}{a}+\frac{k}{b}=1$$

5. Find the equation of the S.L. which makes the 150° with the position X- axis in the positive direction and which passes through the point (−2, −1)

Sol: θ = 150°, A(–2, –1)

$m = \tan 150^\circ = \tan(90 + 60) = -\cot 60^\circ = -\frac{1}{\sqrt{3}}$

The equation of the S.L. with slope $-\frac{1}{\sqrt{3}}$ and passing through the point (–2, –1) is

$y - y_1 = m(x - x_1)$

$$y+1=\frac{-1}{\sqrt{3}}(x+2)$$

$$\sqrt{3}(y+1)=-(x+2)$$

$$\sqrt{3}y+\sqrt{3}=-(x+2)$$

$$\sqrt{3}(y+1)=-x-2$$

$$x+\sqrt{3}y+2+\sqrt{3}=0$$

Exercise:

Find the slopes of the lines (i) parallel to and (ii) perpendicular to the line passing through (6, 3) and (–4, 5)

6. The angle made by a S.L. with the positive X-axis in the positive direction is 60º and the Y- intercept cul off by it is 3. Find the equation of the S.L

Sol: $\theta = 60^\circ$, C = 3

$\therefore$ Required equation y = mx + c. (m = $\tan 60^\circ = \sqrt{3}$)

$y = \sqrt{3}x + 3$

$\Rightarrow \sqrt{3}x - y + 3 = 0$

7. Find the equation of the S.L. passing through the origin and making equal angles with the coordinate axes.

Sol: O(0, 0)

Since making equal angles with the coordinate axes

$\theta = 45^\circ, 135^\circ$.

$m = \tan 45^\circ$ or $\tan 135^\circ$

$m = 1$ or $\tan 135^\circ = \tan(180^\circ - 45^\circ) = -\tan 45^\circ = -1$

$m = \pm 1$

$\therefore$ Equation of S.L. with slope ±1 and passing through origin is

$(y - 0) = \pm 1(x - 0)$

$y = \pm x$

$x + y = 0$ or $x - y = 0$

8. Find the equation of the S.L. passing through the point (2, 3) and making non - zero intercepts on the axes of coordinatas whose sum is zero

Sol: A(2, 3)

sum of the intercepts = zero

$a + b = 0 \Rightarrow b = -a$

Equation of a S.L. in intercept form

$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{a} + \frac{y}{-a} = 1$

$\Rightarrow \frac{x - y}{a} = 1 \Rightarrow x - y = a$ .....(1)

But it is passing through A(2, 3)

$2 - 3 = a \Rightarrow a = -1$

$\therefore$ From (1)

$x - y = -1$

$x - y + 1 = 0$

9. Find the equation of the S.L. passing through $(-4,\ 5)$ and cutting off equal and non zero intercepts on the coordinate axes.

Sol: A = (–4, 5)

According to the problem a = b

equation of a S.L. in intercept form

$$\boxed{\frac{x}{a}+\frac{y}{b}=1}$$

$$\Rightarrow \frac{x}{a}+\frac{y}{a}=1$$

$$\Rightarrow \frac{x+y}{a}=1 \Rightarrow x+y=a \qquad \ldots(1)$$

But it is passing through A(–4, 5)

$-4+5=a \Rightarrow a=1$

$\therefore$ Equation of the required S.L. is

$x+y=1$

$x+y-1=0$

10. Find the equation of the S.L. passing through A(–1, 3) and (i) parallel (ii) perpendincular to the S.L. passing through B(2, –5) and C(4, 6)

Sol: A(–1, 3), B(2, –5), C(4, 6)

Slope of BC = $\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{6-(-5)}{4-2}=\dfrac{6+5}{2}=\dfrac{11}{2}$

(i) Slope of a parallel line to BC also $=\dfrac{11}{2}$

equaton of S.L. with slope $\dfrac{11}{2}$ and passing through $A(-1, 3)$ is

$$y-3=\frac{11}{2}(x-(-1)$$

$$2(y-3)=11(x+1)$$

$$2y-6=11x+11$$

$$11x-2y+17=0$$

(ii) Slope of a line which is perpendicular to BC is $=\dfrac{-1}{m}$

$$=-\frac{1}{\left(\frac{11}{2}\right)}=\frac{-2}{11}$$

equation of a S.L. with slope $\frac{-2}{11}$ and passing through A(–1, 3) is

$$y-3=\frac{-2}{11}(x-(-1)$$

$$11(y-3)=-2(x+1)$$

$$11y-33=-2x-2$$

$$2x+11y-31=0$$

The Straight Line

Exercise 3b

Note :

1 The equation of the straight line, whose distance from the origin is P and the normal ray of which drawn from the origin makes an angle α with the positive divection of the X-axis measured counter clock - wise is

$$x\cos\alpha + y\sin\alpha = P$$

It is called Normal Form

2. The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of the X-axis measured counter - clock wise is

$$(x-x_1):\cos\theta=(y-y_1):\sin\theta$$

3. $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}$ = r say

$x=x_1+r\cos\theta,\ y=y_1+r\sin\theta$ is called parameteric from of the point P.

$|r|$ denotes the distance of the point (x_1, y_1) from the point (x, y) on the straight line.

4. General form of a straight line is $ax + by + c = 0$

Its slope $(m)=\frac{-a}{b}$

Questions

1. Transform the equation $x + y + 1 = 0$ into normal form

Sol: $x + y + 1 = 0$

Normal form of $ax + by + c = 0$ is

$$\frac{-ax}{\sqrt{a^2+b^2}}+\frac{-(b)y}{\sqrt{a^2+b^2}}=\frac{c}{\sqrt{a^2+b^2}}, \qquad \text{where } c>0$$

Divide with $\sqrt{a^2+b^2}=\sqrt{(1)^2+(1)^2}=\sqrt{1+1}=\sqrt{2}$

$$\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}+\frac{1}{\sqrt{2}}=0$$

$$\left(\frac{-1}{\sqrt{2}}\right)x+\left(\frac{-1}{\sqrt{2}}\right)y=\frac{1}{\sqrt{2}} \qquad \ldots\ldots(1)$$

Compare with x cosα + ysinα = P

$$\cos\alpha=\frac{-1}{\sqrt{2}},\ \sin\alpha=\frac{-1}{\sqrt{2}},\ \mathrm{P}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \alpha=\frac{5\pi}{4}$$

∴ Equation of the required straight line

$$x\cos\left(\frac{5\pi}{4}\right)+y\sin\left(\frac{5\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

2. Transform the equation 4*x* – 3*y* + 12 = 0 into (a) slope - intercept form (b) intercept form and (c) normal form

Sol. : L = 4*x* – 3*y* + 12 = 0

(a) Slope - intercept form ($y = mx + c = 0$)

$$\Rightarrow 4x+12=3y$$

$$y=\frac{4x+12}{3}$$

$$y=\left(\frac{4}{3}\right)x+4$$

$$\Rightarrow m=\frac{4}{3},\ c=4$$

(b) Intercept form $\frac{x}{a}+\frac{y}{b}=1$

$$\Rightarrow 4x-3y+12=0$$

$$-4x+3y=12$$

$$\frac{-4x}{12}+\frac{3y}{12}=\frac{12}{12}$$

$$\frac{x}{(-3)}+\frac{y}{(4)}=1$$

$$\Rightarrow a=-3,\ b=4$$

(c) Normal form $(x\cos\alpha+y\sin\alpha=\mathrm{P})$

$$4x-3y+12=0$$

$-4x+3y=12$

Divide with $\sqrt{a^2+b^2}=\sqrt{(-4)^2+(3)^2}=\sqrt{16+9}=5$

$$\frac{-4x}{5}+\frac{3y}{5}=\frac{12}{5}$$

$$\cos\alpha=\frac{-4}{5},\ \sin\alpha=\frac{3}{5},\ \mathrm{P}=\frac{12}{5}$$

3. Transform the equation $\sqrt{3}x+y=4$ into (a) slope intercept form (b) intercept form and (c) normal form.

Sol. : L = $\sqrt{3}x+y=4$

(a) Slope - intercept form $y=mx+c$

$\sqrt{3}x+y=4$

$\Rightarrow y=-\sqrt{3}x+4$

$\Rightarrow m=-\sqrt{3},\ c=4$

(b) Intercept form $\frac{x}{a}+\frac{y}{b}=1$

$\sqrt{3}x+y=4$

$$\Rightarrow \frac{\sqrt{3}}{4}x+\frac{y}{4}=\frac{4}{4}$$

$$\frac{x}{\left(4/\sqrt{3}\right)}+\frac{y}{(4)}=1$$

$$\Rightarrow a=\frac{4}{\sqrt{3}},\ b=4$$

(c) Normal form $(x\cos\alpha+y\sin\alpha=\mathrm{P})$

$\sqrt{3}x+y=4$

Divide with $\sqrt{a^2+b^2}=\sqrt{(\sqrt{3})^2+(1)^2}=\sqrt{3+1}=2$

$$\frac{\sqrt{3}x}{2}+\frac{y}{2}=\frac{4}{2}$$

$$x\left(\frac{\sqrt{3}}{2}\right)+y\left(\frac{1}{2}\right)=2$$

$$\cos\alpha=\frac{\sqrt{3}}{2},\ \sin\alpha=\frac{1}{2},\ \mathrm{P}=2$$

$$\Rightarrow \alpha = 30^\circ = \frac{\pi}{6}$$

$$\therefore x\cos\frac{\pi}{6} + y\sin\frac{\pi}{6} = 2$$

Home Work

II 3i, iv, v, vi

4. A straight line parallel to the line $y = \sqrt{3}x$ passer through Q(2, 3) and cuts the line $2x + 4y - 27 = 0$ at P. Find the length PQ

Sol: $y = \sqrt{3}x$ ....(1) $\quad 2x + 4y - 27 = 0$ ....(2)

Slope of a equation which is parallel to (1) also $\sqrt{3}$.

$$m = \sqrt{3} = \tan 60^\circ \Rightarrow \theta = 60^\circ$$

$$\theta = 60^\circ \text{ and } Q(2, 3)$$

$$P = (x, y) = (x_1 + r\cos\theta,\ y_1 + r\sin\theta)$$

$$P = (2 + r\cos\theta,\ 3 + r\sin\theta)$$

$$= \left(2 + r\left(\frac{1}{2}\right),\ 3 + r\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= \left(2 + \frac{r}{2},\ 3 + \frac{\sqrt{3}r}{2}\right)$$

But P is on equation (2)

$$2\left(2 + \frac{r}{2}\right) + 4\left(3 + \frac{\sqrt{3}r}{2}\right) - 27 = 0$$

$$2\left(\frac{4 + r}{2}\right) + 4\left(\frac{6 + \sqrt{3}r}{2}\right) = 27$$

$$4 + r + 12 + 2\sqrt{3}r = 27$$

$$(2\sqrt{3} + 1)r = 11$$

$$\Rightarrow r = \frac{11}{2\sqrt{3}+1} = \frac{11}{2\sqrt{3}+1} \times \frac{2\sqrt{3}-1}{2\sqrt{3}-1} = \frac{11(2\sqrt{3}-1)}{12-1}$$

$$\Rightarrow r = \frac{11(2\sqrt{3}-1)}{11} = 2\sqrt{3} - 1$$

$$\therefore PQ = |r| = 2\sqrt{3} - 1$$

5. If the area of the triangle formed by the straight lines $x = 0$, $y = 0$ and $3x + 4y = a$ ($a > 0$) is 6, find the value of a.

Sol: $3x+4y=a$

$$\frac{3x}{a}+\frac{4y}{a}=\frac{a}{a} \Rightarrow \frac{x}{\left(\frac{a}{3}\right)}+\frac{y}{\left(\frac{a}{4}\right)}=1 \qquad \text{.......... (1)}$$

Area of the triangle with coordinates axes and $\frac{x}{a}+\frac{y}{b}=1$ is $\Delta=\frac{1}{2}|ab|$

$$\therefore\ \Delta=\frac{1}{2}\left|\frac{a}{3}\times\frac{a}{4}\right|=6 \Rightarrow \frac{a^2}{12}=12$$

$$\Rightarrow a^2=144 \Rightarrow a=\sqrt{144}$$

$\therefore\ a = 12.$

6. Transform the equation $\frac{x}{a}+\frac{y}{b}=1$ into the normal form when $a > 0$ and $b > 0$. If the perpendicular distance of the straight line from the origin is P, deduce that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

Sol: $\frac{x}{a}+\frac{y}{b}=1$

Divide with $\sqrt{a^2+b^2}=\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2}$

$$\frac{\left(x/a\right)}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}+\frac{\left(y/b\right)}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

$$x\left(\frac{1}{a\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right)+y\left(\frac{1}{b\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right)=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

According to the problem $\mathrm{P}=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} \Rightarrow \frac{1}{\mathrm{P}}=\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$

Squaring on Both Sides

$$\frac{1}{P^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

The Straight Line

Exercise 3c

Note :

1 $L_1 = a_1x + b_1y + c_1 = 0,\ L_2 = a_2x + b_2y + c_2 = 0$ then intersecting point of L_1, L_2 is

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

2. $L = ax + by + c = 0$ & two points $A(x_1, y_1), B(x_2, y_2)$ then notation of $L_{11} = ax_1 + by_1 + c_1$ and $L_{22} = ax_2 + by_2 + c_2$.

(a) If A & B are same side to L = 0

$\Rightarrow$ L_1 & L_2 has same sign

(b) If A & B are opposite side to L = 0

$\Rightarrow$ L_{11} & L_{22} has opposite signs.

3. (a) X-axis divides line segment $\overline{AB}$ in the ratio $= -y_1 : y_2$

(a) Y-axis divides the line segment $\overline{AB}$ in the ratio $= -x_1 : x_2$

4. $L_1 = 0, L_2 = 0, L_3 = 0$ are concurrent lines

$$\Leftrightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \qquad \text{OR}$$

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

Questions

1. Find the ratio in which the straight line 2x + 3y = 5 divides the line segment joining the (0, 0) and (–2, 1). State whether the points lie on the same side or neither side of the S.L.

Sol: $L \equiv 2x + 3y = 5,\ A(0, 0), B(-2, 1)$

$L_{11} \equiv 2(0) + 3(0) - 5$

$L_{11} = -5 < 0$

$L_{22} \equiv 2(-2) + 3(1) - 5 = -4 + 3 - 5 = -6$

$L_{22} = -6 < 0$

$L = 0$ divides the line segment $\overline{AB}$ in the ratio $= -L_{11} : L_{22}$

Ratio = $-(-5) : (-6) = -5:6$

$L_{11} < 0, L_{22} < 0$ & $L_{11}, L_{22} < 0$

$\therefore$ A, B are lie on the same side of the line $L = 0$

2. Find the value of k, if the lines $2x - 3y + k = 0$, $3x - 4y - 13 = 0$ and $8x - 11y - 33 = 0$ are concurrent.

Sol: $L_1 \equiv 2x - 3y + k = 0$, $L_2 \equiv 3x - 4y - 13 = 0$, $L_3 \equiv 8x - 11y - 33 = 0$

L_1, L_2, L_3 are concurent

$$\Rightarrow \begin{vmatrix} 2 & -3 & k \\ 3 & -4 & -13 \\ 8 & -11 & -33 \end{vmatrix} = 0$$

$2(132 - 143) + 3(-99 + 104) + k(-33 + 32) = 0$

$2(-11) + 3(5) + k(-1) = 0$

$\therefore k = -7.$

Home work I_5 & II_7

3. A triangle of area 24 sq. units is formed by a S.L. and the coordinate axes in the first quadrant. Find the equation of the S.L. if it passes through (3, 4).

Sol: $x = 0, y = 0$, A(3, 4)

intercept form of a S.L. $\frac{x}{a} + \frac{y}{b} = 1$ ....(1)

Area of the triangle with coordinate axes & eqn (1) $\Delta = 24$ sq. units.

$\frac{1}{2}|ab| = 24 \Rightarrow ab = 48 \Rightarrow b = \frac{48}{a}$ ....(2)

But equation (1) is passing through A

$$\frac{3}{a} + \frac{4}{b} = 1 \Rightarrow \frac{3}{a} + \frac{4}{\left(\frac{48}{a}\right)} = 1$$

From equation (2)

$$\Rightarrow \frac{3}{a} + \frac{4a}{48} = 1 \Rightarrow \frac{36 + a^2}{12a} = 1$$

$$\Rightarrow 36 + a^2 = 12a$$

$\Rightarrow a^2 - 12a + 36 = 0$

$\Rightarrow (a-6)^2 = 0$

$\therefore$ $a = 6$.

From equation (2) $b = \frac{48}{6} = 8$

From (1) : Equation of required S.L.

$\frac{x}{6} + \frac{y}{8} = 1 \Rightarrow \frac{4x+3y}{24} = 1$

$4x + 3y = 24$

$4x + 3y - 24 = 0$.

4. If $3a + 2b + 4c = 0$, then show that the equation $ax + by + c = 0$ represents a family of concurrent straight lines and find the point of concurrency

Sol: $3a + 2b + 4c = 0$

Divide with 4

$\frac{3a}{4} + \frac{2b}{4} + \frac{4c}{4} = 0$

$a\left(\frac{3}{4}\right) + b\left(\frac{1}{2}\right) + c = 0$ ....(1)

$\therefore$ Each member of the family of S.Lines given by $ax + by + c = 0$ passes through the fixed point $\left(\frac{3}{4}, \frac{1}{2}\right)$. Hence the set of lines $ax + by + c = 0$ for parametric values of a, b and c is a family of concurrent lines and the point of concurrency is $\left(\frac{3}{4}, \frac{1}{2}\right)$.

H.W page No. 55 Example 4

5. Find the point on the straight line $3x + y + 4 = 0$ which is equidistant from the points (–5, 6) and (3, 2).

Sol: A(–5, 6), B(3, 2), $L \equiv 3x + y + 4 = 0$

Let P(a, b) be the required print

AP = BP

$\sqrt{(a+5)^2 + (b-6)^2} = \sqrt{(a-3)^2 + (b-2)^2}$

Squaring on B.S.

$a^2 + 25 + 10a + b^2 + 36 - 12b = a^2 + 9 - 6a + b^2 + 4 - 4b$

$10a - 12b + 61 + 6a + 4b - 13 = 0$

$16a - 8b + 48 = 0$

Divide with $2a - b + 6 = 0$ ....(1)

But P is on L = 0

$3a + b + 4 = 0$ ....(2)

Now (1) + (2)

$2a - b + 4 = 0$

$\underline{3a + b + 4 = 0}$

$5a + 10 = 0 \Rightarrow 5a = -10$

$\therefore a = -2$

From (1) $2(-2) - b + 6 = 0 \Rightarrow -4 + 6 = b$

$\therefore b = 2$

$\therefore$ Required point P = (–2, 2).

6. A straight line through P(3, 4) makes an angle of 60º with the positive direction of the X-axis. Find the coordinates of the points on the line which are 5 units away from P.

Sol: P(3, 4), $r = 5$, $\theta = 60^\circ$

Parametri form of a point

$(x, y) = (x_1 \pm r\cos\theta,\ y_1 \pm r\sin\theta)$

The point on the line which is at a distance of 5 units from P

$(x, y) = (3 \pm 5\cos 60^\circ,\ 4 \pm 5\sin 60^\circ)$

$$= \left[3 \pm 5\left(\frac{1}{2}\right),\ 4 \pm 5\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \left[\frac{6 \pm 5}{2},\ \frac{8 \pm 5\sqrt{3}}{2}\right]$$

$$= \left(\frac{6+5}{2},\ \frac{8+5\sqrt{3}}{2}\right), \left(\frac{6-5}{2},\ \frac{8-5\sqrt{3}}{2}\right)$$

$\therefore$ Required points $= \left(\frac{11}{2},\ \frac{8+5\sqrt{3}}{2}\right), \left(\frac{1}{2},\ \frac{8-5\sqrt{3}}{2}\right)$

7. A S.L through $Q(\sqrt{3}, 2)$ makes an angle $\pi/6$ with the positive direction of the X-axis. If the S.L. intersects the line $\sqrt{3}x - 4y + 8 = 0$ at P, find the distance PQ

Sol: $\sqrt{3}x - 4y + 8 = 0$ .....(1)

$Q(\sqrt{3}, 2)$, $\theta = \frac{\pi}{6} = \frac{180}{6} = 30 \Rightarrow m = \tan 30^o = \frac{1}{\sqrt{3}}$

Equation of a S.L. with slope $\frac{1}{\sqrt{3}}$ and passing through $Q(\sqrt{3}, 2)$

$y - 2 = \frac{1}{\sqrt{3}}\left(x - \sqrt{3}\right)$

$\sqrt{3}(y-2) = 1\left(x - \sqrt{3}\right)$

$\sqrt{3}y - 2\sqrt{3} = x - \sqrt{3}$

$x - \sqrt{3}y + \sqrt{3} = 0$ ....(2)

For P : (1) $-\sqrt{3}$(2)

$\sqrt{3}x - 4y + 8 = 0$

$\sqrt{3}x - 3y + 3 = 0$

$- \quad + \quad -$

$-y + 5 = 0 \Rightarrow y = 5.$

from (1) $\sqrt{3}x - 4(5) + 8 = 0 \Rightarrow \sqrt{3}x = 12 \Rightarrow x = \frac{12}{\sqrt{3}} = \frac{12 \times \sqrt{3}}{3} = 4\sqrt{3}$

$P = \left(4\sqrt{3}, 5\right)$

$\therefore PQ = \sqrt{\left(\sqrt{3} - 4\sqrt{3}\right)^2 + (2-5)^2}$

$= \sqrt{\left(-3\sqrt{3}\right)^2 + (-3)^2} = \sqrt{9(3)+9} = \sqrt{36}$

$\therefore$ PQ = 6.

H.W. Page 54 : Example 3

The Straight Line

Excrcise 3d

Note :

1. Let θ be the angle between the line L_1 & L_2 then $\theta = \cos^{-1}\left(\left|\frac{a_1a_2 + b_1b_2}{\sqrt{\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right)}}\right|\right)$

2. $L_1 \perp L_2 \Leftrightarrow a_1a_2 + b_1b_2 = 0$ OR $m_1.m_2 = -1$

3. Perpendicular distance from $P(x_1, y_1)$ to $L \equiv ax+by+c=0$ is $(d) = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

4. Distance between the parallel lines $L_1 \equiv ax+by+c_1=0$ & $L_2 \equiv ax+by+c_2=0$

is $= \left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|$

Questions

1. Find the value of *k*, if the angle between the straight lines 4*x* – *y* + 7 = 0 and *kx* – 5*y* – 9 = 0 is 45°

Sol: $L_1 \equiv 4x-y+7=0,\ L_2 \equiv kx-5y-9=0,\ \theta = 45^o$

$$\cos\theta = \frac{|a_1a_2+b_1b_2|}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}$$

$$\cos 45^o = \left|\frac{4k+(-1)(-5)}{\sqrt{(4^2+(-1)^2)(k^2+(-5)^2)}}\right|$$

$$\frac{1}{\sqrt{2}} = \frac{4k+5}{\sqrt{(16+1)(k^2+25)}}$$

$$\frac{1}{2} = \frac{16k^2+25+40k}{17(k^2+25)}$$

$$17k^2+425 = 32k^2+50+80k$$

$$15k^2+80k-375=0$$

$$3k^2+16k-75=0$$

$$3k^2+25k-9k-75=0$$

$$k(3k+25)-3(3k+25)=0$$

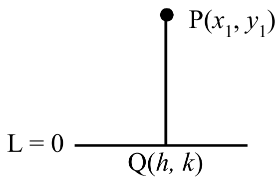
$$(3k+25)(k-3)=0$$

$3k + 25 = 0$ OR $k - 3 = 0$

$3k = -25$ OR $k = 3$

$\therefore\ k = 3$ OR $\frac{-25}{3}$

2. Find the value of k, if the straight lines $y - 3kx + 4 = 0$ and $(2k - 1)x - (8k - 1)y - 6 = 0$ are perpendicular



Sol: $L_1 \equiv y - 3kx + 4 = 0 \Rightarrow L_1 \equiv 3kx - y - 4 = 0$ .... (1)

$L_2 \equiv (2k - 1)x - (8k - 1)y - 6 = 0$ .... (2)

$L_1 \perp L_2 \Leftrightarrow a_1a_2 + b_1b_2 = 0$

$3k(2k-1) + (-1)[-(8k-1)] = 0$

$6k^2 - 3k + 8k - 1 = 0$

$6k^2 + 5k - 1 = 0$

$6k^2 + 6k - k - 1 = 0$

$6k(k+1) - 1(k+1) = 0$

$(k+1)(6k-1) = 0$

$k + 1 = 0$ OR $6k - 1 = 0$

$k = -1$ OR $k = \frac{1}{6}$

3. Find the length of the perpendicular distance from the point $(-2, -3)$ to the line $5x - 2y + 4 = 0$.

Sol: $L \equiv 5x - 2y + 4 = 0$, $P(-2, -3)$

$$(d) = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

Distance between parallel line (1) & (2) $(d) = \left|\frac{5(-2) - 2(-3) + 4}{\sqrt{(5)^2 + (-2)^2}}\right| = \frac{|-10 + 6 + 4|}{\sqrt{25 + 4}}$

$\therefore d = 0.$

Home Work $I_{9, 10}$

4. Find the distance between parallel lines $3x + 4y - 3 = 0$, $6x + 8y - 1 = 0$

Sol: $3x + 4y - 3 = 0$ $\qquad$ $6x + 8y - 1 = 0$ ..... (2)

$2\times(1) \Rightarrow 6x + 8y - 6 = 0$ ..... (1)

distance between parallel lines $= \left|\frac{-6 - (-1)}{\sqrt{(6)^2 + (8)^2}}\right|$

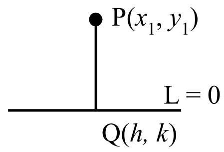
$$= \frac{|-6 + 1|}{\sqrt{36 + 64}} = \left|\frac{-5}{\sqrt{100}}\right| = \left|\frac{-5}{10}\right| = \left|\frac{-1}{2}\right|$$

$\therefore d = \frac{1}{2}$

5. If Q(h, k) is the foot of the perpendicular from P(x_1, y_1) on the straight line $ax + by + c = 0$, then

$(h - x_1) : a = (k - y_1) : b = -(ax_1 + by_1 + c) : (a^2 + b^2)$

Sol: $L \equiv ax + by + c = 0$, P(x_1, y_1), Q(h, k)



Slope of $L(m_1) = \frac{-a}{b}$

Slope of PQ $(m_2) = \frac{k - y_1}{h - x_1}$

$L \perp PQ \Leftrightarrow m_1 m_2 = -1$

$$\left(\frac{-a}{b}\right)\left(\frac{k - y_1}{h - x_1}\right) = -1$$

$$\frac{k - y_1}{h - x_1} = \frac{b}{a} \Rightarrow \frac{k - y_1}{b} = \frac{h - x_1}{a}$$

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \lambda \text{ say}$$

$\frac{h - x_1}{a} = \lambda$ $\quad\quad$ $\frac{k - y_1}{b} = \lambda$

$h - x_1 = a\lambda$ $\quad\quad$ $k - y_1 = b\lambda$

$h = x_1 + a\lambda$ $\quad\quad$ $k = y_1 + b\lambda$

But Q is on L = 0

$$a(x_1 + a\lambda) + b(y_1 + b\lambda) + c = 0$$

$$ax_1 + a^2\lambda + by_1 + b^2\lambda + c = 0$$

$$(a^2 + b^2)\lambda = -ax_1 - by_1 - c = \lambda \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$$

$$\therefore \frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-(ax_1 + by_1 - c)}{a^2 + b^2}$$

6. Find the foot of the perpendicular from (–1, 3) on the straight line $5x - y - 18 = 0$

Sol: Let Q(h, k) be the foot of the perpendicular from (–1, 3) to $5x - y - 18 = 0$

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-(ax_1 + by_1 - c)}{a^2 + b^2}$$

$$\frac{h - (-1)}{5} = \frac{k - 3}{-1} = \frac{-(-5 - 3 - 18)}{(5)^2 + (1)^2}$$

$$\frac{h+1}{5} = \frac{k-3}{-1} = \frac{26}{25+1}$$

$$\frac{h+1}{5} = \frac{k-3}{-1} = \frac{26}{26} = 1$$

$\frac{h+1}{5} = 1$ $\quad\quad$ $\frac{k-3}{-1} = 1$

$h + 1 = 5$ $\quad\quad$ $k - 3 = -1$

$h = 5 - 1$ $\quad\quad$ $k = -1 + 3$

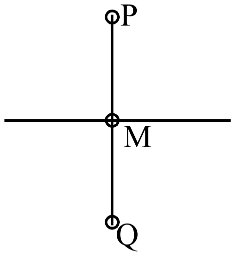
$h = 4$ $\quad\quad$ $k = 2$

$\therefore$ $(h, k) = (4, 2)$

Home Work II_5

7. If Q(h, k) is the image of the point P(x_1, y_1) w.r.t. the straight line $ax + by + c = 0$ then $(h - x_1) : a = (k - y_1) : b = -2(ax_1 + by_1 + c) : (a^2 + b^2)$

Sol: $L \equiv ax + by + c = 0$ P(x_1, y_1) Q(h, k)



Slope of L $= 0$ $(m_1) = \frac{-a}{b}$

Slope of PQ $(m_2) = \frac{k - y_1}{h - x_1}$

$PQ \perp L \Leftrightarrow m_1 m_2 = -1$

$$\left(\frac{-a}{b}\right)\left(\frac{k-y_1}{h-x_1}\right) = -1$$

$$\frac{k-y_1}{h-x_1} = \frac{b}{a} \Rightarrow \frac{k-y_1}{b} = \frac{h-x_1}{a}$$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \lambda \text{ let } \quad ....(1)$$

$\frac{h-x_1}{a} = \lambda$ $\quad\quad$ $\frac{k-y_1}{b} = \lambda$

$h - x_1 = a\lambda$ $\quad\quad$ $k - y_1 = b\lambda$

$h = x_1 + a\lambda$ $\quad\quad$ $k = y_1 + b\lambda$ $\quad\quad$ ....(2)

Since M is the midpoint of P & Q

$$M = \left(\frac{x_1+h}{2}, \frac{y_1+k}{2}\right)$$

It is on L = 0

$$a\left(\frac{x_1+h}{2}\right)+b\left(\frac{y_1+k}{2}\right)+c=0$$

$$\frac{ax_1+ah+by_1+bk+2c}{2}=0$$

$$ax_1+a(x_1+a\lambda)+by_1+b(y_1+b\lambda)+2c=0$$

$$ax_1+ax_1+a^2\lambda+by_1+by_1+b^2\lambda+2c=0$$

$$\lambda(a^2+b^2)=-2ax_1-2by_1-2c$$

$$\therefore\ \lambda=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

From (1)

$$\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

8. Find the image of (1, –2) w.r.t. the straight line 2x – 3y + 5 = 0.

Sol: (h, k) is the image of (1, –2) w.r.t the line $2x - 3y + 5 = 0$

$$\frac{h-1}{2}=\frac{k+2}{-3}=\frac{-2(2(1)-3(-2)+5)}{(2)^2+(-3)^2}$$

$$\frac{h-1}{2}=\frac{k+2}{-3}=\frac{-2(2+6+5)}{4+9}=\frac{-2(13)}{13}=-2$$

$$\frac{h-1}{2}=-2 \qquad \frac{k+2}{-3}=-2$$

$$h-1=-4 \qquad k+2=6$$

$$h=-3 \qquad k=4$$

$\therefore$ (h, k) = (–3, 4)

H om e W ork II_8

9. Find the equations of the straight lines passing through the point (–3, 2) and making an angle of 45º with the straight line 3x – y + 4 = 0

Sol: P(–3, 2) , L ≡ $3x - y + 4 = 0$, θ = 45º

Let m be the slope of a line which is passing through P

$y - 2 = m(x + 3)$ .....(*)

$y - 2 = mx + 3m$

$mx - y + (2 + 3m) = 0$ .....(1)

According to the problem, 45º is the angle between

L = 0 & (1)

$$\cos 45^{\circ} = \frac{3(m)+(-1)(-1)}{\sqrt{\left((3)^2+(-1)^2\right)\left(m^2+(-1)^2\right)}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{3m+1}{\sqrt{(9+1)(m^2+1)}}$$

Squaring on B.S.

$$\frac{1}{2} = \frac{9m^2+1+6m}{10(m^2+1)}$$

$$10(m^2+1) = 2(9m^2+1+6m)$$

$$10m^2+10 = 18m^2+2+12m$$

$$8m^2+12m-8=0$$

$$2m^2+3m-2=0$$

$$2m^2+4m-1m-2=0$$

$$2m(m+2)-1(m+2)=0$$

$$(m+2)((2m-1)=0$$

$m + 2 = 0$ OR $2m - 1 = 0$

$m = -2$ OR $m = \frac{1}{2}$

Case (i):

$m = -2$ then, from equation (*)

$$y-2=-2(x+3)$$

$$y-2=-2x-6$$

$$2x+y+4=0$$

Case (ii):

$m = \frac{1}{2}$ then, from equation (*)

$$y-2=\frac{1}{2}(x+3)$$

$$2y-4=x+3$$

$$x-2y+7=0$$

∴ Equation of required s. lines

$2x + 4y + 4 = 0,$

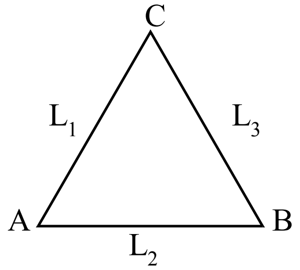
$x - 2y + 7 = 0$

10. Find the angles of the triangle whose sides are $x + y - 4 = 0$, $2x + y - 6 = 0$ and $5x + 3y - 15 = 0$.

Sol: $L_1 \equiv x + y - 4 = 0$

$L_2 \equiv 2x + y - 6 = 0$

$L_3 \equiv 5x + 3y - 15 = 0$



$$\cos A = \frac{|1(2)+1(1)|}{\sqrt{(1^2+1^2)(2^2+1^2)}} = \frac{|2+1|}{\sqrt{(1+1)(4+1)}} = \frac{3}{\sqrt{10}}$$

$$A = \cos^{-1}\left(\frac{3}{\sqrt{10}}\right) = 18.45^\circ$$

$$\cos B = \frac{|2(5)+1(3)|}{\sqrt{(2^2+1^2)(5^2+3^2)}} = \frac{|10+3|}{\sqrt{(4+1)(25+9)}} = \frac{13}{\sqrt{170}}$$

$$B = \cos^{-1}\left(\frac{13}{\sqrt{170}}\right) = 4.43^\circ$$

$$\cos C = \frac{|5(1)+3(1)|}{\sqrt{(5^2+3^2)(1^2+1^2)}} = \frac{|5+3|}{\sqrt{(25+9)(1+1)}} = \frac{8}{\sqrt{68}} = \frac{8}{2\sqrt{17}} = \frac{4}{\sqrt{17}}$$

$$C = \cos^{-1}\left(\frac{4}{\sqrt{17}}\right) = 14.02^\circ$$

$A + B + C = 36.9^\circ$ which is in correct.

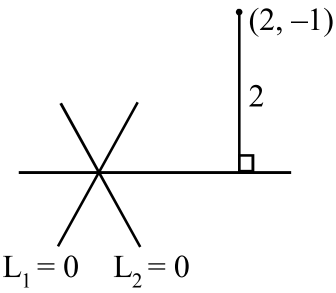
$\because$ Sum of the angles in triangle is 180°.

$\Rightarrow A = \pi - \cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$

11. Find the equations of the straight lines passing through the point of intersection of the lines $3x + 2y + 4 = 0$, $2x + 5y = 1$ and whose distance from $(2, -1)$ is 2.

Sol: $L_1 \equiv 3x + 2y + 4 = 0$, $L_2 \equiv 2x + 5y = 1$, $A = (2, -1)$, $d = 2$

For the point of intersection of L_1 & L_2



$2L_1 - 3L_2$

$6x + 4y + 8 = 0$

$\underline{6x + 15y - 3 = 0}$

$-11y + 11 = 0 \Rightarrow 11y = 11 \Rightarrow y = 1$

From L_1 $3x + 2(1) + 4 = 0 \Rightarrow 3x = -6 \Rightarrow x = -2$

P(2, –1)

Let m be the slope of a line which is passing through P

$y - 1 = m(x + 2)$ ....(*)

$y - 1 = mx + 2m$

$mx - y + (1+2m) = 0$ ....(1)

Given that perpendicular distance from P to equation (1) = 2

$$\left|\frac{m(2)-(-1)+1+2m}{\sqrt{m^2+(-1)^2}}\right| = 2$$

$$\frac{2m+1+1+2m}{\sqrt{m^2+1}} = 2 \Rightarrow \frac{4m+2}{\sqrt{1+m^2}} = 2$$

$$2(2m+1) = 2\sqrt{1+m^2} \Rightarrow 2m+1 = \sqrt{1+m^2}$$

Squaring on both sides

$$4m^2+1+4m = 1+m^2$$

$$3m^2+4m = 0$$

$$m(3m+4) = 0$$

$$m = 0 \text{ OR } m = \frac{-4}{3}$$

Case (i): $m = 0$ then, from (*)

$$y-1 = 0(x+3)$$

$$y = 1$$

Case (ii): $m = \frac{-4}{3}$ then, from (*)

$$y-1 = -\frac{4}{3}(x+2)$$

$$3y-3 = -4x-8$$

$$4x+3y+5 = 0$$

∴ Equation of required straight lines $y = 1$, $4x+3y+5=0$.

12. Find the equation of the line perpendicular to the line 3x + 4y + 6 = 0 and making intercept –4 on the X-axis.

Sol: $L_1 \equiv 3x + 4y + 6 = 0$, X-intercept $= -4$

Equation of any line which is perpendicualr to L = 0

$$4x - 3y + k = 0 \quad .....(*)$$

$$4x - 3y = -k$$

$$\frac{4x}{(-k)} - \frac{3y}{(-k)} = \frac{-k}{(-k)}$$

$$\frac{x}{\left(\frac{-k}{4}\right)} + \frac{y}{\left(\frac{+k}{3}\right)} = 1 \quad .....(1)$$

X-intercept of (1) $= \frac{-k}{4}$

According to the problem $\frac{-k}{4} = -4 \Rightarrow k = 16$

∴ Equation of required straight line $4x - 3y + 16 = 0$

The Straight Line

Excrcise 3e

Note :

1. The medians of a triangle are concerrent the concurrent point of medians is the centroid
2. The altitudes of a triangle are concurrent. The cencurrent point of the altitudes is ortho-center.
3. The internal bisectors of the angles of a triangle are concurrent. The concurrent point is called incenter.
4. The perpenducular bisectors of the sides of a triangle are concurrent. The point of concurrency is the 'circumeenter' of the triangle.

Questions

1. Find the equation of the straight line passing through the origin and also through the point of intersection of the lines $2x - y + 5 = 0$ and $x + y + 1 = 0$.

Sol: $O(0, 0)$, $L_1 \equiv 2x - y + 5 = 0$, $L_2 \equiv x + y + 1 = 0$

From $L_1 + L_2$

$$\begin{array}{r} 2x - y + 5 = 0 \\ \underline{x + y + 1 = 0} \\ 3x + 6 = 0 \Rightarrow 3x = -6 \Rightarrow x = -2 \end{array}$$

From $L_1 : 2(-2) - y + 5 = 0 \Rightarrow 1 - y = 0 \Rightarrow y = 1$

intersecting point of L_1 & L_2 $A(-2, 1)$

Equation of requirest straight line passing through O & A

$$y - 0 = \left(\frac{1-0}{-2-0}\right)(x - 0)$$

$$-2(y) = 1(x)$$

$$x + 2y = 0$$

2. Find the equation of the straight line parallel to the line $3x + 4y = 7$ and passing through the point of intersection of the lines $x - 2y - 3 = 0$ and $x + 3y - 6 = 0$

Sol: $L_1 \equiv 3x + 4y = 7$, $L_2 \equiv x - 2y - 3 = 0$, $L_2 \equiv x + 3y - 6 = 0$

To find the point intersection of L_2, L_3

$$\begin{array}{r} x - 2y - 3 = 0 \\ \underline{x + 3y - 6 = 0} \\ - \quad - \quad + \quad \\ -5y + 3 = 0 \end{array}$$

$$\Rightarrow 5y = 3 \Rightarrow y = \frac{3}{5}$$

From L_2 $\quad x - 2\left(\frac{3}{5}\right) - 3 = 0$

$$x = \frac{6}{5} + 3 = \frac{6+15}{5} = \frac{21}{5}$$

Equation of any line parallel to $A = \left(\frac{21}{5}, \frac{3}{5}\right)$

$L_1 \equiv 3x + 4y = 7$ Equation of any line parallel to $L_1 = 0$ is $3x + 4y + k = 0$

But it is passing through A

$$3\left(\frac{21}{5}\right) + 4\left(\frac{3}{5}\right) + k = 0$$

$$k + \frac{63+12}{5} = 0 \Rightarrow k = \frac{-75}{5} = -15$$

$\therefore$ Equation of required straight line $3x + 4y - 15 = 0$

3. Find the equation of the straight line making non - zero equal intercepts on the coordinate axes and passing through the point of intersection of the lines $2x - 5y + 1 = 0$ and $x - 3y - 4 = 0$

Sol: $L_1 \equiv 2x - 5y + 1 = 0$, $L_2 \equiv x - 3y - 4 = 0$

To find the point intersection of L_1, L_2

$$\begin{array}{l} 2x - 5y + 1 = 0 \\ \underline{2x - 6y - 8 = 0} \\ \quad - \quad + \quad + \\ \qquad y + 9 = 0 \Rightarrow y = -9 \end{array}$$

From L_1: $2x - 5(-9) + 1 = 0$

$2x + 45 + 1 = 0$

$2x = -46$

$x = -23$

$A = (-23, -9)$

Since equal intercepts on the coordinate axes

$\Rightarrow a = b$

Equation of S.L. in intercept form

$$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{a} + \frac{y}{a} = 1 \quad (\because a = b)$$

But it is passing through A

$-23 - 9 = a$

$-32 = a$

Equation of a required straight line $x + y = -32 \Rightarrow x + y + 32 = 0$.

4. Find the length of the perpendicular drawn from the point of intersection of the lines $3x + 2y + 4 = 0$ and $2x + 5y - 1 = 0$ to the streight line $7x + 24y - 15 = 0$

Sol: $L_1 \equiv 3x + 2y + 4 = 0$, $L_2 \equiv 2x + 5y - 1 = 0$, $L_2 \equiv 7x + 24y - 15 = 0$

Point of intersection of L_1 & L_2

$$\frac{x}{2(-1)-5(4)} = \frac{y}{4(2)-(-1)(3)} = \frac{1}{3(5)-2(2)}$$

$$\frac{x}{-2-20} = \frac{y}{8+3} = \frac{1}{15-4}$$

$$\frac{x}{-22} = \frac{y}{11} = \frac{1}{11}$$

A = (–2, 1)

Perpendicular distance from A to $L_3 = 0$

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$= \left|\frac{7(-2)+24(1)-15}{\sqrt{(7)^2+(24)^2}}\right| = \left|\frac{-14+24-15}{\sqrt{49+576}}\right|$$

$$= \left|\frac{-5}{\sqrt{625}}\right| = \frac{5}{25}$$

$$\therefore d = \frac{1}{5}$$

5. The base of an equilareral triangle is $x + y - 2 = 0$ and the opposite vertex is (2, –1). Find the equations of the remaining sides.

Sol: $L \equiv x + y - 2 = 0$, A(2, –1)

Let 'm' be the slope of a line which is passing through A.

Equation of a line with slope m and passing through A is

equation of the line passing through A and intersecting L = 0 at B

$y + 1 = m(x - 2)$ ....(*)

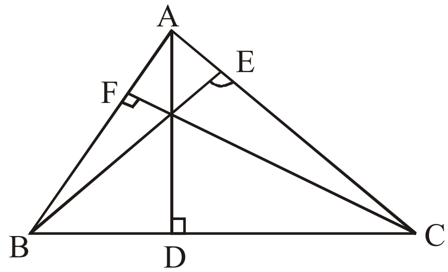
$y + 1 = mx - 2m$

$mx - y - (1 + 2m) = 0$ ....(1)

Since ΔABC is equilatiral triangle

$\Rightarrow \underline{|A} = \underline{|B} = \underline{|C}$

Angle between L L = 0, (1)



$$\cos 60^{\circ} = \frac{1(m)+1(-1)}{\sqrt{((1)^2+(1)^2)(m^2+(-1)^2)}}$$

Squaring on both sides

$$\frac{1}{2} = \frac{m-1}{\sqrt{(1+1)(1+m^2)}}$$

$$\frac{x}{-2-20} = \frac{y}{8+3} = \frac{1}{15-4}$$

$$\frac{1}{4} = \frac{m^2+1-2m}{2(1+m^2)}$$

$$(1+m^2) = 2m^2+2-4m$$

$$2m^2+2-4m-1-m^2 = 0$$

$$m^2-4m+1=0$$

$$m = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$$

$$= \frac{4\pm\sqrt{16-4}}{2} = \frac{4\pm\sqrt{12}}{2}$$

$$= \frac{4\pm 2\sqrt{3}}{2} = \frac{2(2\pm\sqrt{3})}{2}$$

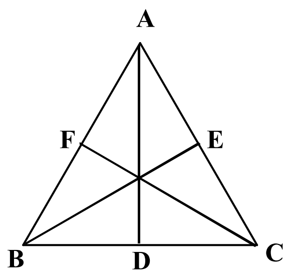
$$m = 2\pm\sqrt{3}$$

From (*) : Equation of required S.L. $y+1=(2\pm\sqrt{3})(x-2)$.

Home work II_2

6. Find the orthocenter of the triangle whose vertices are (–5, –7), (13, 2) and (–5, 6)

Sol: A(–5, –7), B(13, 2), C(–5, 6)



Slope of BC $= \frac{6-2}{-5-13} = \frac{4}{-18} = \frac{-2}{9}$

Let AD be the perpendicular drawn from A to BC

Altitude drawn from CF to AB

Let BE = be the perpendicular drawn from B To $\overline{AC}$

AD ⊥ BC

Slope of AD $= \frac{-1}{m} = \frac{-1}{\left(\frac{-2}{9}\right)} = \frac{9}{2}$

Equation of a line with Slope $\frac{9}{2}$ and passing through A is

$$y+7=\frac{9}{2}(x+5)$$

$$2y+14=9x+45$$

$$9x-2y+31=0 \qquad .....(1)$$

Slope of AC $\frac{6+7}{-5-(-5)}=\frac{13}{0}=\infty$

BE $\perp$ AC $\Rightarrow$ Slope of BE = 0

Equation of BE $\quad y-2=0(x-13)$

$y-2=0$

$y=2 \qquad ....(2)$

Intersecting point of (1), (2) is our required orthocenter

From (1) & (2) $\quad 9x-2(2)+31=0$

$9x+27=0 \Rightarrow 9x=-27 \Rightarrow 9x=-3.$

$\therefore$ O = (–3, 2)

7. If the equations of the sides of a triangle are $7x+y-10=0$, $x-2y+5=0$ and $x+y+2=0$, find the orthocenter of the triangle.

Sol: $L_1 \equiv 7x+y-10=0$, $L_2 \equiv x-2y+5=0$, $L_2 \equiv x+y+2=0$

For A $2L_1+L_2$

$$\begin{array}{l} 14x+2y-20=0 \\ \underline{\;\;x-2y+\;5=0} \\ 15x\qquad -15=0 \Rightarrow x=1. \end{array}$$

From L_1 $7(1)+y-10=0 \Rightarrow y-3=0 \Rightarrow y=3 \quad \Rightarrow$ A = (1, 3)

For B L_1-L_3

$$\begin{array}{l} 7x+y-10=0 \\ \underline{\;\;x+y+\;\;2=0} \\ -\quad-\quad- \\ 6x\qquad -12=0 \Rightarrow x=2. \end{array}$$

from L_1 $7(2)+y-10=0 \Rightarrow y+4=0 \Rightarrow y=-3 \Rightarrow$ B = (2, –4)

For C L_2-L_3

$$\begin{array}{l} x-2y+\;5=0 \\ x+y+\;\;2=0 \\ \quad -3y+3=0 \Rightarrow y=1. \end{array}$$

L_2 from $x-2(1)+5=0 \Rightarrow x+3=0 \Rightarrow x=-3$

$\Rightarrow$ C = (–3, 1)

Slope of BC $= \frac{-1}{(1)} = -1$

AD ⊥ BC ⇒ Slope of AD $= \frac{-1}{(-1)} = 1$

Equation of AD $y - 3 = 1(x - 1)$

$y - 3 = (x - 1)$

$x - y + 2 = 0$ ....(1)

Slope of AC $= \frac{-1}{(-2)} = \frac{1}{2}$

BE ⊥ AC ⇒ Slope of BE $= \frac{-1}{(1/2)} = -2$

Equation of BE $y + 4 = -2(x - 2)$

$y + 4 = -2x + 4$

$2x + y = 0$ ....(2)

Intersecting point of (1) & (2) is our required orthocenter (1) + (2)

$x - y + 2 = 0$

$\underline{2x + y \quad = 0}$

$3x \quad + 2 = 0 \Rightarrow x = -2/3$

From (1) $\frac{-2}{3} - y + 2 = 0 \Rightarrow \frac{-2+6}{3} = y \Rightarrow y = \frac{4}{3}$

∴ required orthocenre $O = \left(\frac{-2}{3}, \frac{4}{3}\right)$

Home Work I_3, II_4 & III_1

8. Find the circumcenter of the triangle whose vertices are (–2, 3), (2, –1) and (4, 0)

Sol: A(–2, 3), B(2, –1), C(4, 0)

Let $S = (a, b)$ be the circumcenter of the ΔABC

We know that SA = SB = SC

From : SA = SB

$\sqrt{(-2-a)^2 + (3-b)^2} = \sqrt{(2-a)^2 + (-1-b)^2}$

$4 + a^2 + 4a + 9 + b^2 - 6b = 4 + a^2 - 4a + 1 + b^2 + 2b$

$4a - 6b + 13 + 4a - 2b - 5 = 0$

$8a - 8b + 8 = 0$

Divide with 8 $a - b + 1 = 0$ ....(1)

SA = SB

$\sqrt{(-2-a)^2+(3-b)^2}=\sqrt{(4-a)^2+(0-b)^2}$

Squaring on B.S

$4+a^2+4a+9+b^2-6b=16+a^2-8a+b^2$

$4a-6b+13+8a-16=0$

$12a-6b-3=0$

$4a-2b-1=0$ ....(2)

Intersecting point of (1) & (2) is our required circumcenter

4(1) – (2) $\quad 4a-4b+4=0$

$\underline{4a-2b-1=0}$

$-\quad +\quad +$

$-2b+5=0 \Rightarrow b=5/2$

From (1) $a-\frac{5}{2}+1=0 \Rightarrow a=\frac{5}{2}-1 \Rightarrow a=\frac{5-2}{2}=\frac{3}{2}$

$\therefore$ required circumcentre S = $\left(\frac{3}{2},\frac{5}{2}\right)$

Home Work (1) page 73 Example (2) 4 II_{5ii} (3) III_2 (4) I_{13}

9. If p and q are the lengths of the from the origin to the straight lines $x\sec\alpha+y\operatorname{cosec}\alpha=a$ and $x\cos\alpha-y\sin\alpha=a\cos2\alpha$ Prove that $4p^2+q^2=a^2$.

Sol: $L_1 \equiv x\sec\alpha+y\operatorname{cosec}\alpha-a=0,\ L_2 \equiv x\cos\alpha-y\sin\alpha-a\cos2\alpha=0$

p = perpendicular distance from (0, 0) to $L_1=0$

$$=\left|\frac{(0)\sec\alpha+(0)\operatorname{cosec}\alpha-a}{\sqrt{(\sec\alpha)^2+(\operatorname{cosec}\alpha)^2}}\right|=\left|\frac{-a}{\sqrt{\frac{1}{\cos^2\alpha}+\frac{1}{\sin^2\alpha}}}\right|$$

$$=\frac{a}{\sqrt{\frac{\sin^2\alpha+\cos^2\alpha}{\cos^2\alpha.\sin^2\alpha}}}=\frac{a}{\sqrt{\frac{1}{(\sin\alpha\cos\alpha)^2}}}=\frac{a}{\left(\frac{1}{\sin\alpha\cos\alpha}\right)}$$

p = a sinα cosα

Now q = perpendicular distance from (0, 0) to $L_2=0$

$$=\left|\frac{(0)\cos\alpha-(0)\sin\alpha-a\cos2\alpha}{\sqrt{(\cos\alpha)^2+(-\sin\alpha)^2}}\right|=\left|\frac{0-0-a\cos2\alpha}{\sqrt{\cos^2\alpha+\sin^2\alpha}}\right|$$

$$=\left|\frac{-a\cos2\alpha}{\sqrt{1}}\right|$$

$\therefore$ q = a cos2α

Now $$4p^2 + q^2 = 4(a\sin\alpha\cos\alpha)^2 + (a\cos 2\alpha)^2$$

$$= a^2(2\sin\alpha\cos\alpha)^2 + a^2\cos^2 2\alpha$$

$$= a^2[\sin^2 2\alpha + \cos^2 2\alpha]$$

$$= a^2(1) \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$\therefore$ $4p^2 + q^2 = a^2$

Unit **4**

Pair of Straight Lines

Key Concepts

→ If a, b, h are real numbers, not all zero, then $H \equiv ax^2 + 2hxy + by^2 = 0$ is called a homogeneous equation of second degree in x and y

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is called a general equation of second degree in x and y.

→ If a, b, h are not all zero, then the equation $H \equiv ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines $\Leftrightarrow h^2 \geq ab$.

→ Let the equation $ax^2 + 2hxy + by^2 = 0$ represent a pair of straight lines. Then the angle θ between the lines is given by

$$\cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}$$

(i) If $h^2 = ab$ then the lines are coincide

(ii) If $a + b = 0$ then the lines are perpendicular

→ If the second degree equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ in the two variables x and y represents a pair of straight lines

⇔ (i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(ii) $h^2 \geq ab,\ g^2 \geq ac,\ f^2 \geq bc$

Long Answer Questions (7 Marks)

1. If θ is an angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$ then prove that

$$\cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}$$

Sol: Let $ax^2 + 2hxy + by^2 = 0$ represents the lines

$l_1x + m_1y = 0$ ....(1)

$l_2x + m_2y = 0$ ....(2)

$\therefore\ ax^2+2hxy+by^2=(l_1x+m_1y)(l_2x+m_2y)$

$\Rightarrow\ ax^2+2hxy+by^2=l_1l_2x^2+(l_1m_2+l_2m_1)xy+m_1m_2y^2$

by comparing like term coefficients on both sides

$l_1l_2=a,\ l_1m_2+l_2m_1=2h,\ m_1m_2=b$

If θ is an angle between (1) and (2) then

$$\cos\theta=\frac{|l_1l_2+m_1m_2|}{\sqrt{l_1^2+m_1^2}\sqrt{l_2^2+m_2^2}}$$

$$=\frac{|l_1l_2+m_1m_2|}{\sqrt{(l_1^2+m_1^2)(l_2^2+m_2^2)}}$$

$$=\frac{|l_1l_2+m_1m_2|}{\sqrt{l_1^2l_2^2+l_1^2m_1^2+l_2^2m_1^2+m_1^2m_2^2}}$$

$$=\frac{|l_1l_2+m_1m_2|}{\sqrt{(l_1l_2-m_1m_2)^2+2l_1l_2m_1m_2+(l_1m_2+l_2m_1)^2-2l_1m_2m_2l_1}}$$

$$=\frac{|a+b|}{\sqrt{(a-b)^2+(2h)^2}}\qquad [\because l_1l_2=a,\ m_1m_2=b,\ l_1m_2+l_2m_1=2h]$$

$$=\frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}$$

2. Prove that the product of perpendiculars from (α,β) to the pair of straight lines $ax^2+2hxy+by^2=0$ is

$$\frac{|a\alpha^2+2h\alpha\beta+b\beta^2|}{\sqrt{(\alpha-\beta)^2+4h^2}}$$

Sol: $ax^2+2hxy+by^2=0$ Let the pair of straight lines represents the lines

$l_1x+m_1y=0\qquad \ldots.(1)$

$l_2x+m_2y=0\qquad \ldots.(2)$

$\therefore\ ax^2+2hxy+by^2=(l_1x+m_1y)(l_2x+m_2y)$

$=l_1l_2x^2+(l_1m_2+l_2m_1)xy+m_1m_2y^2$

by comparing like term coefficients on both sides

$l_1l_2=a,\ l_1m_2+l_2m_1=2h,\ m_1m_2=b$

Let d_1 = perpendicular distance from a point (α, β) to $l_1x + m_1y = 0$

$$d_1 = \frac{|l_1\alpha + m_1\beta|}{\sqrt{l_1^2 + m_1^2}}$$

similarly

Let d_2 = perpendicular distance from a point (α, β) to $l_2x + m_2y = 0$

$$d_2 = \frac{|l_2\alpha + m_2\beta|}{\sqrt{l_2^2 + m_2^2}}$$

∴ Product of perpendicular distances $= d_1 \times d_2$

$$= \frac{|l_1\alpha + m_1\beta|}{\sqrt{l_1^2 + m_1^2}} \times \frac{|l_2\alpha + m_2\beta|}{\sqrt{l_2^2 + m_2^2}}$$

$$= \frac{|(l_1\alpha + m_1\beta)(l_2\alpha + m_2\beta)|}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}}$$

$$= \frac{|l_1l_2\alpha^2 + (l_1m_1 + l_2m_1)\alpha\beta + m_1m_2\beta^2|}{\sqrt{l_1^2l_2^2 + l_1^2m_2^2 + l_2^2m_1^2 + m_1^2m_2^2}}$$

$$= \frac{|l_1l_2\alpha^2 + (l_1m_1 + l_2m_1)\alpha\beta + m_1m_2\beta^2|}{\sqrt{(l_1l_2 - m_1m_2)^2 + 2l_1l_2m_1m_2 + (l_1m_2 + l_2m_1)^2 - 2l_1l_2m_1m_2}}$$

$$= \frac{|l_1l_2\alpha^2 + (l_1m_1 + l_2m_1)\alpha\beta + m_1m_2\beta^2|}{\sqrt{(l_1l_2 - m_1m_2)^2 + (l_1m_2 + l_2m_1)^2}}$$

$$= \frac{|a\alpha^2 + 2h\alpha\beta + b\beta^2|}{\sqrt{(a-b)^2 + (2h)^2}} \qquad [\because l_1l_2 = a,\ m_1m_2 = b,\ l_1m_2 + l_2m_1 = 2h]$$

$$= \frac{|a\alpha^2 + 2h\alpha\beta + b\beta^2|}{\sqrt{(a-b)^2 + 4h^2}}$$

3. Show that the area of a triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is $\left|\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}\right|$

Sol: Let the equation $ax^2 + 2hxy + by^2 = 0$ represents the lines

$l_1x + m_1y + n_1 = 0$ ....(1)

$l_2x + m_2y + n_2 = 0$ ....(2)

$\therefore \quad ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$

$$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2$$

Let the given line be $lx + my + n = 0$ ....(3)

$l_1l_2 = a,\ l_1m_2 + l_2m_1 = 2h,\ m_1m_2 = b$

Let A be the intersection point of (1) and (3)

$$\frac{x}{m_1n - o} = \frac{y}{o - nl_1} = \frac{1}{l_1m - lm_1}$$

$$\Rightarrow x = \frac{m_1n}{l_1m - lm_1}, \quad y = \frac{-nl_1}{l_1m - lm_1}$$

$$\therefore \text{A} = \left(\frac{m_1n}{l_1m - lm_1}, \frac{-nl_1}{l_1m - lm_1}\right)$$

Similarly, Let B be the intersection point of (2) and (3)

$$\frac{x}{m_2n - o} = \frac{y}{o - nl_2} = \frac{1}{l_2m - lm_2}$$

$$\therefore \text{B} = \left(\frac{m_2n}{l_2m - lm_2}, \frac{-nl_2}{l_2m - lm_2}\right)$$

clearly 0(0,0) be the point of intersection of (1) and (2)

[Area of a triangle whose vertices (0, 0), (x_1, y_1), (x_2, y_2) is $= \frac{1}{2}|x_1y_2 - x_2y_1|$]

$$\therefore \text{ Area of } \Delta\text{OAB } = \frac{1}{2}\left|\frac{m_1n}{l_1m - lm_1} \times \frac{(-nl_2)}{l_2m - lm_2} - \frac{m_2n}{l_2m - lm_2} \times \frac{(-nl_1)}{l_1m - lm_1}\right|$$

$$= \frac{1}{2}\left|\frac{l_1m_2n^2 - m_1l_2n^2}{(l_1m - lm_1)(l_2m - lm_2)}\right|$$

$$= \frac{1}{2}\left|\frac{n^2(l_1m_2 - m_1l_2)}{l_1l_2m^2 - (l_1m_2 + l_2m_1)lm + m_1m_2l^2}\right|$$

$$= \frac{1}{2}\left|\frac{n^2\sqrt{(l_1m_2 - l_2m_1)^2}}{l_1l_2m^2 - (l_1m_2 + l_2m_1)lm + m_1m_2l^2}\right|$$

$$= \frac{1}{2}\left|\frac{n^2\sqrt{(l_1m_2 + l_2m_1)^2 - 4l_1m_2l_2m_1}}{l_1l_2m^2 - (l_1m_2 + l_2m_1)lm + m_1m_2l^2}\right|$$

$$= \frac{1}{2}\left|\frac{n^2\sqrt{(2h)^2-4ab}}{am^2-2hlm+bl^2}\right| \qquad [\because l_1l_2 = a,\ m_1m_2 = b,\ l_1m_2 + l_2m_1 = 2h]$$

$$= \frac{1}{2}\left|\frac{n^2\sqrt{4h^2-4ab}}{am^2-2hlm+bl^2}\right|$$

$$= \left|\frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}\right|$$

4. Show that the straight lines represented by $(x + 2a)^2 - 3y^2 = 0$ and $x = 0$ from an **equilateral triangle.**

Sol: Given equation of pair of straight lines si $(x + 2a)^2 - (\sqrt{3}\,y)^2 = 0$

$$\Rightarrow \left(x+2a-\sqrt{3}y\right)\left(x+2a+\sqrt{3}y\right)=0$$

$\therefore$ The lines represented by the given equation of pair of straight lines are

$$x+\sqrt{3}y+2a=0 \qquad ....(1)$$

$$x-\sqrt{3}y+2a=0 \qquad ....(2)$$

Given line be $x - a = 0$ ....(3)

Let A be the angle between (1) and (3) then

$$\cos A = \frac{\left|1.1+\sqrt{3}.0\right|}{\sqrt{1+3}\sqrt{1+0}} = \frac{1}{2} \Rightarrow A = 60^\circ$$

Again B be the angle between (2) and (3) then

$$\cos B = \frac{\left|1.1+(-\sqrt{3}).0\right|}{\sqrt{1+3}\sqrt{1+0}} = \frac{1}{2} \Rightarrow B = 60^\circ$$

$\therefore$ Since (1) and (2) are different lines and two angles are 60, 60°

$\therefore$ Third angle is also 60°

$\therefore$ Given lines form an equilateral triangle.

5. Find the centroid and area of a triangle formed by the lines $\mathbf{2y^2 - xy - 6x^2 = 0}$ **and $x + y + 4 = 0$**

Sol: Given equation of straight lines $2y^2 - xy - 6x^2 = 0$ .....(1)

$$\Rightarrow 2y^2-4xy+3xy-6x^2=0$$

$$\Rightarrow 2y(y-2x)+3x(y-2x)=0$$

$$\Rightarrow (y-2x)(2y+3x)=0$$

$\therefore$ The lines represented by the given equation $2y^2 - xy - 6x^2 = 0$ are given by

$$\Rightarrow 2x - y = 0 \qquad .....(2)$$

$$\text{and } 3x + 2y = 0 \qquad .....(3)$$

and given straight line $x + y + 4 = 0$ .....(4)

Clearly O(0, 0) is point of intersection of (2) and (3)

Let A be the point of intersection of (2) and (4)

$$\frac{x}{-4-0} = \frac{y}{0-8} = \frac{1}{2+1}$$

$$\therefore \text{A} = \left(\frac{-4}{3}, \frac{-8}{3}\right)$$

B be the point of intersection of (3) and (4)

$$\frac{x}{8-0} = \frac{y}{0-12} = \frac{1}{3-2}$$

$$\therefore \text{B} = (8, -12)$$

Centroid of ΔABC $= \left(\frac{0 - 4/3 + 8}{3}, \frac{0 - 8/3 - 12}{3}\right) = \left(\frac{20}{9}, \frac{-44}{9}\right)$

[Area of a triangle whose vertices are O(0, 0), A(x_1, y_1), B(x_2, y_2) is $= \frac{1}{2}|x_1y_2 - x_2y_1|$]

$\therefore$ Required area of triangle $= \frac{1}{2}\left|\left(\frac{-4}{3}\right)(-12) - 8\left(\frac{-8}{3}\right)\right|$

$$= \frac{1}{2}\left|\frac{48}{3} + \frac{64}{3}\right| = \frac{112}{2\times 3} = \frac{56}{3} \text{ Sq.units}$$

6. Show that the lines represented by $(lx + my)^2 - 3(mx - ly) = 0$ and $lx + my = 0$ form an equilateal triangle with area $\frac{n^2}{\sqrt{3}(l^2 + m^2)}$

Sol: Given straight line $lx + my = 0$ .....(1)

Slope of (1) $m_1 = \frac{-l}{m}$

Le the equation of line $y = m_2x$ which makes an angle of 60° with (1)

$$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$$\tan 60^o = \left|\frac{-l/m - y/x}{1+\left(-l/m\right) y/x}\right| \qquad \left[\because m_2 = y/x\right]$$

$$\Rightarrow \sqrt{3} = \left|\frac{\frac{-lx-my}{mx}}{\frac{mx-ly}{mx}}\right| = \left|\frac{-(lx+my)}{mx-ly}\right|$$

squaring both sides

$$3(mx-ly)^2 = (lx+my)^2$$

$$\Rightarrow (lx+my)^2 - 3(mx-ly)^2 = 0 \qquad \ldots(2)$$

which is given equation of pair of straight in the data

$\therefore (lx+my)^2 - 3(mx-ly)^2 = 0$ and $lx + my + n = 0$ form a equilateral triangle

Also O $(0,\ 0)$ is point of intersection of lines in (2)

$\therefore$ Perpendicular distance to (1) from O(0,0) $\text{P} = \frac{|n|}{\sqrt{l^2+m^2}}$

Area of that equilateral triangle

$$\frac{\text{P}^2}{\sqrt{3}} = \frac{n^2}{\sqrt{3}(l^2+m^2)}$$

7. If (α, β) **is the centroid of the triangle formed by the lines** $\boldsymbol{ax^2 + 2hxy + by^2 = 0}$ **and** $\boldsymbol{lx + my = 0}$ **then prove that** $\frac{\alpha}{bl-hm} = \frac{\beta}{am-hl} = \frac{2}{3(bl^2-2hlm+am^2)}$

Sol: Let $ax^2 + 2hxy + by^2 = 0$ represent the lines

$$l_1x + m_1y = 0 \qquad \ldots(1)$$

$$l_2x + m_2y = 0 \qquad \ldots(2)$$

$$\therefore \quad ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$$

$$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2$$

comparing both sides like terms

$l_1l_2 = a,\ l_1m_2 + l_2m_1 = 2h,\ m_1m_2 = b$

Given equation $lx + my = 0 \qquad \ldots(3)$

Clearly O(0, 0) is point of intersection of (1) and (2)

Let A be the point of intersection of (1) and (3)

$$\frac{x}{-m_1 - 0} = \frac{y}{0 + l_1} = \frac{1}{ml_1 - lm_1}$$

$$\therefore \text{A} = \left(\frac{-m_1}{l_1 m - lm_1}, \frac{l_1}{l_1 m - lm_1} \right)$$

Again let B be the point of intersection of (2) and (3)

$$\frac{x}{-m_2 - 0} = \frac{y}{0 + l_2} = \frac{1}{ml_2 - lm_2}$$

$$\therefore \text{B} = \left(\frac{-m_2}{l_2 m - lm_2}, \frac{l_2}{l_2 m - lm_2} \right)$$

centroid of ΔOAB = (α, β)

$$\therefore (\alpha, \beta) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

$$\Rightarrow (\alpha, \beta) = \frac{1}{3}\left[\frac{-m_1}{l_1 m - lm_1} - \frac{m_2}{l_2 m - lm_2}, \frac{l_1}{l_1 m - lm_1} + \frac{l_2}{l_2 m - lm_2} \right]$$

$$\therefore \ \alpha = \frac{1}{3}\left[\frac{-m_1}{l_1 m - lm_1} - \frac{m_2}{l_2 m - lm_2} \right], \beta = \left[\frac{l_1}{l_1 m - lm_1} + \frac{l_2}{l_2 m - lm_2} \right]$$

Now

$$\alpha = \frac{1}{3}\left[\frac{-m_1(l_2 m - lm_2) - m_2(l_1 m - lm_1)}{(l_1 m - lm_1)(l_2 m - lm_2)} \right]$$

$$= \frac{1}{3}\left[\frac{-m_1 l_2 m + lm_1 m_2 - l_1 m m_2 + lm_1 m_2}{l_1 l_2 m^2 - (l_1 m_2 + l_2 m_1)lm + m_1 m_2 l^2} \right]$$

$$= \frac{1}{3}\left[\frac{2m_1 ml - m(l_1 m_2 + l_2 m_1)}{l_1 l_2 m^2 - (l_1 m_2 + l_2 m_1)lm + m_1 m_2 l^2} \right]$$

$$= \frac{1}{3}\left[\frac{2bl - m(2h)}{am^2 - 2hlm + bl^2} \right]$$

$$\Rightarrow \frac{\alpha}{bl - hm} = \frac{2}{3(bl^2 - 2hlm + am^2)}$$

similarly we can prove $\dfrac{\beta}{am - hl} = \dfrac{2}{3(bl^2 - 2hlm + am^2)}$

$$\therefore \ \frac{\alpha}{bl - hm} = \frac{\beta}{am - hl} = \frac{2}{3(bl^2 - 2hlm + am^2)}$$

8. Prove that the equation $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$ represents a pair of perpendicular lines.

Sol: compare the given equation $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$ with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$a = 2,\ h = \frac{3}{2},\ b = -2,\ g = \frac{3}{2},\ f = \frac{1}{2},\ c = 1$$

Now $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 2(-2)(1) + 2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right) - 2\left(\frac{1}{2}\right)^2 - (-2)\left(\frac{3}{2}\right)^2 - 1\left(\frac{3}{2}\right)^2$$

$$= -4 + \frac{9}{4} - \frac{1}{2} + \frac{9}{2} - \frac{9}{4}$$

$$= -4 + \frac{8}{2} = 0$$

also $h^2 - ab = \frac{9}{4} - 2(-2) = \frac{9}{4} + 4 = \frac{9+16}{4} = \frac{25}{4} > 0$

$$g^2 - ac = \frac{9}{4} - 2(1) = \frac{9}{4} - 2 = \frac{9-8}{4} = \frac{1}{4} > 0$$

$$f^2 - bc = \frac{1}{4} - (-2) = \frac{1}{4} + 2 = \frac{1+8}{4} = \frac{9}{4} > 0$$

$\therefore$ given equation $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$ represents a pair of straight line also coeff. x^2 + coeff. $y^2 = 2 - 2 = 0$

$\therefore$ The lines in the pair of straight lines are perpendicular.

9. Find the values of k, if the lines joining the origin to the points of intersection of the curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line x+2y=k are mutually perpendicular

Sol: Given straight line equation $x + 2y = k \Rightarrow \frac{x+2y}{k} = 1$ ..... (1)

Given curve equation $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ ..... (2)

To get the required equation by homogenising (2) with the help of equation (1)

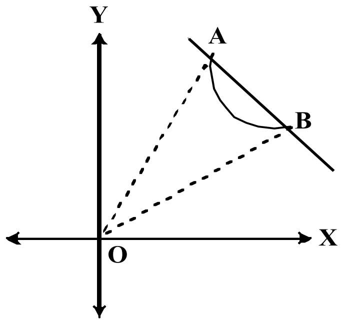
$\therefore 2x^2 - 2xy + 3y^2 + (2x - y)(1) - 1(1^2) = 0$

$$\Rightarrow 2x^2 - 2xy + 3y^2 + (2x - y)\frac{(x+2y)}{k} - 1\frac{(x+2y)^2}{k^2} = 0$$

$$\Rightarrow k^2(2x^2 - 2xy + 3y^2) + k(2x^2 + 4xy - xy - 2y^2) - (x^2 + 4y^2 + 4xy) = 0$$

$$\Rightarrow x^2(2k^2 + 2k - 1) + xy(-2k^2 + 3k - 4) + y^2(3k^2 - 2k - 4) = 0$$

If the two lines in above equation are perpendicular then



coeff. x^2 + coeff. y^2 = 0

$\Rightarrow 2k^2 + 2k - 1 + 3k^2 - 2k - 4 = 0$

$\Rightarrow 5k^2 - 5 = 0$

$\Rightarrow 5k^2 = 5$

$\Rightarrow k^2 = 1$

$\Rightarrow k = \pm 1$

10. Find the angle between the lines joining the origin to the points of intersection of the curve $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$ and the line $3x - y + 1 = 0$

Sol: Given straight line equation $3x - y + 1 = 0 \Rightarrow y - 3x = 1$ ..... (1)

Given curve equation $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$ ..... (2)

to get the equations of the lines OA, OB is obtained by homogenising the equation (2) by equation (1)

$\therefore x^2 + 2xy + y^2 + 2x + 2y - 5(1^2) = 0$

$$\Rightarrow x^2 + 2xy + y^2 + (2x + 2y)(y - 3x) - 5(y - 32)^2 = 0$$

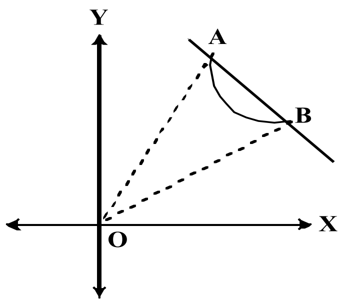
$$\Rightarrow x^2 + 2xy + y^2 + 2xy - 6x^2 + 2y^2 - 6xy - 5(y^2 + 9x^2 - 6xy) = 0$$

$$\Rightarrow x^2 + 2xy + y^2 + 2xy - 6x^2 + 2y^2 - 6xy - 5y^2 - 45x^2 + 30xy = 0$$

$$\Rightarrow -50x^2 + 28xy - 2y^2 = 0$$

$$\Rightarrow 25x^2 - 14xy + y^2 = 0$$

If θ is the angle between the lines in the above equation then



$$\cos\theta = \frac{|a + b|}{\sqrt{(a - b)^2 + 4h^2}}$$

$a = 25, h = -7, b = 1$

$$\therefore \cos\theta = \frac{|25 + 1|}{\sqrt{(25 - 1)^2 + 4(-7)^2}}$$

$$= \frac{26}{\sqrt{276 + 196}}$$

Unit **5**

Three Dimensional Coordinates

Key concepts

→ If $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ are two points in the space then distance between P and Q

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

→ The point dividing the segment $\overline{AB}$ where $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ in the ratio m:n is

(i) $$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

(ii) Mid point of $\overline{AB}$

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

→ Centroid of a triangle with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

→ Centroid of a tetrahedron whose vertices are
(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

Very Short Answer Questions (2 Marks)

1. Find x if the distance between (5, −1, 7) and (*x*, 5, 1) and (x, 5, 1) is 9 units.

Sol: Let the given points be A(5, –1, 7), B(*x*, 5, 1)

from problem, we have AB = 9

$\Rightarrow AB^2 = 81$

$\Rightarrow (x-5)^2 + (5+1)^2 + (1-7)^2 = 81$

$\Rightarrow (x-5)^2 + 36 + 36 = 81$

$\Rightarrow (x-5)^2 = 9$

$\Rightarrow (x-5) = \pm 3$

$\Rightarrow x = 5 \pm 3$

$\Rightarrow x = 8, 2$

2. Show that the points (1, 2, 3), (2, 3, 1), (3, 1, 2) form an equilateral triangle.

Sol: Let given points be A(1, 2, 3), B(2, 3, 1), C(3, 1, 2)

$$AB = \sqrt{(2-1)^2 + (3-2)^2 + (1-3)^2} = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$$

$$BC = \sqrt{(3-2)^2 + (1-3)^2 + (2-1)^2} = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

$$CA = \sqrt{(1-3)^2 + (2-1)^2 + (3-2)^2} = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$$

AB = BC = CA

$\therefore$ Δ ABC is an equilateral triangle

3. find the centroid of the triangle whose vertices are (5, 4, 6), (1, −1, 3), and (4, 3, 2)

Sol: Let given points be A $(x_1, y_1, z_1) = (5, 4, 6)$

B $(x_2, y_2, z_2) = (1, -1, 3)$

C $(x_3, y_3, z_3) = (4, 3, 2)$

$$\text{Centroid of } \Delta\, ABC = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

$$= \left(\frac{5+1+4}{3}, \frac{4-1+3}{3}, \frac{6+3+2}{3}\right)$$

$$= \left(\frac{10}{3}, 2, \frac{11}{3}\right)$$

4. Find the centroid of the tetrahedron whose vertices are (2, 3, −4), (−3, 3, −2) (−1, 4, 2) and (3, 5, 1)

Sol: Let the given points of a tetrahedron be

A $(x_1, y_1, z_1) = (2, 3, -4)$

B $(x_2, y_2, z_2) = (-3, 3, -2)$

C $(x_3, y_3, z_3) = (-1, 4, 2)$

D $(x_4, y_4, z_4) = (3, 5, 1)$

Centroid of a tetrahedron

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

$$= \left(\frac{2-3-1+3}{4}, \frac{3+3+4+5}{4}, \frac{-4-2+2+1}{4}\right) = \left(\frac{1}{4}, \frac{15}{4}, \frac{-3}{4}\right)$$

5. Find the fourth vertex of the parallelogram whose consecutive vertices are (2, 4, −1), (3, 6, −1) and (4, 5, 1)

Sol: Let ABCD be parallelogram where A(2, 4, –1), B(3, 6, –1), C(4, 5, 1) and D (a, b, c)

$\therefore$ mid point of AC = mid point of BD

$$\Rightarrow \left(\frac{2+4}{2}, \frac{4+5}{2}, \frac{-1+1}{2}\right) = \left(\frac{3+a}{2}, \frac{6+b}{2}, \frac{-1+c}{2}\right)$$

$$\Rightarrow \left(\frac{6}{2}, \frac{9}{2}, 0\right) = \left(\frac{a+3}{2}, \frac{b+6}{2}, \frac{c-1}{2}\right)$$

$$\Rightarrow \frac{a+3}{2} = \frac{6}{2}, \frac{b+6}{2} = \frac{9}{2}, \frac{c-1}{2} = 0$$

$\therefore a = 3, b = 3, c = 1 \Rightarrow 4^{th}$ vertex $D(a, b, c) = (3, 3, 1)$

6. Find the coordinates of the vertex 'C' of Δ ABC if its centroid is the origin and the vertices A, B are (1, 1, 1) and (–2, 4, 1) respectively.

Sol: Let given points be A(1, 1, 1), B(–2, 4, 1), C(x, y, z)

Given that centroid of Δ ABC = (0, 0, 0)

$$\Rightarrow \left(\frac{1-2+x}{3}, \frac{1+4+y}{3}, \frac{1+1+z}{3}\right) = (0,0,0)$$

$$\Rightarrow \left(\frac{x-1}{3}, \frac{y+5}{3}, \frac{z+2}{3}\right) = (0,0,0)$$

$$\Rightarrow \frac{a+3}{2} = \frac{6}{2}, \frac{b+6}{2} = \frac{9}{2}, \frac{c-1}{2} = 0$$

$\Rightarrow -1+x = 0, \quad 5+y = 0, \quad z+2 = 0$

$\Rightarrow x = 1, \quad y = -5, \quad z = -2 \Rightarrow$ third vertex $C(x, y, z) = (1, -5, -2)$

7. If (3, 2, –1), (4, 1, 1), (6, 2, 5) are three vertices and (4, 2, 2) is the centroid of a tetrahedron, find the fourth vertex.

Sol: Let the given points be A(3, 2, –1), B(4, 1, 1), C(6, 2, 5), D(x, y, z)

Given that centroid of ABCD tetrahedron = (4, 2, 2)

$$\Rightarrow \left(\frac{3+4+6+x}{4}, \frac{2+1+2+y}{4}, \frac{-1+1+5+z}{4}\right) = (4,2,2)$$

$$\Rightarrow \left(\frac{13+x}{4}, \frac{5+y}{4}, \frac{5+z}{4}\right) = (4,2,2) \Rightarrow \frac{13+x}{4} = 4, \frac{5+y}{4} = 2, \frac{5+z}{4} = 2$$

$\Rightarrow x = 3, \quad y = 3, \quad z = 3 \Rightarrow 4^{th}$ vertex $D(x, y, z) = (3, 3, 3)$

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Unit **6**

Direction Cosines and Direction Ratios

Key concepts

→ A ray $\overrightarrow{OP}$ passing through origin Oand making angles α, β, γ respectively with $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ then the numbers cosα, cosβ, cosγ are called the direction cosines of the ray $\overrightarrow{OP}$. Usually they are denoted by (l, m, n) where $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

→ If (l, m, n) are direction cosines of $\overrightarrow{OP}$. then direction cosines of $\overrightarrow{PO}$ are $(-l, -m, -n)$

→ If (l, m, n) are direction cosines of a line then $l^2 + m^2 + n^2 = 1$

→ Triad of numbers (a, b, c) proportional to direction cosines of a line are called its direction ratios.

→ Direction cosines of a line whose direction ratios (a, b, c) are

$$\pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

→ Direction ratios of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2)

$(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ and its direction cosines are

$$\left(\frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}\right)$$

→ If θ is angle between two lines whose direction cosines are (l_1, m_1, n_1), (l_2, m_2, n_2) then $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

→ If θ is angle between two lines whose direction ratios are (a_1, b_1, c_1), (a_2, b_2, c_2) then

$$\cos\theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Long Answer Questions (7 Marks)

1. Find the direction cosines of two lines which are connected by the relations $l + m + n = 0$ and $mn - 2nl - 2lm = 0$

Sol: Given that $l + m + n = 0$ ....(1)

$mn - 2nl - 2lm = 0$ ....(2)

$l = -m - n$

From (1) $l = -m + n$ substituting l value in (2) we get

$$mn - 2n(-m - n) - 2(-m - n)m = 0$$

$$\Rightarrow mn + 2mn + 2n^2 + 2m^2 + 2mn = 0$$

$$\Rightarrow 2m^2 + 4mn + mn + 2n^2 = 0$$

$$\Rightarrow 2m(m + 2n) + n(m + 2n) = 0$$

$$\Rightarrow (m + 2n)(2m + n) = 0$$

$$\Rightarrow m + 2n = 0 \text{ or } 2m + n = 0$$

Now $2m + n = 0 \Rightarrow n = -2m$

from (1) $l = -m - (-2m) = m$

$\therefore l : m : n = m : m : -2m = 1:1:-2$

directional cosines $\left(\frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, \frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, \frac{-2}{\sqrt{1^2 + 1^2 + (-2)^2}}\right)$

$$= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

Now $m + 2n = 0 \Rightarrow m = -2n$

from (1) $l = -m - n = 2n - n = n$

$\therefore l : m : n = n : -2n : n = 1 : -2 : 1$

Direction cosines of a line is $\left(\frac{1}{\sqrt{1^2 + (-2)^2 + 1^2}}, \frac{-2}{\sqrt{1^2 + (-2)^2 + 1^2}}, \frac{1}{\sqrt{1^2 + (-2)^2 + 1^2}}\right)$

$$= \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

Thus the d.c's of the two lines are $= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right) = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

2. Find the direction cosines of two lines which are connected by the relations $l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$

Sol: Given that $l - 5m + 3n = 0$ ....(1)

$7l^2 + 5m^2 - 3n^2 = 0$ ....(2)

From (1) $l = 5m - 3n$ substitute this value in (2)

$$7(5m-3n)^2 + 5m^2 - 3n^2 = 0$$

$$\Rightarrow 7(25m^2 + 9n^2 - 30mn) + 5m^2 - 3n^2 = 0$$

$$\Rightarrow 175m^2 + 63n^2 - 210mn + 5m^2 - 3n^2 = 0$$

$$\Rightarrow 180m^2 - 210mn + 60n^2 = 0$$

$$\Rightarrow 6m^2 - 7mn + 2n^2 = 0$$

$$\Rightarrow (3m-2n)(2m-n) = 0$$

$\Rightarrow 3m - 2n = 0$ లేదా $2m - n = 0$

Now $3m - 2n = 0 \Rightarrow m = \frac{2}{3}n$

From (1) $l = 5m - 3n = \frac{10}{3}n - 3n = \frac{10n - 9n}{3} = \frac{n}{3}$

$\therefore l : m : n = \frac{n}{3} : \frac{2}{3}n : n = \frac{1}{3} : \frac{2}{3} : 1 = 1 : 2 : 3$

Direction cosines $\left(\frac{1}{\sqrt{1^2+2^2+3^2}}, \frac{2}{\sqrt{1^2+2^2+3^2}}, \frac{3}{\sqrt{1^2+2^2+3^2}}\right)$

$$= \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

similarly $2m - n = 0 \Rightarrow n = 2m$

(1) నుండి $l = 5m - 3(2m) = 5m - 6m = -m$

$\therefore l : m : n = -m : m : 2m = -1:1:2$

Direction cosines of another line $\left(\frac{-1}{\sqrt{(-1)^2+1^2+2^2}}, \frac{1}{\sqrt{(-1)^2+1^2+2^2}}, \frac{2}{\sqrt{(-1)^2+1^2+2^2}}\right)$

$$= \left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

3. Find the angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$

Sol: Given equations $l + m + n = 0$ ....(1)

$l^2 + m^2 - n^2 = 0$ ....(2)

From (1) $l = -m - n$ substitute this value in (2)

$$(-m-n)^2 + m^2 - n^2 = 0$$

$$\Rightarrow m^2 + n^2 + 2mn + m^2 - n^2 = 0$$

$$\Rightarrow 2m^2 + 2mn = 0$$

$$\Rightarrow 2m(m+n) = 0$$

$$\Rightarrow m = 0 \text{ or } m + n = 0$$

Now if $m = 0$ From (1) $l = 0 - n = -n$

$\therefore\ l : m : n = -n : 0 : n = -1 : 0 : 1$

direction ratios of a line $(a_1, b_1, c_1) = (-1,0,1)$

$m + n = 0 \Rightarrow m = -n$

$l = -m - n = -(-n) - n = n - n = 0$

$\therefore l : m : n = 0 : -n : n = 0 : -1 : 1$

direction ratios of another line $(a_2, b_2, c_2) = (0,-1,1)$

Let θ be the angle between two lines then

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{|0\times(-1) + 0\times(-1) + 1\times 1|}{\sqrt{1+0+1}\sqrt{1+0+1}}$$

$$= \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^{\circ} = \frac{\pi}{3}$$

4. Find the angle between the lines whose direction cosines are given by the equations $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$

Sol: Given equation $3l + m + 5n = 0$ ....(1)

$6mn - 2nl + 5lm = 0$ ....(2)

From (1) $m = -3l - 5n$ substitute this value in (2)

$$6(-3l-5n)n - 2nl + 5l(-3l-5n) = 0$$

$$\Rightarrow -18ln - 30n^2 - 2nl - 15l^2 - 25ln = 0$$

$$\Rightarrow -15l^2 - 45ln - 30n^2 = 0$$

$$\Rightarrow l^2 + 3ln + 2n^2 = 0$$

$$\Rightarrow (l+2n)(l+n) = 0$$

$\Rightarrow l + 2n = 0$ లేదా $l + n = 0$

Now $l + 2n = 0 \Rightarrow l = -2n$

From (1) $m = -3l - 5n = 6n - 5n = n$

$\therefore\ l : m : n = -2n : n : n = -2:1:1$

Direction ratios of a line $(a_1, b_1, c_1) = (-2, 1, 1)$

also $l + n = 0 \Rightarrow l = -n$

from (1) $m = -3l - 5n = -3(-n) - 5n = 3n - 5n = -2n$

$\therefore\ l : m : n = -n : -2n : n = -1:-2:1 = 1:2:-1$

direction ratios of another line $(a_2, b_2, c_2) = (1, 2, -1)$

If θ is angle between two lines then

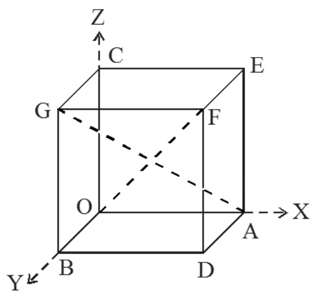
$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{|(-2)\times 1 + 1\times 2 + 1\times(-1)|}{\sqrt{(-2)^2 + 1^2 + 1^2}\sqrt{1^2 + 2^2 + (-1)^2}}$$

$$= \frac{|-1|}{\sqrt{6}\sqrt{6}} = \frac{1}{6}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{1}{6}\right)$$

5. Find the angle between two diagonals of a cube

Sol: Let 'O', One of the vertices of the cube, be taken as the origin and the three coterminus edges $\overline{OA}, \overline{OB}, \overline{OC}$ as coordinate axes.



let $\overline{OA} = \overline{OB} = \overline{OC} = a$

Coordinates of vertices of cube O(0, 0, 0), A(a, 0, 0), B(0, a, 0)

C(0, 0, a), D(a, a, 0), E (a, 0, a), F(a, a, a), G (0, a, a)

Also four diagonals of cube are $\overline{OF}, \overline{AG}, \overline{BE}, \overline{CD}$

Now direction ratio's of diagonal $\overline{OF}$ are $(a - 0, a - 0, a - 0) = (a, a, a)$

direction ratio's of diagonal $\overline{AG}$ $(0 - a, a - 0, a - 0) = (-a, a, a)$

If θ is the angle between $\overline{OF}, \overline{AG}$ then by

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{|a(-a) + a(a) + a(a)|}{\sqrt{a^2 + a^2 + a^2}\sqrt{a^2 + a^2 + a^2}}$$

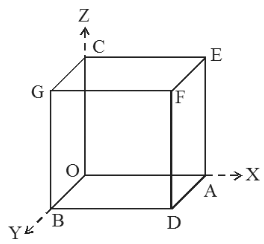
$$= \frac{a^2}{\sqrt{3}a.\sqrt{3}a} = \frac{1}{3}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

6. If a ray makes angles α, β, γ, δ with the four diagonals of a cube find $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$.

Sol: Let 'O', One of the vertices of the cube, be taken as the origin and the three coterminus edges $\overline{OA}, \overline{OB}, \overline{OC}$ as coordinate axes.

let $\overline{OA} = \overline{OB} = \overline{OC} = a$



Coordinates of vertices of cube O(0, 0, 0), A(a, 0, 0), B(0, a, 0), C(0, 0, a), D(a, a, 0), E (a, 0, a), F(a, a, a), G (0, a, a)

Also four diagonals of cube are $\overline{OF}, \overline{AG}, \overline{BE}, \overline{CD}$

Now direction ratio's of diagonal $\overline{OF}$ are $(a-0, a-0, a-0) = (a, a, a)$

direction ratio's of diagonal $\overline{AG}$ $(0-a, a-0, a-0) = (-a, a, a)$

direction ratio's of diagonal $\overline{BE}$ $(a-0, a-0, a-0) = (a, -a, a)$

direction ratio's of diagonal $\overline{CD}$ $(a-0, a-0, 0-a) = (a, a, -a)$

If any ray having direction ratio's (l, m, n) and makes an angle with four diagonals $\overline{OF}, \overline{AG}, \overline{BE}, \overline{CD}$ are α, β, γ, δ then

$$\cos\alpha = \frac{|al+am+an|}{\sqrt{a^2+a^2+a^2}\sqrt{l^2+m^2+n^2}} = \frac{|a(l+m+n)|}{\sqrt{3a^2}\sqrt{l^2+m^2+n^2}}$$

$$\therefore \cos^2\alpha = \frac{a^2(l+m+n)^2}{3a^2(l^2+m^2+n^2)} = \frac{(l+m+n)^2}{3(l^2+m^2+n^2)}$$

similarly

$$\cos^2\beta = \frac{(-l+m+n)^2}{3(l^2+m^2+n^2)}, \cos^2\gamma = \frac{(l-m+n)^2}{3(l^2+m^2+n^2)}, \cos^2\delta = \frac{(l+m-n)^2}{3(l^2+m^2+n^2)}$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{(-l+m+n)^2+(l-m+n)^2+(l+m-n)^2+(l+m+n)^2}{3(l^2+m^2+n^2)}$$

simplification

$$= \frac{4l^2+4m^2+4n^2}{3(l^2+m^2+n^2)} = \frac{4}{3}\frac{(l^2+m^2+n^2)}{(l^2+m^2+n^2)} = \frac{4}{3}$$

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Unit **7**

The Plane

Key concepts

→ A plane is a surface with at least three non collinear points such that the line joining any two points on the surface lies entirely on it.

→ Equation of the plane in normal form is $lx + my + nz = p$ where (l, m, n) are direction cosines of the normal to the plane and 'p' is the perpendicular distance to the plane from the origin.

→ General equation of the plane is $ax + by + cz + d = 0$ where (a, b, c) are direction ratios of the normal to the plane.

→ Equation of the plane in intercept form $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ where a, b, c are intercepts on X, Y, Z – axes respectively.

→ Angle between two planes is the angle between their normals. If θ is the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ then

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(i) If $a_1a_2 + b_1b_2 + c_1c_2 = 0$ then planes are perpendicular.

(ii) If $\frac{a_1}{a_1} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ then the planes are parallel.

Very Short Answer Questions (2 Marks)

1. Reduce the equation $x + 2y - 3z - 6 = 0$ of the plane to the normal form.

Sol: Given equation of the plane $x + 2y - 3z - 6 = 0$

dividing on both sides by $\sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$

$$\frac{x+2y-3z}{\sqrt{14}} = \frac{6}{\sqrt{14}}$$

$$\Rightarrow \frac{1}{\sqrt{14}}x + \frac{2}{\sqrt{14}} - \frac{3}{\sqrt{14}} = \frac{6}{\sqrt{14}}$$

2. Find the equation of the plane whose intercepts on X, Y, Z – axes are 1, 2, 4 respectively.

Sol: Let the intercepts on X, Y, Z – axes are $a = 1$, $b = 2$, $c = 4$ respectively then equation of the plane in intercept form is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$\Rightarrow \frac{x}{1}+\frac{y}{2}+\frac{z}{4}=1$$

$$\Rightarrow 4x+2y+z=4$$

3. Write the equation of the plane $4x - 4y + 2z + 5 = 0$ in the intercept form.

Sol: Given equation of the plane $4x - 4y + 2z + 5 = 0$

$$\Rightarrow 4x-4y+2z=-5$$

$$\Rightarrow \frac{4x-4y+2z}{5}=\frac{-5}{5}$$

$$\Rightarrow \frac{-4}{5}x+\frac{4}{5}y-\frac{2}{5}z=1$$

$$\Rightarrow \frac{x}{-5/4}+\frac{y}{5/4}+\frac{z}{-5/2}=1$$

4. Find the angle between the planes $x + 2y + 2z - 5 = 0$ and $3x + 3y + 2z - 8 = 0$

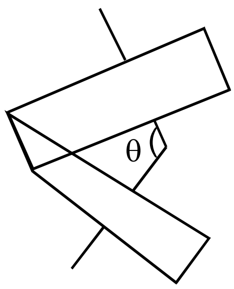
Sol: Given equation of the plane $x + 2y + 2z - 5 = 0$ .....(1)

$$a_1 = 1, b_1 = 2, c_1 = 2$$

and given another plane equation $3x + 3y + 2z - 8 = 0$ .....(2)

$$a_2 = 3, b_2 = 3, c_2 = 2$$

Let θ be the angle between the planes (1) and (2)



$$\text{then } \cos\theta=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$=\frac{|1\times 3+2\times 3+2\times 2|}{\sqrt{1^2+2^2+2^2}\sqrt{3^2+3^2+2^2}}$$

$$=\frac{|3+6+4|}{\sqrt{1+4+4}\sqrt{9+9+4}}=\frac{13}{3\sqrt{22}}$$

$$\therefore \theta=\cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$$

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Unit **8**

Limits

Key concepts

* $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a} = na^{n-1}$
* $\underset{x \to 0}{Lt} \dfrac{\tan x}{x} = 1$ (x is in radians)
* $\underset{x \to \infty}{Lt} \left(1 + \dfrac{1}{x}\right)^x = e$
* $\underset{x \to 0}{Lt} \dfrac{x^n - a^n}{x^m - a^m} = \dfrac{n}{m} a^{n-m}$
* $\underset{x \to 0}{Lt} \dfrac{\tan ax}{x} = a$ (x is in radians)
* $\underset{x \to 0}{Lt} \dfrac{\sin x}{x} = 1$ (x is in radians)
* $\underset{x \to 0}{Lt} (1+x)^{1/x} = e$
* $\underset{x \to 0}{Lt} \left(\dfrac{a^x - 1}{x}\right) = \log_e a$
* $\underset{x \to 0}{Lt} \dfrac{\sin ax}{x} = a$ (x is in radians)
* $\underset{x \to 0}{Lt} \dfrac{e^x - 1}{x} = 1$

Very Short Answer Questions (2 Marks)

1. $\underset{x \to 0}{Lim} \dfrac{e^{x+3} - c^3}{x}$

A. $= \underset{x \to 0}{Lim} \dfrac{e^x . e^3 - e^3}{x}$ $\boxed{a^{m+n} = a^m . a^n}$

$= \underset{x \to 0}{Lim} \; e^3 \dfrac{(e^x - 1)}{x}$

$= e^3 \underset{x \to 0}{Lim} \dfrac{e^x - 1}{x}$

$= e^3 .1 = e^3$

2. $\underset{x \to 0}{Lim} \dfrac{e^x - \sin x - 1}{x}$

A. $\underset{x \to 0}{Lim} \left(\dfrac{e^x - 1}{x} - \dfrac{\sin x}{x}\right)$

$$= \lim_{x \to 0} \frac{e^x - 1}{x} - \lim_{x \to 0} \frac{\sin x}{x}$$

$= 1 - 1 = 0$

3. $\underset{x \to 0}{Lt} \dfrac{\sin ax}{x \cos x}$

A. $$\underset{x \to 0}{Lt} \frac{\sin ax}{x \cos x} = \frac{\underset{ax \to 0}{Lt} \frac{\sin ax}{ax} a}{\underset{x \to 0}{Lt} \cos x} = \frac{a}{1} = a$$

4. $\underset{x \to \infty}{Lt} \dfrac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$

A. $\underset{x \to \infty}{Lt} \dfrac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$

$$= \underset{x \to \infty}{Lt} \frac{x^3\left(11 - \frac{3}{x^2} + \frac{4}{x^3}\right)}{x^3\left(13 - \frac{5}{x} - \frac{7}{x^3}\right)}$$

As $x \to \infty, \frac{1}{x}, \frac{1}{x^2}$ and $\frac{1}{x^3} \to 0$

$$= \underset{x \to \infty}{Lt} \frac{11 - \frac{3}{x^2} + \frac{4}{x^3}}{13 - \frac{5}{x} - \frac{7}{x^3}} = \frac{11 - 0 + 0}{13 - 0 - 0} = \frac{11}{13}$$

5. $\underset{x \to 1}{Lt} \dfrac{\sin(x-1)}{(x^2 - 1)}$

A. $\underset{x \to 1}{Lt} \dfrac{\sin(x-1)}{(x^2 - 1)}$

$$\underset{x \to 1}{Lt} \frac{\sin(x-1)}{(x-1)} . \underset{x \to 1}{Lt} \frac{1}{x+1}$$

Put $y \equiv x - 1$ so that as $x \to 1, y \to 0$

$$\boxed{\underset{x \to 1}{Lt} \frac{\sin(x-1)}{(x-1)} = \underset{y \to 0}{Lt} \frac{\sin y}{y} = 1}$$

$$= 1 . \frac{1}{1+1} = \frac{1}{2}$$

6. $\underset{x \to 0}{Lim} \dfrac{\sin a x}{\sin b x} \quad b \neq 0, \; a \neq b$

A. $\underset{x \to 0}{Lim} \dfrac{\sin a x}{\sin b x} = \underset{x \to 0}{Lim} \dfrac{\dfrac{\sin ax}{ax} \times ax}{\dfrac{\sin bx}{bx} \times bx}$

$$= \frac{\underset{ax \to 0}{Lim} \dfrac{\sin ax}{ax}}{\underset{bx \to 0}{Lim} \dfrac{\sin bx}{bx}} \frac{a}{b}$$

$$= \frac{1}{1} \frac{a}{b} = \frac{a}{b}$$

7. $\underset{x \to \infty}{Lt} \dfrac{8|x|+3x}{3|x|-2x}$

A. $x \to \infty \Rightarrow x > 0 \therefore |x| = x$

$\underset{x \to \infty}{Lt} \dfrac{8|x|+3x}{3|x|-2x} = \underset{x \to \infty}{Lt} \dfrac{8x+3x}{3x-2x}$

$$= \underset{x \to \infty}{Lt} \frac{11x}{x} = 11$$

8. $\underset{x \to 0}{Lim} \dfrac{a^x - 1}{b^x - 1}$

A. $\underset{x \to 0}{Lim} \dfrac{a^x - 1}{b^x - 1} = \underset{x \to 0}{Lim} \dfrac{\dfrac{a^x - 1}{x}}{\dfrac{b^x - 1}{x}}$

$$= \frac{\underset{x \to 0}{Lim} \dfrac{a^x - 1}{x}}{\underset{x \to 0}{Lim} \dfrac{b^x - 1}{x}}$$

$$= \frac{\log_e a}{\log_e b}$$

$$= \log_b a$$

9. $\underset{x\to 0}{Lt}\dfrac{e^{7x}-1}{x}$

A. As $x\to 0$

$\Rightarrow 7x\to 0$

$\underset{7x\to 0}{Lt}\dfrac{e^{7x}-1}{7x}\times 7=7\left(\because \underset{x\to 0}{Lt}\dfrac{e^{x}-1}{x}=1\right)$

10. $\underset{x\to 0}{Lt}\left(\dfrac{\sqrt{1+x}-1}{x}\right)$

A. For $0<|x|<1$, we have

$\dfrac{\sqrt{x+1}-1}{x}=\dfrac{\sqrt{1+x}-1}{x}.\dfrac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$=\dfrac{1+x-1}{x\left(\sqrt{1+x}+1\right)}=\dfrac{x}{x\left(\sqrt{1+x}+1\right)}$

$\Rightarrow \underset{x\to 0}{Lt}\dfrac{\sqrt{1+x}-1}{x}=\underset{x\to 0}{Lt}\dfrac{1}{\sqrt{1+x}+1}$

$=\dfrac{1}{1+1}=\dfrac{1}{2}$

11. $\underset{x\to 0}{Lim}\dfrac{e^{\sin x}-1}{x}$

A. $\underset{x\to 0}{Lim}\dfrac{e^{\sin x}-1}{x}=\underset{x\to 0}{Lim}\dfrac{\dfrac{e^{\sin x}-1}{\sin x}}{\dfrac{x}{\sin x}}$

$=\underset{x\to 0}{Lim}\dfrac{e^{\sin x}-1}{\sin x}\ \underset{x\to 0}{Lim}\dfrac{\sin x}{x}$

$=\underset{\sin x\to 0}{Lim}\dfrac{e^{\sin x}-1}{\sin x}.1$

$= 1.1 = 1$

12. $\underset{x\to 0}{Lim}\dfrac{\log(1+5x)}{x}$

A. $\underset{x\to 0}{Lim}\dfrac{\log(1+5x)}{x}=\underset{x\to 0}{Lim}\dfrac{1}{x}.\log(1+5x)=\underset{x\to 0}{Lim}\log(1+5x)^{\frac{1}{x}}$ $\boxed{m\log x=\log x^{m}}$

$=\log\left(\underset{5x\to 0}{Lim}(1+5x)^{\frac{1}{5x}}\right)^{5}$

$= \log_e e^5$

$= 5\log_e e$

$= 5 \quad [\because \log_e e = 1]$

13. Show that $\underset{x \to 0}{Lt}\left(\frac{2|x|}{x} + x + 1\right) = 3$

A. $x \to 0+ \Rightarrow x > 0$

$|x| = x$

$= \underset{x \to 0}{Lt}\left(\frac{2|x|}{x} + x + 1\right)$

$= \underset{x \to 0}{Lt}\left(\frac{2x}{x} + x + 1\right)$

$= 2 + 0 + 1 = 3$

Short Answer Questions (4 Marks)

1. $\underset{x \to 3}{Lim} \frac{x^2 - 8x + 15}{x^2 - 9}$

A. $\underset{x \to 3}{Lim} \frac{x^2 - 5x - 3x + 15}{(x-3)(x+3)}$

$= \underset{x \to 3}{Lim} \frac{(x-5)\cancel{(x-3)}}{\cancel{(x-3)}(x+3)}$

$= \frac{-\cancel{2}}{\cancel{6}\,3}$

$= \frac{-1}{3}$

2. Compute $\underset{x \to 0}{Lt}\left[\frac{(1+x)^{\frac{1}{3}} - (1-x)^{\frac{1}{3}}}{x}\right]$

A. $\underset{x \to 0}{Lt} \frac{(1+x)^{\frac{1}{3}} - (1-x)^{\frac{1}{3}}}{x}$

$= \underset{x \to 0}{Lt} \frac{(1+x)^{\frac{1}{3}} - 1 - (1-x)^{\frac{1}{3}} + 1}{x}$

$= \underset{x \to 0}{Lt} \frac{(1+x)^{\frac{1}{3}} - 1 - \left((1-x)^{\frac{1}{3}} - 1\right)}{x}$

$$= \underset{(1+x)\to 1}{Lt} \frac{(1+x)^{\frac{1}{3}}-1}{(1+x)-1} + \underset{(1-x)\to 1}{Lt} \frac{(1-x)^{\frac{1}{3}}-1^{\frac{1}{3}}}{(1-x)-1}$$

$$= \frac{1}{3}1^{-2/3} + \frac{1}{3}1^{-2/3} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

3. $\underset{x\to a}{Lt} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

A. $\underset{x\to a}{Lt} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} = \underset{x\to a}{Lt} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \times \dfrac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}} \times \dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}$

$$= \underset{x\to a}{Lt} \frac{(a+2x-3x)}{3a+x-4x} \cdot \frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{3a+x}+2\sqrt{x}}$$

$$= \underset{x\to a}{Lt} \frac{a-x}{3(a-x)} \quad \underset{x\to a}{Lt} \frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{3a+x}+2\sqrt{x}}$$

$$= \frac{1}{3}.\frac{\cancel{2}\sqrt{3a}}{\cancel{4}\sqrt{a}} \qquad = \frac{1}{2\sqrt{3}}$$

4. $\underset{x\to 0}{Lim} \dfrac{\cos ax - \cos bx}{x^2}$

A. $\underset{x\to 0}{Lim} \dfrac{\cos ax - \cos bx}{x^2} = \underset{x\to 0}{Lim} \dfrac{-2\sin\frac{ax+bx}{2}\sin\frac{ax-bx}{2}}{x^2}$ $\boxed{\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}}$

$$= -2 \underset{x\to 0}{Lim} \frac{\sin\frac{(a+b)x}{2}}{x} \; \frac{\sin\frac{(a-b)x}{2}}{x}$$

$$= -2 \underset{x\to 0}{Lim} \frac{\sin\frac{(a+b)x}{2}}{\frac{(a+b)}{2}x}\left(\frac{a+b}{2}\right) \underset{x\to 0}{Lim} \frac{\sin\frac{(a-b)x}{2}}{\frac{(a-b)}{2}x}\left(\frac{a-b}{2}\right)$$

$$= -2 \underset{\frac{a+b}{2}x\to 0}{Lim} \frac{\sin\left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x}\frac{(a+b)}{2} \underset{\frac{a-b}{2}x\to 0}{Lim} \frac{\sin\left(\frac{a-b}{2}\right)x}{\frac{(a-b)}{2}x}\left(\frac{a-b}{2}\right)$$

$$= -2.1\left(\frac{a+b}{2}\right).1\frac{(a-b)}{2}$$

$$= -\frac{(a+b)(a-b)}{2}$$

$$= \frac{b^2 - a^2}{2}$$

5. $\lim_{x \to 0} \frac{1-\cos 2mx}{\sin^2 nx} (m, n \in -z)$

A. $\lim_{x \to 0} \frac{1-\cos 2mx}{\sin^2 nx} = \lim_{x \to 0} \frac{2\sin^2 mx}{\sin^2 nx}$

$$= 2\frac{\left(\lim_{x \to 0} \frac{\sin mx}{mx}\right)^2}{\left(\lim_{x \to 0} \frac{\sin nx}{nx}\right)^2} = \frac{2m^2}{n^2}$$

6. $\lim_{x \to 0} \frac{\sin(a+bx) - \sin(a-bx)}{x}$

A. Since $\boxed{\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}}$, $\lim_{x \to 0} \frac{\sin(a+bx) - \sin(a-bx)}{x}$

$$= \lim_{x \to 0} 2\cos\frac{a+bx+a-bx}{2} \frac{\sin\frac{(a+bx)-(a-bx)}{2}}{x}$$

$$= 2\cos\frac{2a}{2} \lim_{x \to 0} \frac{\sin bx}{bx}.b$$

$$= 2\cos a \times b \qquad = 2b\cos a$$

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Unit **9**

Differentiation

Key concepts

* $\frac{d}{dx}(uv) = u\frac{du}{dx} + v\frac{dv}{dx}$
* $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
* $\frac{d}{dx}(x^n) = nx^{n-1}$
* $\frac{d}{dx}(x) = 1$
* $\frac{d}{dx}(\sin x) = \cos x$
* $\frac{d}{dx}(\cos x) = -\sin x$
* $\frac{d}{dx}(\tan x) = \sec^2 x$
* $\frac{d}{dx}(\cot x) = -co\sec^2 x$
* $\frac{d}{dx}(\sec x) = \sec x \tan x$
* $\frac{d}{dx}(co\sec x) = -\cos ecx \cot x$
* $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
* $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
* $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
* $\frac{d}{dx}(co\sec^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$
* $\frac{d}{dx}(e^x) = e^x$
* $\frac{d}{dx}(a^x) = a^x \log a$
* $\frac{d}{dx}(\log x) = \frac{1}{x}$
* $\frac{d}{dx}(\sinh x) = \cosh x$
* $\frac{d}{dx}(\cosh x) = \sinh x$
* $\frac{d}{dx}(\tanh x) = \sec h^2 x$
* $\frac{d}{dx}(\coth x) = -\cos ech^2 x$
* $\frac{d}{dx}(\sec hx) = -\sec hx \tanh x$
* $\frac{d}{dx}(\cos echx) = -\cos echx \coth x$
* $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$
* $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
* $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$
* $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$
* $\frac{d}{dx}(\sec h^{-1} x) = \frac{-1}{|x|\sqrt{1-x^2}}$
* $\frac{d}{dx}(\cos ech^{-1} x) = \frac{-1}{|x|\sqrt{x^2+1}}$

Very Short Answer Questions (2 Marks)

1. $\mathbf{y = \log(\sin(\log x)),}$ **then find** $\mathbf{\frac{dy}{dx}}$

A. $v = \log x, u = \sin v \quad y = \log u$

$$\frac{dy}{du} = \frac{1}{u}; \frac{du}{dv} = \cos v; \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv};\frac{dv}{dx}$$

$$= \frac{1}{\sin(\log x)}.\cos(\log x)\frac{1}{x} = \frac{\cot(\log x)}{x}$$

2. $\mathbf{f(x) = \log(\sec x + \tan x),}$ **then find** $\mathbf{f'(x)}$

A. $u = \sec x + \tan x$ and $y = \log u$

$$\frac{dy}{du} = \frac{1}{u}, \frac{du}{dx} = \sec x.\tan x + \sec^2 x$$

$$= \sec x(\sec x + \tan x)$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$= \frac{1}{\sec x + \tan x}.\sec x(\sec x + \tan x) = \sec x$$

3. Find the derivatives of the following functions.

(i) $\frac{d}{dx}\log(\tan 5x)$ (ii) $\frac{d}{dx}\cos[\log x + e^x]$ (iii) $x = e^{\sinh y}$ (iv) $\frac{d}{dx}\sin(\cos(x^2))$

A. (i) $$\frac{d}{dx}\log(\tan 5x) = \frac{1}{\tan 5x}\frac{d}{dx}\tan 5x$$

$$= \frac{1}{\tan 5x}\sec^2 5x\frac{d}{dx}5x$$

$$= 5\frac{\sec^2 5x}{\tan 5x}$$

(ii) $$\frac{d}{dx}\cos[\log x + e^x]$$

$$= -\sin[\log x + e^x]\frac{d}{dx}[\log x + e^x]$$

$$= -\sin[\log x + e^x]\left[\frac{1}{x} + e^x\right]$$

(iii) $x = e^{\sinh y}$

$$\frac{dx}{dy} = \frac{d}{dy}e^{\sinh y}$$

$$\frac{dx}{dy} = e^{\sinh y}\frac{d}{dy}\cosh y$$

$$\frac{dx}{dy} = e^{\sinh y}\cosh y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^{\sinh y}\cosh y}$$

(iv) $\frac{d}{dx}\sin(\cos(x^2))$

$$\cos(\cos(x^2))\frac{d}{dx}\cos x^2$$

$$\cos(\cos(x^2))(-\sin x^2)\frac{d}{dx}x^2$$

$$-\cos(\cos(x^2))\sin(x^2)2x$$

4. $\mathbf{f(x) = 1 + x + x^2 + ... + x^{100}}$ **, then find** $\mathbf{f^1(1)}$

A. Differentiate both sides with respect to ‘x’

$$f'(x) = 1 + 2x + 3x^2..... + 100x^{99}$$

$$f'(1) = 1 + 2 + 3.......... + 100$$

$$= \frac{100 \times 101}{2} = 5050\left(\Sigma x = \frac{x(x+1)}{2}\right)$$

5. $\mathbf{f(x) = xe^x \sin x}$ **then find** $\mathbf{f^1(x)}$

A. Differentiate both sides with respect to ‘x’

$$\frac{d}{dx}f(x) = \frac{d}{dx}xe^x \sin x$$

$$f^1(x) = xe^x\frac{d}{dx}\sin x + xsisnx\frac{d}{dx}e^x + \sin x\frac{dx}{dx}.e^x$$

$$= xe^x \cos x + x\sin xe^x + e^x \sin x$$

6. $\mathbf{f(x) = e^x,\ g(x) = \sqrt{x}}$**, Find the derivatives of the f(x) with respect to g(x)**

A. $f(x) = e^x \quad g(x) = 5x$

$$\Rightarrow \frac{d}{dx}f(x) = \frac{d}{dx}e^x$$

$$\frac{d}{dx}g(x) = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$\therefore \frac{df(x)}{dg(x)} = \frac{\frac{df(x)}{dx}}{\frac{dg(x)}{dx}} = \frac{e^x}{\frac{1}{2\sqrt{x}}} = 2\sqrt{x}e^x$$

7. $\mathbf{x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}}$ **then show that** $\mathbf{\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}}$

A. Differentiate both sides with respect to 'x'

$$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx} = 0$$

$$x^{-\frac{1}{3}} + y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

8. **y = log (cosh2x) then find** $\mathbf{\frac{dy}{dx}}$.

A. Differentiate both sides with respect to 'x'

$$\frac{d}{dx}y = \frac{d}{dx}\log(\cosh 2x)$$

$$= \frac{1}{\cosh 2x}\frac{d}{dx}\cosh 2x$$

$$= \frac{\sinh 2x}{\cosh 2x}\frac{d}{dx}2x$$

$$= 2\tanh 2x$$

9. $\mathbf{x = a\cos^3 t,\ y = a\sin^3 t}$ **then find** $\mathbf{\frac{dy}{dx}}$

A. Differentiate both sides with respect to 'x'

$$\frac{dx}{dt} = -3a\cos^2 t\sin t;\ \frac{dy}{dt} = 3\sin^2 t\cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$$= \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t}$$

$$= -\tan t$$

Short Answer Questions (4 Marks)

1. Find the derivatives ot the following functions from the first principles.

A. (i) $f(x) = \sin 2x$

$f(x+h) = \sin 2(x+h) = \sin(2x+2h)$

First principle

$$f^1(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$$

$$f^1(x) = \underset{h\to 0}{Lt} \frac{\sin(2x+2h)-\sin 2x}{h}$$

$$\boxed{\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}}$$

$$= \underset{h\to 0}{Lt}\, 2\frac{\cos\left(\frac{2x+2h+2h}{2}\right)\sin\left(\frac{2x+2h-2x}{2}\right)}{h}$$

$$= 2\underset{h\to 0}{Lt}\cos\frac{4x+2h}{2}\underset{h\to 0}{Lt}\frac{\sin\frac{2h}{2}}{h}$$

$$= 2\cos\frac{4x}{2}.1$$

$$\therefore \frac{d}{dx}\sin 2x \quad = 2\cos 2x$$

[**Practice:** sin x, cos x]

(ii) $f(x) = \tan 2x$

$f(x+h) = \tan 2(x+h) = \tan(2x+2h)$

First principle

$$f^1(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{Lim}\frac{\tan(2x+2h)-\tan 2x}{h}$$

$$\underset{h\to 0}{Lim}\frac{\frac{\sin(2x+2h)}{\cos(2x+2h)}-\frac{\sin 2x}{\cos 2x}}{h}$$

$$\underset{h\to 0}{Lim}\frac{\frac{\sin(2x+2h)\cos 2x-\cos(2x+2h)\sin 2x}{\cos(2x+2h)\cos 2x}}{h}$$

$$\operatorname*{Lim}_{h\to 0}\frac{\sin[2x+2h-2h]}{h\cos(2x+2h)\cos 2x}$$

$$\operatorname*{Lim}_{2h\to 0}\frac{\sin 2h}{2h}\times 2\operatorname*{Lim}_{h\to 0}\frac{1}{\cos(2x+2h)\cos 2x}$$

$$1\times 2\frac{1}{\cos^2 2x} \qquad 2\sec^2 2x$$

(iii) $f(x)=ax^2+bx+c$

First principle

$$f^1(x)=\operatorname*{Lt}_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\operatorname*{Lim}_{h\to 0}\frac{[a(x+h)^2+b(x+h)+c]-(ax^2+bx+c)}{h}$$

$$=\operatorname*{Lim}_{h\to 0}\frac{a(x^2+2hx+h^2)+b(x+b)-(ax^2+bx+c)}{h}$$

$$=\operatorname*{Lim}_{h\to 0}\frac{ax^2+2hax+ah^2+bx+bh+c-ax^2-bx-c}{h}$$

$$=\operatorname*{Lim}_{h\to 0}\frac{h(2x+b+ah)}{h} \qquad \therefore \frac{d}{dx}f(x)=2ax+b$$

2. $\mathbf{x^3+y^3-3axy=0}$ **then find** $\mathbf{\frac{dy}{dx}}$

A. Differentiate both sides with respect to ‘x’

$$\frac{dy}{dx}\left[x^3+y^3-3axy\right]=0$$

$$3x^2+3y^2\frac{dy}{dx}-3a\left[x\frac{dy}{dx}+y\frac{d}{dx}x\right]=0$$

$$x^2+y^2\frac{dy}{dx}-ax\frac{dy}{dx}-ay=0$$

$$(y^2-ax)\frac{dy}{dx}=ay-x^2$$

$$\therefore\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$$

3. $\mathbf{y = e^{a\sin^{-1}x}}$ **then show that** $\mathbf{\dfrac{dy}{dx} = \dfrac{ay}{\sqrt{1-y^2}}}$

A. $y = e^{a\sin^{-1}x}$.......(1)

Differentiate both sides with respect to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}e^{a\sin^{-1}x}$$

$$= e^{a\sin^{-1}x}\frac{d}{dx}a\ \sin^{-1}x$$

$$\frac{dy}{dx} = e^{a\sin^{-1}x}\frac{a}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{ya}{\sqrt{1+x^2}} \text{ from (1)}$$

4. Find the derivative of $\mathbf{\tan^{-1}\left(\dfrac{a-x}{1+ax}\right)}$

A. put $a = \tan A \Rightarrow A = \tan^{-1} a$

$= \tan B \Rightarrow B = \tan^{-1} c$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{\tan A - \tan B}{1+\tan A\ \tan B}\right)\right)$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\tan(A-B)\right)\right)$$

$$\frac{d}{dx}(A-B)$$

$$\frac{d}{dx}\left(\tan^{-1} a - \tan^{-1} x\right)$$

$$0 - \frac{1}{1+x^2}$$

$$= -\frac{1}{1+x^2}$$

5. $\mathbf{x^y = e^{x-y}}$ **then show that** $\mathbf{\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}}$

A. $x^y = e^{x-y}$

take log both sides

$$\log_e x^y = \log_e e^{x-y}$$

$y\log x = (x-y)\log_e e$

$x = y + y\log x$

$x = y(1+\log x)$

$y = \dfrac{x}{(1+\log x)}$

Differentiate both sides with respect to 'x'

$\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{x}{1+\log x}$

$= \dfrac{(1+\log x)1 - x\left(\dfrac{1}{x}\right)}{(1+\log x)^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{1+\log x - 1}{(1+\log x)^2}$

6. siny = xsin(a + y) then show that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$

A. $x = \dfrac{\sin y}{\sin(a+y)}$

Differentiate both sides with respect to 'y'

$\dfrac{dx}{dy} = \dfrac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$

$\dfrac{dx}{dy} = \dfrac{\sin[a+y-y]}{\sin^2(a+y)}$

$\therefore \dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$

Long Answer Questions (8 Marks)

1. $f(x) = \dfrac{x\cos x}{\sqrt{1+x^2}}$, then find f'(x).

A. Differentiate both sides with respect to 'x'

$\dfrac{d}{dx}f(x) = \dfrac{d}{dx}\dfrac{x\cos x}{\sqrt{1+x^2}}$

$f'(x) = \dfrac{\sqrt{1+x^2}\dfrac{d}{dx}[x\cos x] - x\cos x\dfrac{d}{dx}\sqrt{1+x^2}}{\left[\sqrt{1+x^2}\right]^2}$

$$= \frac{\sqrt{1+x^2}\left[x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}x\right] - x\cos x\frac{1}{2\sqrt{1+x^2}}\frac{d}{dx}(1+x^2)}{1+x^2}$$

$$= \frac{\sqrt{1+x^2}\left[x(-\sin x) + \cos x\right] - \frac{x\cos x}{2\sqrt{1+x^2}}(0+2x)}{1+x^2}$$

$$= \frac{(1+x^2)\left[(-x\sin x + \cos x\right] - x^2\cos x}{\sqrt{1+x^2}\,(1+x^2)}$$

$$= -\frac{x(1+x^2)\sin x + \cos x + x^2\cos x - x^2\cos x}{(1+x^2)^{3/2}}$$

$$= (1+x^2)^{3/2}[\cos x - x(1+x^2)\sin x]$$

Practice Problem: Find the value of $\frac{d}{dx}\left[\frac{x(1+x^2)}{\sqrt{1-x^2}}\right]$.

2. $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$**, then show that** $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

A. Let $x = \sin A \quad y = \sin B \Rightarrow A = \sin^{-1} x \Rightarrow B = \sin^{-1} y$

$\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

Put $x = \sin\theta, y = \sin\phi$

$\therefore \sqrt{1-\sin^2\theta} + \sqrt{1-\sin^2\phi} = a(\sin\theta - \sin\phi)$

$\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$2\cos\frac{\theta+\phi}{2}.\cos\frac{\theta-\phi}{2}$

$= \left[2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right]$

$\therefore \cos\frac{\theta-\phi}{2} = a.\sin\frac{\theta-\phi}{2}$

$\tan\frac{\theta-\phi}{2} = \frac{1}{2}; \frac{\theta-\phi}{2} = \tan^{-1}\left(\frac{1}{2}\right)$

$\phi = \theta - 2\tan^{-1}\left(\frac{1}{a}\right);$

$\sin^{-1} y = \sin^{-1} x - 2\tan^{-1}\left(\frac{1}{a}\right)$

Differentiating w.r.to x

$$\frac{1}{\sqrt{1-y^2}}.\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

3. Find the derivation of $\log\left(\frac{x^2+x+2}{x^2-x+2}\right)$.

A. $\frac{d}{dx}\log\left(\frac{x^2+x+2}{x^2-x+2}\right)$

$$=\frac{d}{dx}\left[\log(x^2+x+2)-\log(x^2-x+2)\right]$$

$$=\frac{1}{x^2+x+2}\frac{d}{dx}(x^2+x+2)-\frac{1}{x^2-x+2}\frac{d}{dx}(x^2-x+2)$$

$$=\frac{2x+1}{x^2+x+2}-\frac{2x-1}{x^2-x+2}$$

$$=\frac{(x^2-x+2)2x+1-(2x-1)(x^2+x+2)}{[(x^2+2)+x]\,[(x^2+2)-x]}$$

$$=-\frac{4x^2+2x^2+4}{x^4+4+4x^2-x^2}$$

$$=\frac{4-2x^2}{x^4+3x^2+4}$$

4. $y=\frac{x^3\sqrt{2+3x}}{(2+x)(1-x)}$ then find $\frac{dy}{dx}$

A. Take 'log' both sides

$$\log y=\log\frac{x^3\sqrt{2+3x}}{(2+x)(1-x)}$$

$$=\log x^3+\log\sqrt{2+3x}-\log(2+x)-\log(1-x)$$

$$\log y=3\log x+\frac{1}{2}(2+3x)-\log(2+x)-\log(1-x)$$

Differentiate both sides with respect to 'x'

$$\frac{1}{y}\frac{dy}{dx}=\frac{3}{x}+\frac{3}{2(2+3x)}-\frac{1}{2+x}+\frac{1}{1+x}$$

$$\therefore \frac{dy}{dx}=y\left[\frac{3}{x}+\frac{3}{2(2+3x)}-\frac{1}{2+x}+\frac{1}{1+x}\right]$$

5. $\mathbf{y = x\sqrt{a^2+x^2} + a^2\log(x+\sqrt{a^2+x^2})}$ **then show that** $\mathbf{\frac{dy}{dx} = 2\sqrt{a^2+x^2}}$

A. Differentiate both sides with respect to 'x'

$$\frac{dy}{dx} = x\frac{1}{2\sqrt{a^2+x^2}}2x + \sqrt{a^2+x^2} + \frac{a^2\left[1+\frac{1}{2\sqrt{a^2+x^2}}.2x\right]}{x+\sqrt{a^2+a^2}}$$

$$= \frac{x^2}{\sqrt{a^2+x^2}} + \frac{a^2\left(\sqrt{a^2+x^2}+x\right)}{\left(\sqrt{a^2+x^2}+x\right)\sqrt{a^2+x^2}}$$

$$= \frac{x^2+a^2}{\sqrt{a^2+x^2}} + \sqrt{a^2+x^2}$$

$$= \sqrt{a^2+x^2} + \sqrt{a^2+x^2}$$

$$= 2\sqrt{a^2+x^2}$$

6. Find the derivations of following functions.

A. (i) $\sec\sqrt{\tan x}$

$$\frac{d}{dx}\sec\sqrt{\tan x}$$

$$= \sec\sqrt{\tan x}\,\tan\sqrt{\tan x}\,\frac{d}{dx}\sqrt{\tan x}$$

$$= \sec\sqrt{\tan x}\,\tan\sqrt{\tan x}\,\frac{1}{2\sqrt{\tan x}}\frac{d}{dx}\tan x$$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}}\sec\sqrt{\tan x}\,\tan x\sqrt{\tan x}$$

(ii) $\frac{1-\cos 2x}{1+\cos 2x}$

$$\frac{d}{dx}\frac{1-\cos 2x}{1+\cos 2x} \qquad 1-\cos 2x = 2\sin^2 x,\quad 1+\cos 2x = 2\cos^2 x$$

$$= \frac{d}{dx}\frac{2\sin^2 x}{2\cos^2 x}$$

$$= \frac{d}{dx}(\tan x)^2$$

$$= 2(\tan x)^{2-1}\frac{d}{dx}\tan x$$

$$= 2\tan x\sec^2 x$$

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Unit **10**

Errors and approximations

Key concepts

1. A small change in x is Δx
2. If x is changed as $x + \Delta x$, then change in y is $\Delta y = f(x + \Delta x) - f(x)$
3. Differential in y, $dy = f'(x).\Delta x$
4. Relative error in $y = \dfrac{\Delta y}{y}$
5. Percentage error in $y = \dfrac{\Delta y}{y} \times 100$

Short Answer Questions (4 Marks)

1. Find $dy, \Delta y$ for the following $y = x^2 + 3x + 6$, $x = 10$, $\Delta x = 0.01$

Sol: $\Delta y = f(x + \Delta x) - f(x)$

$\Delta y = f(10 + 0.01) - f(10)$

$\Delta y = f(10.01) - f(10)$

$= (10.01)^2 + 3\times 10.01 + 6 - (10^2 + 3\times 10 + 6)$

$= 100.2001 + 30.03 + 6 - 100 - 30 - 6$

$= 130.2301 - 130$

$= 0.2301$

$\Delta y = f'(x)\Delta x$

$= (2x + 3)\Delta x$

$= (2 \times 10 + 3) \times 0.01$

$= 23 \times 0.01$

$= 0.23$

2. Find $dy, \Delta y$ for the following function $y = \cos x$ when $x = 60^o$, $\Delta x = 1^o$

Sol: $x = 60^o$, $\Delta x = 1^o$

$\Delta y = f(x+\Delta x) - f(x)$

$\Delta y = f(60^o + 1^o) - f(60^o)$

$= f(61^o) - f(60^o)$

$= \cos 61^o - \cos 60^o$

$= 0.4848 - 0.5 = -0.0152$

$\Delta y = f'(x)\Delta x$

$= -\sin x.\Delta x$

$= -\sin 60^o \times 1^o$

$= -0.866 \times 0.0174$ $(1^o = 0.0174$ radians$)$

$= -0.0150$

3. Find $dy, \Delta y$ for the following function $y = e^x + x$, $x = 5$, $\Delta x = 0.02$

Sol: $\Delta y = f(x+\Delta x) - f(x)$

$= f(5+0.02) - f(5)$

$= f(5.02) - f(5)$

$= e^{5.02} + 5.02 - e^5 - 5$

$= e^{5.02} - e^5 + 0.02$

$\Delta y = f'(x)\Delta x$

$= (e^x + 1).\Delta x$

$= (e^5 + 1)(0.02)$

4. Find $dy, \Delta y$ for the following function $y = 5x^2 + 6x + 6$, $x = 2$, $\Delta x = 0.001$

Sol: $\Delta y = f(x+\Delta x) - f(x)$

$= f(2+0.001) - f(2)$

$= f(2.001) - f(2)$

$= 5(2.001)^2 + 6\times 2.001 + 6 - (5\times 2^2 + 6\times 2 + 6)$

$= 5(4.004001) + 12.006 + 6 - 20 - 12 - 6$

$= 0.026005$

$\Delta y = f'(x)\Delta x$

$= (10x + 6).\Delta x$

$= (10 \times 2 + 6)\ (0.001)$

$= 26 \times 0.001 = 0.026$

5. If the increase in the side of a square is 2%, find the change in the area of the square.

Sol: Let side of the square = x

Area of the square $A = x^2$

given $\frac{\Delta x}{x} \times 100 = 2$ ......(1)

$$\Rightarrow \frac{\Delta A}{dx} = 2x \Rightarrow \frac{\frac{\Delta A}{dx}}{A} = \frac{2x}{x^2}$$

$$\Rightarrow \frac{\frac{\Delta A}{dx}}{A} \times 100 = 2 \times \left(\frac{\Delta x}{x} \times 100\right)$$

$= 2 \times 2$ from (1)

$= 4$

6. If the increase in the side of the square is 4% find the change in the area of the square.

Sol: Let side of the square = x

Area of the square $A = x^2$

$\frac{\Delta x}{x} \times 100 = 4$ (given)

$A = x^2 \Rightarrow \frac{\Delta A}{dx} = 2x$

$$\Rightarrow \frac{\frac{\Delta A}{dx}}{A} \times 100 \times \Delta x = \frac{2x}{x^2} \times 100 \times \Delta x$$

$$= 2 \times \frac{\Delta x}{x} \times 100$$

$= 2 \times 4$ from (1)

$= 8$

7. The radius of the sphere is measured is 14 c.m., Later it was found that there is an error 0.02cm in measuring the radius, find the approximate error in surface area of the sphere.

Sol: Let radius of the sphere = r

Surface Area $A = 4\pi r^2 \Rightarrow \frac{dA}{dr} = 4\pi \times 2r$

$$dA = \frac{dA}{dr} \times \Delta r$$

$= 4\pi \times 2r \times \Delta r$

$= 8 \times \frac{22}{7} \times 14 \times 0.02$

$= 7.04$ sq.cm.

Geometrical interpretation of the derivative

Key concepts

1. Slope of the tangent at point (x, y) on the curve $y = f(x)$ is $\frac{dy}{dx}$

2. Equation of tangent at point (a,b) on the curve $y = f(x)$ is $y - b = m(x - a)$ $\left(\because m = \frac{dy}{dx}\right)$

3. Slope of the normal $= -\frac{1}{m}$

3. Equation of the normal is $y - b = -\frac{1}{m}(x - a)$

1. Find the slope of the tangents to the curve $y = 3x^2 - x^3$, where it meets the X-axis.

Sol: $y = 3x^2 - x^3$ —— (1)

Equation of x-axis is $y = 0$ ---- (2)

from (1) and (2)

$3x^2 - x^3 = 0$ $(\because y = 0)$

$x^2(3 - x) = 0$

$x^2 = 0$ or $3 - x = 0$

$x = 0$ or $x = 3$

$\therefore$ The given curve intersects x-axis at the points $(0, 0), (3, 0)$

Slope of tangent at (0, 0), m $= \left(\frac{dy}{dx}\right)_{(0,0)}$

$= 6x - 3x^2$

$= 6 \times 0 - 3 \times 0$

$= 0$

Equation of tangent at (0,0) is

$y - 0 = 0(x - 0)$

$\Rightarrow y = 0$

Slope of tangent at (3, 0), m = $\left(\frac{dy}{dx}\right)_{(3,0)}$

$= 6x - 3x^2$

$= 6\times3 - 3\times3^2$

$= 18 - 27 = -9$

Equation of tangent at (3, 0) is

$y - 0 = -9\,(x - 3)$

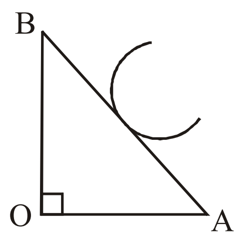
$y = -9x + 27$

$9x + y - 27 = 0$

2. Show that the area of the triangle formed by the tangent at any point on the curve $xy = c\ (C \neq 0)$ with the co-ordinate axes is constant.

Sol: Let P (x_1, y_1) be the point on the curve $xy = c$ and $x_1 \neq 0$, $y_1 \neq 0$.

$y = \frac{c}{x}$



Defferentiating both sides w.r.t. 'x'

$\frac{dy}{dx} = -\frac{c}{x^2}$

Slope of tangent, $m = -\frac{c}{x_1^2}$ (at $P(x_1, y_1)$)

Equation of the tangent at $P(x_1, y_1)$ is

$y - y_1 = -\frac{c}{x_1^2}\,(x - x_1)$

$y \cdot x_1^2 - y_1 x_1^2 = -cx + cx_1$

$cx + yx_1^2 = cx_1 + y_1 x_1^2$

$cx + yx_1^2 = cx_1 + y_1 x_1 x_1$

$cx + yx_1^2 = cx_1 + cx_1 \qquad (\because x_1 y_1 = c)$

$cx + yx_1^2 = 2cx_1$

$cx + yx_1^2 - 2cx_1 = 0$

$$\Rightarrow cx + x_1^2 y = 2cx_1$$

$$\frac{cx}{2cx_1} + \frac{x_1^2 y}{2cx_1} = 1$$

$$\frac{x}{2x_1} + \frac{y}{\frac{2c}{x_1}} = 1$$

area of the triangle $= \frac{1}{2}(2x_1).\left(\frac{2c}{x_1}\right) = 2c$

3. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$

Sol: $y = 3x^4 - 4x$

Differentiate both sides w.r.t. 'x'

$$\frac{dy}{dx} = 3 \times 4x^3 - 4$$

slope of the tangent at $x = 4$ $m = 12 \times 4^3 - 4$

$= 12 \times 64 - 4$

$= 768 - 4 = 764$

4(i). Find the slope of the tangent to the curve $y = x^2 - 3x + 2$ at the point whose x coordinate is 3.

Sol: $y = x^3 - 3x + 2$

Differentiate both sides w.r.t. 'x'

$$\frac{dy}{dx} = 3x^2 - 3$$

slope of the tangent at $x = 3$, $m = 3 \times 3^2 - 3$

$= 27 - 3 = 24$

4(ii). Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$

Sol: $x = a\cos^3\theta,\ \ y = a\sin^3\theta$

Differentiating with respect to θ

$$\frac{dx}{d\theta} = a3\cos^2\theta\,(-\sin\theta)$$

$$\frac{dy}{d\theta} = a \times 3\sin^2\theta\,(\cos\theta)$$

slope of the tangent, $(m) = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta \cdot \sin\theta} = -\tan\theta = -\tan\frac{\pi}{4}$

$$m = -1$$

$\therefore$ slope of the normal $= -\frac{1}{m} = \frac{-1}{-1} = 1$

5. Find the slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^3\theta$, at $\theta = \frac{\pi}{4}$

Sol: $x = 1 - a\sin\theta,\ \ y = b\cos^2\theta$

differentiating with respect to θ

$\frac{dx}{d\theta} = -a\cos\theta$, $\frac{dy}{d\theta} = 2b\cos\theta\,(-\sin\theta)$

slope of the tangent (at$\theta = \frac{\pi}{2}$) $m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2b\cos\theta\sin\theta}{-a\cos\theta}$

$$= \frac{2b}{a}, \sin\frac{\pi}{2}$$

$$m = \frac{2b}{a}$$

slope of the normal at $\theta = \frac{\pi}{2}$ $= -\frac{1}{m}$

$$= -\frac{1}{2b/a}$$

$$= \frac{-a}{2b}$$

6. Find the equation of tangent and normal to the curve $y = x^2 - 4x + 2$ at point (4, 2)

Sol: $y = x^2 - 4x + 2$

differentiating with respect to x

$$\frac{dy}{dx} = 2x - 4$$

slope of the tangent at (4, 2), $m = 2 \times 4 - 4 = 4$

equation of the tangent is

$$y - y_1 = m\ (x - x_1)$$
$$y - 2 = 4\ (x - 4)$$
$$y - 2 = 4x - 16$$
$$4x - y - 16 + 2 = 0$$
$$4x - y - 14 = 0$$

slope of the normal = $-\frac{1}{m} = -\frac{1}{4}$

equation of the normal is, $y - 2 = -\frac{1}{4}(x - 4)$

$$4y - 8 = -x + 4$$
$$x + 4y - 8 - 4 = 0$$
$$x + 4y - 12 = 8.$$

7. Find the equation of tangent and normal to the curve $y = x^3 + 4x^2$ at $(-1, 3)$

Sol: $y = x^3 + 4x^2$

differentiating with respect to x

$$\frac{dy}{dx} = 3x^2 + 8x$$

slope of the tangent at $(-1, 3)$ $m = 3 \times (-1)^2 + 8(-1)$

$$= 3 - 8$$
$$= -5$$

equation of the tangent is, $y - 3 = -5\ (x + 1)$

$$y - 3 = -5x - 5$$
$$5x + y + 2 = 0$$

slope of the normal $= -\frac{1}{m}$

$$= \frac{-1}{-5} = \frac{1}{5}$$

equation of the normal $y - 3 = \frac{1}{5}\ (x + 1)$

$$5y - 15 = x + 1$$
$$x - 5y - 16 = 0$$

8. Find the tangent and normal to the curve $y = 2e^{-\frac{x}{3}}$ at the point where the curve meets y-axis.

Sol: Equation of y–axis x = 0

$y = 2e^{-\frac{x}{3}}$, the curve meets y- axis is at $(0,\ 2)$ $[\because\ y = 2e^{-\frac{x}{3}} = 2]$

$y = 2e^{-\frac{x}{3}}$

differentiating with respect to x

$$\frac{dy}{dx} = 2e^{-x/3} \times \left(-\frac{1}{3}\right)$$

$$= -\frac{2}{3} \cdot e^{-x/3}$$

slope of the tangent (at $x = 0$) $m = -\frac{2}{3} \cdot e^{-0/3} = -\frac{2}{3}$

equation of the tangent at $(0,\ 2)$ is

$$y - 2 = -\frac{2}{3}(x - 0)$$

$$3y - 6 = -2x$$

$$2x + 3y - 6 = 0$$

slope of the normal $= -\frac{1}{m} = \frac{-1}{-\frac{2}{3}} = \frac{3}{2}$

equation of the normal $y - 2 = \frac{3}{2}\ (x - 0)$

$$2y - 4 = 3x$$

$$3x - 2y + 4 = 0$$

9. If the tangent at any point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ intersects the co-ordinate axes in A and B then show that the length of AB is constant.

Sol: P (x_1, y_1) be the any point the on the curve $x^{2/3} + y^{2/3} = a^{2/3}$

$x_1^{2/3} + y_1^{2/3} = a^{2/3}$ .... (1)

$x^{2/3} + y^{2/3} = a^{2/3}$

differentiating both sides with respect to 'x'

$$\frac{2}{3}\cdot x^{-\frac{1}{3}}+\frac{2}{3}\cdot y^{-\frac{1}{3}}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{y^{1/3}}{x^{1/3}}$$

slope of the tangent at P(x_1, y_1)

$$m=-\left(\frac{y_1}{x_1}\right)^{1/3}$$

equation of the tangent at P(x_1, y_1)

$$y-y_1=-\left(\frac{y_1}{x_1}\right)^{1/3}\cdot(x-x_1)$$

$$y\cdot x_1^{1/3}-y_1\cdot x_1^{1/3}=-x\cdot y_1^{1/3}+y_1^{1/3}\cdot x_1$$

$$x\cdot y_1^{1/3}+y\cdot x_1^{1/3}=x_1y_1^{1/3}+x_1^{1/3}y_1$$

Dividing by $y_1^{1/3}x_1^{1/3}$, we get

$$\frac{x\cdot y_1^{1/3}}{y_1^{1/3}x_1^{1/3}}+\frac{y\cdot x_1^{1/3}}{y_1^{1/3}\cdot x_1^{1/3}}=\frac{x_1y_1^{1/3}}{y_1^{1/3}x_1^{1/3}}+\frac{x_1^{1/3}\cdot y_1}{y_1^{1/3}\cdot x_1^{1/3}}$$

$$\Rightarrow\frac{x}{x_1^{1/3}}+\frac{y}{y_1^{1/3}}=x_1^{2/3}+y_1^{2/3}$$

$$\Rightarrow\frac{x}{x_1^{1/3}}+\frac{y}{y_1^{1/3}}=a^{2/3}\quad\text{(from eqn–1)}$$

above equation intersects x–axis at A $\left(a^{2/3}\cdot x_1^{1/3},0\right)$, y – axis at B$\left(0,a^{2/3}\cdot y_1^{1/3}\right)$

$$\text{AB}=\sqrt{\left(a^{2/3}\cdot x_1^{2/3}\right)^2+\left(a^{2/3}\cdot y_1^{1/3}\right)^2}$$

$$=\sqrt{\left(a^{2/3}\right)^2\left(x_1^{2/3}+y_1^{2/3}\right)^2}$$

$$=\sqrt{a^{4/3}\cdot a^{2/3}}\qquad\text{(from eqn–1)}$$

$$=\sqrt{a^{6/3}}$$

$$=\sqrt{a^2}$$

$= a$ = constant

Angle between two curves and condition for orthogonality of curves

1. Angle between tangents drawn at intersecting points of two curves C_1 and C_2 is angle between the curves C_1 and C_2
2. If slopes of tangents drawn at intersecting points of curves is m_1 and m_2 and angle between the curves is 'θ' then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$
3. If $m_1 = m_2$, the curves have common tangents, and touch each other.
4. If $m_1 . m_2 = -1$ the curves intersects orthogonally.

1. Show that the curves $y^2 = 4(x+1)$, $y^2 = 36(9-x)$ intersect orthogonally.

Sol: $y^2 = 4(x+1)$, $y^2 = 36(9-x)$

$\therefore\ 4(x+1) = 36(9-x)$

$x + 1 = 9(9-x)$

$x + 1 = 81 - 9x$

$x + 9x = 81 - 1$

$10x = 80$

$x = 8$

$y^2 = 4(x+1)$

$y^2 = 4(8+1)$

$y^2 = 4 \times 9 = 36$

$y = \pm 6$

intersecting points of given curves are $(8, 6)$, $(8, -6)$

slope of tangent at 'p' to the curve $y^2 = 4(x+1)$

$$2y \cdot \frac{dy}{dx} = 4$$

$$\Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y} \text{ (first curve)}$$

$$y^2 = 36(9-x) \Rightarrow 2y \cdot \frac{dy}{dx} = -36$$

$$= \frac{dy}{dx} = -\frac{36}{2y} = -\frac{18}{y} \text{ (second curve)}$$

slope of the curve $y^2 = 4(x + 1)$ at (2, 6) $m_1 = \frac{2}{6} = \frac{1}{3}$

slope of the curve $y^2 = 36(9 - x)$ at (2, 6) $m_2 = \frac{-18}{6} = -3$

$$m_1 \times m_2 = \frac{1}{3} \times (-3) = -1$$

The given curves intersect orthogonally. Similarly we can prove at point Q(8,–6)

2. Show that the condition for the orthogonality of curves $ax^2 + bx = 1, a_1x^2 + b_1y^2 = 1$ is $\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$

Sol: Let intersecting point of given curves $ax^2 + bx = 1$, $a_1x^2 + b_1y^2 = 1$ be $P(x_1, y_1)$

$\therefore\ ax_1^2 + by_1^2 - 1 = 0,\quad a_1x_1^2 + b_1y_1^2 - 1 = 0$

$$\begin{array}{cccc} x_1^2 & y_1^2 & 1 & \\ b & -1 & a & b \\ b_1 & -1 & a_1 & b_1 \end{array}$$

$$\frac{x_1^2}{-b + b_1} = \frac{y_1^2}{-a_1 + a} = \frac{1}{ab_1 - a_1b} \qquad \text{.... (1)}$$

$\therefore$ slope of the tangent to the curve $ax^2 + by^2 = 1$ at $P(x_1, y_1)$

$$2ax + 2by^2 . y^1 = 0$$

$$\Rightarrow y_1 = \frac{-2ax}{2by} = \frac{-ax}{by}$$

$$\therefore\ m_1 = \frac{ax_1}{by_1}$$

slope of the tangent to the curve $a_1x^2 + b_1y^2 = 1$ at $P(x_1, y_1)$

$$m_1 = \frac{a_1x_1}{b_1x_1}$$

condition for orthogonality is $m_1\, m_2 = -1$

$$-\frac{ax_1}{by_1} \times \frac{a_1x_1}{b_1y_1} = -1$$

$$\frac{aa_1}{bb_1} \cdot \frac{x_1^2}{y_1^2} = -1$$

$$\frac{x_1^2}{y_1^2} = -\frac{bb_1}{aa_1}$$

$$\frac{b_1 - b}{a - a_1} = \frac{-bb_1}{aa_1}$$ (*from eqn*---(1))

$b_1.aa_1 - b.aa_1 = -abb_1 + ba_1b_1$

$a\,a_1b_1 - ba_1b_1 = aba_1 - abb_1$

$a_1\,b_1\,(a-b) = ab\,(a_1-b_1)$

$$\frac{a-b}{ab} = \frac{a_1 - b_1}{a_1b_1}$$

$$\frac{a}{ab} - \frac{b}{ab} = \frac{a_1}{a_1b_1} - \frac{b_1}{a_1b_1}$$

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{b_1} - \frac{1}{a_1}$$

$$\therefore \quad \frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$$

3. Find the angle between the curves $x + y + 2 = 0$, $x^2 + y^2 - 10y = 0$

Sol: $x + y + 2 = 0$

$\Rightarrow x = -(y + 2)$

$x^2 + y^2 - 10y = 0$

$(-(y + 2))^2 + y^2 - 10y = 0$

$y^2 + 4y + 4 + y^2 - 10y = 0$

$2y^2 - 6y + 4 = 0$

$y^2 - 3y + 2 = 0$

$(y - 1)(y - 2) = 0$

$y = +1$ or $y = 2$

at $y = 1$, $x = -(1 + 2) = -3$

at $y = 2$ $x = -(2 + 2) = -4$

Intersecting points are $(-3, 1)$, $(-4, 2)$

slope of the tangent at $(-3, 1)$ to the curve $x + y + 2 = 0$

$1 + y^1 = 0$

$y^1 = -1$

$m_1 = -1$

$x^2 + y^2 - 10\,y = 0$

differeintating both sides with respect to 'x'

$2x + 2y \,.\, y^1 - 10 \,.\, y^1 = 0.$

$$y^1 = \frac{-2x}{2y-10}$$

$$m_2 = \frac{-2\times(-3)}{2(+1)+10}$$

$$= \frac{6}{2-10} = \frac{6}{-8} = \frac{-3}{4}$$

If angle between two curves is θ, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$$

$$= \left|\frac{-1+3/4}{1+(-1)(-3/4)}\right|$$

$$= \left|\frac{\frac{-4+3}{4}}{\frac{4+3}{4}}\right|$$

$$= \left|\frac{-1}{7}\right|$$

$$= \frac{1}{7}$$

$$\theta = \text{Tan}^{-1}\left(\frac{1}{7}\right)$$

4. Find the angle between the curves $y^2 = 4x$, $x^2 + y^2 = 5$

Sol: $y^2 = 4x$, $x^2 + y^2 = 5$

$\Rightarrow x^2 + 4x - 5 = 0$

$(x + 5)(x - 1) = 0$

$x = -5$ or 1

$x = 1 \Rightarrow y^2 = 4 \times 1$

$\Rightarrow y^2 = 4$

$y = \pm 2$

intersecting points of two curves $(1, 2), (1, -2)$

$y^2 = 4x$

differentiating both sides with respect to 'x'

$2y \cdot y^1 = 4$

$$y^1 = \frac{4}{2y} = \frac{2}{y}$$

slope of the first curve at $(1, 2) = \frac{2}{2} = 1$ (m_1)

$x^2 + y^2 = 5$

differentiating both sides with respect to 'x'

$2x + 2yy^1 = 0 \Rightarrow y^1 = \frac{-x}{y}$

slope of the second curve at $(1, 2)$

$$m_2 = -\frac{1}{2}$$

If angle between two curves 'θ' then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{1 + \frac{1}{2}}{1 + (1)\left(-\frac{1}{2}\right)}\right|$$

$$= \left|\frac{\frac{3}{2}}{\frac{1}{2}}\right| = 3$$

$\tan\theta = 3$

$\theta = \tan^{-1}(3)$

5. Find the angle between two curves $x^2 = 2(y+1)$; $y = \frac{8}{x^2+4}$

Sol: $x^2 = 2(y+1),\ y = \frac{8}{x^2+4}$

$\Rightarrow x^2 + 4 = \frac{8}{y}$

$x^2 = \frac{8}{y} - 4$

$\therefore \frac{8}{y} - 4 = 2(y+1)$

$\frac{8}{y} = 2y + 2 + 4$

$\frac{8}{y} = 2y + 6$

$\frac{4}{y} = y + 3$

$4 = y^2 + 3y$

$y^2 + 3y - 4 = 0$

$(y+4)(y-1) = 0$

$y = -4$ or $y = 1$

$x^2 = 2(1+1)$

$x^2 = 2 \times 2$

$x = \pm 2$

$\therefore$ intersecting points are $(2,1)$ and $(2, -1)$

$x^2 = 2(y+1)$

differeintating both sides with respect to 'x'

$2x = 2y^1$

$y^1 = x$

slope of the curve at $(2, 1)$ $m_1 = 2$

$y = \frac{8}{x^2+4}$

differeintating both sides with respect to 'x'

$$y^1 = -\frac{8}{\left(x^2+4\right)^2} \cdot 2x$$

slope of the second curve at (2, 1) to the curve $y = \frac{8}{x^2+4}$

$$m_2 = \frac{-16 \times 2}{\left(2^2+4\right)^2} = \frac{-32}{64} = -\frac{1}{2}$$

$$m_1 \times m_2 = 2 \times \left(-\frac{1}{2}\right) = -1$$

$\therefore$ the given curves intersect orthogonally.

6. Show that the curves $6x^2 - 5x + 2y = 0$, $4x^2 + 8y^2 = 3$ touch each other at points $\left(\frac{1}{2}, \frac{1}{2}\right)$

Sol: $6x^2 - 5x + 2y = 0$

differeintating both sides with respect to 'x'

$6 \times 2x - 5 + 2y^1 = 0$

$2y^1 = 5 - 12x$

$$y^1 = \frac{5-12x}{2}$$

slope of the tangent at $\left(\frac{1}{2}, \frac{1}{2}\right)$ to the first curve is $m_1 = \frac{5 - 12 \times \frac{1}{2}}{2}$

$$= \frac{5-6}{2} = -\frac{1}{2}$$

$4x^2 + 8y^2 = 3$

differeintating both sides with respect to 'x'

$4 \times 2x + 8 \times 2y.y^1 = 0$

$$y^1 = \frac{-8x}{16y} = \frac{-x}{2y}$$

slope of the tangent at $\left(\frac{1}{2}, \frac{1}{2}\right)$ to the second curve is $m_2 = \frac{-\frac{1}{2}}{2\times\frac{1}{2}} = -\frac{1}{2}$

$\therefore \quad m_1 = m_2$

$\therefore$ the given curves touch each other at $\left(\frac{1}{3}, \frac{1}{2}\right)$

Maxima and Minima

Key concepts

1. 1st derivative test :

Let f be a differential function on an interval D, $c \in D$, and f is defined in some neighbourhood of c, suppose, c is a stationary point of f such that $(c-\delta, c+\delta)$ does not contain any other stationary point for some $\delta > 0$ then.

(i) c is a point of local maximum, if $f'(x)$ changes sign from positive to negative at $x = c$

(ii) c is a point of local minimum, if $f'(x)$ changes sign from negative to positive at $x = c$

(iii) c is neither a point of local maximum nor a points of local minimum $f'(x)$ does not change sign at $x = c$

2. 2nd derivative test :

(i) x=c is a point of local miximum of 'f' if $f'(c)=0$ and $f''(c) < 0,$ and local maximum value of 'f' is $f(c)$

(ii) $x=c$ is a point of local minimum of 'f' if $f^{1}(C) = 0$ and $f^{11}(C) > 0$ and local minimum value 'f' is f (c)

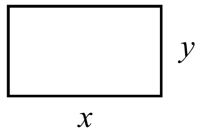
(iii) the test fails if $f'(c) = 0$ and $f''(c) = 0$

1. Find the maximum area of the rectangle that can be formed with fixed perimetre 20.

Sol: Let x and y denote the length and the breadth of a rectangle respectively . Given that the perimeter of the rectangle is 20.

$\therefore 2(x+y) = 20$ .... (1)

$\Rightarrow x + y = 10$ ....(2)



area of the rectangle $A = x.y.$ .... (3)

$A = x.(10-x) \qquad \because x + y = 10$

$A = 10x - x^2$ .... (4)

differentiating both sides with respect to 'x'

$$\frac{dA}{dx} = 10 - 2x \quad \text{.... (5)}$$

$$\frac{dA}{dx} = 0 \Rightarrow 10 - 2x = 0$$

$$\Rightarrow 2x = 10$$

$$\Rightarrow x = 5$$

$\therefore$ x = 5 is the stationary point

differentiating (5) with respect to 'x'

$$\frac{d^2A}{dx^2} = -2$$

$$\frac{d^2A}{dx^2} < 0$$

which is negative, therefore by second derivative test the area A is maximum at x = 5 and hence y = 10 – 5 = 5, and the maximum area is A = x.y = 5 (5) = 25

2. The profit function P(x) of a company selling x items per day is given by $P(x) = (150 - x)x - 1600$. Find the number of items that the company should sell to get maximum profit. Also find the maximum profit.

Sol: $P(x) = (150 - x)\,x - 1600$

$$= 150x - x^2 - 1600$$

differintating both sides with respect to 'x'

$$\frac{dP}{dx} = 150 - 2x \quad \text{..... (1)}$$

$$\frac{dP}{dx} = 0$$

$$\Rightarrow 150 - 2x = 0$$

$$2x = 150$$

$$x = 75$$

Again differintating Eqn .... (1) with respect to 'x'

$$\frac{d^2p}{dx^2} = -2$$

$$\frac{d^2p}{dx^2} < -2$$

$\therefore$ The profit P (x) is maximum for $x = 75$

$\therefore$ The company should sell 75 items a day to make maximum profit.

the maximum profit P (75) = (150 – 75) · 75 – 1600

= 75 × 75 – 1600

= 5625 – 1600

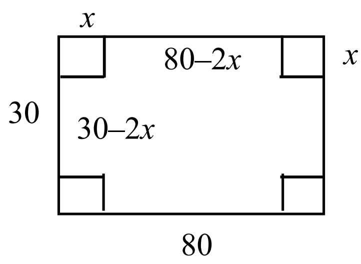
= 4025.

3. From a rectangular sheet of dimensions $30\text{cm} \times 80\text{cm}$ four equal squares of side x cm are removed at the corners, and the sides are then turned up so as to form an open rectangular box. Find the value of x, so that the volume of the box is the greatest.

Sol: length of the box = $80 - 2x$

breadth of the box = $30-2x$

height of the box = x



volume V = $(80 - 2x)(30 - 2x) \cdot x$

$= (2400 - 160x - 60x + 4x^2) \cdot x.$

$V = 4x^3 - 220x^2 + 2400x$

differintating both sides with respect to 'x'

$$\frac{dv}{dx} = 12x^2 - 440x + 2400 \qquad ..... 91)$$

$$\frac{dv}{dx} = 10$$

$12x^2 - 440x + 2400 = 0$

$3x^2 - 110x + 600 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{110 \pm \sqrt{(110)^2 - 4 \times 3 \times 600}}{2 \times 3}$$

$$x = \frac{110 \pm \sqrt{12100 - 7200}}{6}$$

$= \frac{110 \pm \sqrt{4900}}{6}$

$= \frac{110 \pm 70}{6}$

$= \frac{110 + 70}{6}$ $\qquad$ $= \frac{110 - 70}{6}$

$= \frac{180}{6}$ $\qquad$ $= \frac{40}{6}$

$= 30$ $\qquad$ $= \frac{20}{3}$

$x = 30 \quad b = 30 - 2 \times 30$

$= -30$

$b < 0$

$\therefore \quad x \neq 30$

$\therefore \quad x = \frac{20}{3}$

Again differintating Eqn .... (1) with respect to 'x'

$$\frac{d^2v}{dx^2} = 24x - 440$$

For $x = \frac{20}{3}$, $\frac{d^2v}{dx^2} = 24 \times \frac{20}{3} - 440$

$= 160 - 440$

$= -280$

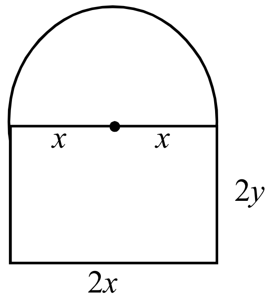
$$\frac{d^2v}{dx^2} < 0$$

Volume of box is maximum at $x = \frac{20}{3}$

4. A window is in the shape of a rectangle surrounded by a semicircle. If the perimeter of the window is 20 ft. Find the maximum area.

Sol: The perimeter of window = 22 + 2y = πx = 20

$2y = 20 - (\pi + 2)\,x.$



$$y = 10 - \frac{\pi + 2}{2} \cdot x$$

area $= \quad 2xy + \frac{\pi x^2}{2}$

$$A = \left(20 - (\pi + 2) \cdot x\right) + \frac{\pi x^2}{2}$$

$$= 20 - (\pi + 2)x^2 + \frac{\pi x^2}{2}$$

differintating both sides with respect to 'x'

$$\frac{dA}{dx} = 20 - (\pi + 2) \cdot 2x + \frac{\pi}{2} \times 2x \qquad \text{..... 91)}$$

$$\frac{dA}{dx} = 0$$

$20 - (\pi + 2) + x = 0$

$x\,(\pi - 2\pi - 4) = -20$

$x\,(-\pi - 4) = -20$

$x\,(\pi + 4) = 20$

$$x = \frac{20}{\pi + 4}$$

Again differintating Eqn .... (1) with respect to 'x'

$$\frac{d^2A}{dx^2} = -\ (\pi + 2) \times 2 + \pi$$

$= -2\pi - 4 + \pi$

$= -\pi - 4$

$$\frac{d^2A}{dx^2} < 0$$

area of the window maximum at $x = \dfrac{20}{\pi + 4}$

$$A = 2 \times y + \frac{\pi}{2}\ .\ x^2$$

$2y = 20 - (\pi + 2)x$

$$2y = 20 - (\pi + 2) \times \frac{20}{\pi + 4}$$

$$2y = \frac{20\pi + 80 - 20\pi - 40}{\pi + 4}$$

$$= \frac{40}{\pi + 4}$$

$$A = 2xy + \frac{\pi}{2} \cdot x^2$$

$$= \frac{20}{\pi + 4} \cdot \frac{40}{\pi + 4} + \frac{\pi}{2}\left(\frac{20}{\pi + 4}\right)^2$$

$$= \frac{1600 + 400 \cdot \pi}{2(\pi + 4)^2}$$

$$= \frac{200(4 + \pi)}{(\pi + 4)2}$$

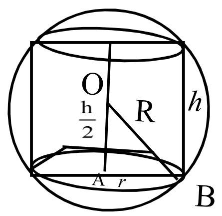
$$= \frac{200}{\pi + 4}$$

5. If the curved surface of right circular cylinder inscribed in a sphere of radius r is maximum, show that the height of the cylinder is $\sqrt{2}$ r.

Sol: From Δ OAB

$$OA^2 + AB = OB^2$$

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$



$$r^2 = R^2 - \frac{h^2}{4}$$

lateral surface area of cylinder = $2\pi rh$

$$= 2\pi \cdot \sqrt{R^2 - \frac{h^2}{4}} \cdot h$$

$$A = \frac{2\pi \cdot h}{2} \cdot \sqrt{4R^2 - h^2}$$

$$A = \pi \cdot h\sqrt{4R^2 - h^2}$$

differintating both sides with respect to 'h'

$$\frac{dA}{dh} = \pi\left\{1.\sqrt{4R^2 - h^2} + \frac{h.1.(-2h)}{2\sqrt{4R^2 - h^2}}\right\}$$

$$\frac{dA}{dh} = \pi\cdot\left\{\frac{4R^2 - h^2 - h^2}{\sqrt{4R^2 - h^2}}\right\}$$

$$= \pi\cdot\left\{\frac{4R^2 - 2h^2}{\sqrt{4R^2 - h^2}}\right\}$$

$$\frac{dA}{dh} = 0$$

$$\pi\cdot\left\{\frac{4R^2 - 2h^2}{\sqrt{4R^2 - h^2}}\right\} = 0 \qquad 4R^2 - 2h^2 = 0$$

$$h = \sqrt{2}\cdot R.$$

Again differintating Eqn .... (1) with respect to 'h'

$$\frac{d^2A}{dh^2} = 2\pi\cdot\left\{\frac{\sqrt{4R^2 - h^2}\cdot(-2h) - \frac{(2R^2 - h^2)}{2\sqrt{4R^2 - h^2}}\cdot(-2h)}{4R^2 - h^2}\right\}$$

$$= 2\pi\cdot\left\{\frac{-8R^2h + 2h^3 + 2R^2h - h^3}{(4R^2 - h^2)\left(\sqrt{4R^2 - 4^2}\right)}\right\}$$

$$= 2\pi\cdot\left\{\frac{-8R^2h + 2h^3 + 2R^2h - h^3}{4R^2 - h^2\cdot\sqrt{4R^2 - h^2}}\right\}$$

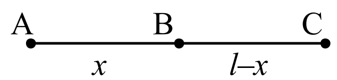
$$= \frac{-4\pi h}{\sqrt{4R^2 - h^2}} < 0,\ \text{at } h = \sqrt{2}R \qquad \frac{d^2A}{dh^2} < 0$$

$\therefore$ Lateral surface area of cylinder is maximum when h = $\sqrt{2}R$

6. A wire of length l is cut into two parts which are bent respectively in the form of a square and a circle. What are the lengths of the pieces of the wire respecrtively so that the sum of the areas is the least.

Sol: Let square is formed with the wire of length x

Let circle is formed with wire of length is $l-x$



side of the square = $\frac{x}{4}$

area = $\left(\frac{x}{4}\right)^2$

circumfrance of the circle, $2\pi r = l - x$

$$r = \frac{l-x}{2\pi}$$

area of the circle = $\pi r^2 = \pi \cdot \left(\frac{l-x}{2\pi}\right)^2$

sum of the areas, $A = \frac{x^2}{16} + \frac{(l-x)^2}{4\pi}$

differintating both sides with respect to 'x'

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{2(l-x)}{4\pi}\cdot(-1) \qquad \text{..... (1)}$$

$$\frac{dA}{dx} = 0$$

$$= \frac{2x}{16} - \frac{2(l-x)}{4\pi} = 0 \Rightarrow \frac{x}{8} - \frac{l-x}{2\pi} = 0$$

$$= \frac{\pi \cdot x - 4l + 4x}{8\pi} = 0 \Rightarrow x(\pi + 4) = 4l$$

$$x = \frac{4l}{\pi + 4}$$

Again differintating Eqn .... (1) with respect to 'x'

$$\frac{dA^2}{dx^2} = \frac{1}{8} - \frac{(-1)}{2\pi}$$

$$= \frac{1}{8} + \frac{1}{2\pi} > 0$$

$\therefore$ At $x = \dfrac{4l}{\pi + 4}$ we get minimum value of **A**

$$l - x = l - \frac{4l}{\pi + 4} \qquad = \frac{\pi l + 4l - 4l}{\pi + 4} \qquad = \frac{\pi l}{\pi + 4}$$

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