The following notation is used throughout the question paper.

- For a real number x, $[x]$ denotes the greatest integer $\leq x$. For example, $[\sqrt{5}] = 2$
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{Q}$ denotes the set of rational numbers.
- $\mathbb{N}$ denotes the set of natural numbers.
- $|S|$ denotes the cardinality of a set S.

1. Let α, β be roots of the equation x^2+3x+1. Then $\dfrac{\alpha^3}{\beta^3}+\dfrac{\beta^3}{\alpha^3}$ equals

   (A) 320. (B) 321.
   (C) 322. (D) 323.

2. Which of the following statements is the negation of "for all sets X and for all sets Y, if $X \subseteq Y$, then $X \setminus Y = \emptyset$"?

   (A) For all sets X, there is a set Y such that if $X \subseteq Y$ then $X \setminus Y = \emptyset$.
   (B) There is a set X such that for all sets Y, if $X \subseteq Y$ then $X \setminus Y = \emptyset$.
   (C) There are sets X and Y for which $X \subseteq Y$ and $X \setminus Y = \emptyset$.
   (D) There are sets X and Y for which $X \subseteq Y$ and $X \setminus Y \neq \emptyset$.

3. Let $n = aaaaaaaaabcd$ be a 12-digit number divisible by 45 where $a \neq 0$ and a, b, c, d need not be distinct. How many such numbers are possible?

   (A) 180 (B) 189
   (C) 198 (D) 207

4. The rank of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & a & 0 \\ 1 & 0 & 0 & 0 & b \end{pmatrix}$$

(A) depends on the value of a but not on the value of b.

(B) depends on the value of b but not on the value of a.

(C) depends on the values of both a and b.

(D) is independent of the values of both a and b.

5. Express the output `f(n)` of the following code fragment as a mathematical function of `n` (where `n` is a natural number).

```
f(n){
    if n is 0
        return 0;
    return 2 * f(n-1) + 1;
}
```

(A) $\texttt{f(n)} = 2^{\texttt{n}+1} - 2$ (B) $\texttt{f(n)} = 2^{\texttt{n}} - 1$

(C) $\texttt{f(n)} = \texttt{n}! - 1$ (D) $\texttt{f(n)} = \texttt{n} \cdot 2^{\texttt{n}-1}$

6. Let $p(x) := ax^2 + bx + c$ be a polynomial where a, b, c are drawn uniformly at random with replacement from the set $\{0, 1, 2\}$. What is the probability that the equation $p(x) = 0$ has two distinct real roots?

(A) $\frac{2}{27}$ (B) $\frac{4}{27}$

(C) $\frac{1}{27}$ (D) $\frac{21}{27}$

7. Let $a < 500$ be a positive integer. Let there be a box containing balls numbered $a, a+1, \ldots, 500$. Suppose that for each x, $a \leq x \leq 500$, the probability that the ball numbered x is picked is

$$\frac{xa}{(500+a)(500-a+1)}.$$

Then the value of a is

(A) 1 (B) 2

(C) 3 (D) 4

8. For a function $f : \mathbb{R} \to \mathbb{R}$ and a subset A of $\mathbb{R}$, define

$$f(A) = \{f(x) \mid x \in A\} \text{ and}$$

$$f^{-1}(A) = \{x \in \mathbb{R} \mid f(x) \in A\}.$$

Consider the following statements.

(I) $f(X \cap Y) = f(X) \cap f(Y)$ for all subsets X, Y of $\mathbb{R}$.

(II) $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$ for all subsets X, Y of $\mathbb{R}$.

Which of the following is correct?

(A) (I) is always true; (II) need not be true.

(B) (II) is always true; (I) need not be true.

(C) Both (I) and (II) are always true.

(D) Neither (I) nor (II) is always true.

9. Let A be a 4×4 upper triangular matrix with diagonal entries $(-1, 1, \frac{1}{2}, -\frac{1}{2})$. Then A^{-1} equals

(A) $-5A - 4A^3$ (B) $5A - 4A^3$

(C) $4A + A^2$ (D) $A^2 + A^3$

10. A connected component of a simple undirected graph $G = (V, E)$ is a subset of vertices $U \subseteq V$ such that between any pair of vertices in U there is a path between these two vertices. Let $|V| = n$ and $|E| = m$ with $0 \leq m \leq (n-1)$. Then the number of connected components in G is

(A) at least $(n - m)$
(B) at most $(n - m)$
(C) exactly $(n - m)$
(D) none of the above

11. Consider the following routine for driving a robot:

```
Drive(n) {
  if n < 2
     STOP
  else {
     Move NORTH n kms.
     Move WEST n kms.
     Move SOUTH n kms.
     Move EAST (n-1) kms.
     Move NORTH 1 km.
     Drive(n-2)
   }
}
```

Let M(n) denote the distance (in kms) traversed by the robot when we call Drive(n). What is the value of M(n) when $\mathtt{n} \in \mathbb{N}$?

(A) $4 \cdot \left[\frac{\mathtt{n}}{2}\right] \cdot \left[\frac{\mathtt{n}+3}{2}\right]$
(B) $4 \cdot \left[\frac{\mathtt{n}}{2}\right] \cdot \left[\frac{\mathtt{n}+2}{2}\right]$
(C) $3 \cdot \left[\frac{\mathtt{n}}{2}\right] \cdot \left[\frac{\mathtt{n}+2}{2}\right]$
(D) $3 \cdot \left[\frac{\mathtt{n}}{2}\right] \cdot \left[\frac{\mathtt{n}+3}{2}\right]$

12. Let a, b and c be the sides of a triangle such that $c^2 = a^2+b^2-ab$. Then which of the following is *always true?*

(A) $a \leqslant c$ and $b \leqslant c$
(B) $a \geqslant c$ and $b \geqslant c$
(C) $a \leqslant c \leqslant b$ or $b \leqslant c \leqslant a$
(D) None of the above.

13. For any non-negative integers `x` and `y`, consider the following function

```
f(x,y){
  if y is 0
     return 0;
  else
       if y is even
          return 2f(x, [y/2]);
       else
          return 2f(x, [y/2]) + x;
}
```

Which of the following is true:

(A) `f(x+1, y+1)` $=$ `2f(x,y) + x + y`
(B) `f(x+1, y+1)` $=$ `2f(x,y) + x + y + 1`
(C) `f(x+1, y+1)` $=$ `f(x,y) + x + y`
(D) `f(x+1, y+1)` $=$ `f(x,y) + x + y + 1`

14. If the matrix $A = \begin{pmatrix} a & 1 \\ 2 & 3 \end{pmatrix}$ has 1 as an eigenvalue, then the determinant of A is

(A) 2 (B) 3
(C) 4 (D) 5

15. Let $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. Define $f_n(x)$ inductively by setting

$$f_1(x) = f(x)$$

and

$$f_{n+1}(x) = f(f_n(x)) \quad \text{for} \quad n > 1.$$

If $f_7(x) = 128x + 381$, then a^b equals

(A) $\frac{1}{32}$ (B) $\frac{1}{8}$
(C) 8 (D) 32

16. For a real number a, the value of $\int_a^{a+1} [x] dx$ is

(A) 1 (B) a
(C) $[a]$ (D) $[a] + 1$

17. Diplomats of 4 countries C1, C2, C3 and C4 are having some serious negotiation with a fifth country C5. If country C1 buys fighter jets from country C5, then country C2 cannot buy helicopters from country C5. If country C3 sells cars to country C5, then country C5 will buy petroleum from country C4. Now if country C1 does not buy fighter jets from country C5 or country C5 buys petroleum from country C4 then the people of country C4 are happy.

Which of the following is necessarily TRUE?

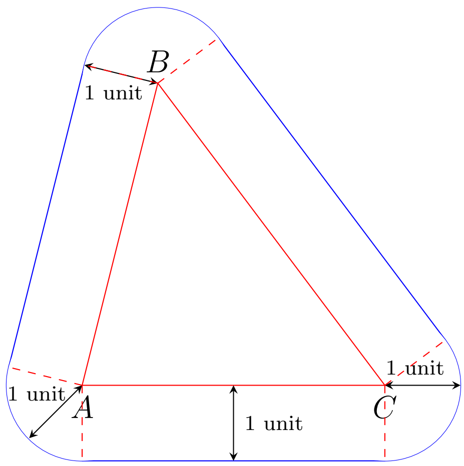
(A) If the people of country C4 are happy then it means country C5 buys petroleum from country C4.
(B) If country C5 buys petroleum from country C4 then country C1 buys fighter jets from country C5.
(C) If the people of country C4 are not happy then country C2 does not buy helicopters from country C5 and country C3 does not sell cars to country C5.
(D) None of the above.

18. The coefficient of $a^6b^4c^2$ in the expansion of $(ac+bd)^6(1+\frac{a}{d})^8$ is

(A) $\frac{12\times 6!8!}{(4!)^4}$ (B) $\frac{11!}{3!3!4!}$

(C) 1551 (D) $\frac{11!}{3!4!5!}$

19. Let ABC be a triangular park with vertices A, B and C located at the coordinates $(0,0)$, $(1,4)$ and $(4,0)$, respectively. A pavement of uniform width 1 unit has to be built all around the park as shown in the figure below. What is the total area of the pavement?



(A) $(9+\sqrt{17})+\pi$ (B) $(9+\sqrt{17})+2\pi$

(C) $(17+\sqrt{17})+\pi$ (D) $(17+\sqrt{17})+2\pi$

20. Let $a \in \mathbb{R}$ be such that

$$n = \frac{1-2a}{4+a} - \frac{\sqrt{|a|-3}+\sqrt{3-|a|}}{3-a}$$

is an integer. Then the value of n is

(A) 1 (B) -1

(C) 5 (D) 7

21. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 3x - 2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then

(A) f is not continuous at any real number.

(B) f is continuous exactly at $x = -1$ and $x = -2$.

(C) f is continuous at all rational numbers.

(D) f is continuous exactly at $x = 1$ and $x = 2$.

22. Suppose P is a real polynomial of degree 5 such that the equation $P(x) = 0$ has exactly 4 distinct real roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let P' denote the derivative of P. The equation

$$P'(x) = 0$$

(A) has exactly 3 distinct real solutions.

(B) has exactly 4 distinct real solutions, none of which belongs to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

(C) has exactly 4 distinct real solutions, at least two of which belong to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

(D) has exactly 4 distinct real solutions, exactly one of which belongs to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

23. Let A be the centre of the circle $x^2 + y^2 - 4x - 6y - 12 = 0$. Tangents are drawn at the points $B = (2, 8)$ and $D = (5, -1)$ on the circle. Those tangents meet at C. Then the area of the quadrilateral $ABCD$ is

(A) 60.

(B) 75.

(C) 90.

(D) 105.

24. The sum of all 3-digit positive numbers which are divisible by 7 but not by 14 is

(A) 62×548. (B) 63×547.
(C) 64×546. (D) 65×545.

25. Let $a < b$ be real numbers. Let f be a real-valued continuous function on $[a, b]$ which is differentiable on (a, b) and which satisfies the condition $f(a) = f(b) = 0$. Which of the following is true?

(A) For every $\alpha \in \mathbb{R}$, there exists $x_0 \in (a, b)$ such that $\alpha f(x_0) + f'(x_0) = 0$.

(B) For $\alpha \in (0, 1)$, there does not exist $x_0 \in (a, b)$ such that $\alpha f(x_0) + f'(x_0) = 0$.

(C) If there exists $x_0 \in (a, b)$ for which $\alpha f(x_0) + f'(x_0) = 0$, then α must be a non-zero integer.

(D) If there exists $x_0 \in (a, b)$ and $\alpha \in \mathbb{R}$ for which $\alpha f(x_0) + f'(x_0) = 0$, then α must be zero.

26. Consider an equilateral triangle ABC where each side is 1 unit long. Let Δ be the collection of points lying inside or on the boundary of ABC. Which of the following statements is true?

(A) Any set of 5 points in Δ will have two points whose distance is at most 1/2 unit.

(B) Any set of 5 points in Δ will have two points whose distance is strictly less than 1/2 unit.

(C) Any set of 6 points in Δ will have two points whose distance is strictly less than 1/2 unit.

(D) None of the above.

27. Consider the regular expressions $R_1 = a^*(ba^*ba^*)^*$ and $R_2 = a^*(ba^*b)^*a^*$ over the alphabet $\{a, b\}$. Let $L(R_i)$ denote the languages corresponding to the regular expressions R_i, $i = 1, 2$. Consider the string $w = bbaaabb$. Which of the following statements is correct?

(A) $w \in L(R_1)$ and $w \in L(R_2)$.
(B) $w \notin L(R_1)$ and $w \in L(R_2)$.
(C) $w \in L(R_1)$ and $w \notin L(R_2)$.
(D) $w \notin L(R_1)$ and $w \notin L(R_2)$.

28. What is the minimum value of ϵ such that the inequality

$$\frac{x + \epsilon}{1 - x - \epsilon} \geq \frac{2x}{1 - x}$$

holds for all real $x \in [0, 1/2]$?

(A) $1/2$
(B) $3 - 2\sqrt{2}$
(C) 0
(D) $3 + 2\sqrt{2}$

29. The value of $\lim_{x \to 1} \dfrac{\sqrt{x+8} - \sqrt{8x+1}}{\sqrt{5-x} - \sqrt{7x-3}}$ is

(A) $\frac{1}{2}$
(B) $\frac{2}{3}$
(C) $\frac{7}{12}$
(D) undefined

30. Let $T(n)$ be the count of n-digit numbers (using digits 0, 1, …, 9, with first digit non-zero) such that no three consecutive digits are the same. Then which of the following is a correct recurrence relation for $T(n)$? Note that $T(1) = 9$ and $T(2) = 90$.

(A) $T(n) = 9T(n-1) + 9T(n-2)$
(B) $T(n) = 9T(n-1) + 8T(n-2) + 9$
(C) $T(n) = 9T(n-1) + 9$
(D) $T(n) = 9T(n-1) - 9T(n-2)$