1. (a) If $a, b$ and $c$ are positive numbers such that $a < b < c$, then show that

$$\frac{a^2}{c} < \frac{a^2 + b^2 + c^2}{a + b + c} < \frac{c^2}{a}.$$

(b) Consider the numbers $0, 1, 2, \cdots, n$ included in a list with frequencies $1, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$, respectively. Find the mean of the numbers in the list.

(c) A man decided to visit four places, $P_1, P_2, P_3$ and $P_4$ in a random order. Find the probability that he visits $P_1$ before $P_2$, and $P_2$ before $P_3$.

$[7 + 4 + 4 = 15]$

2. (a) Evaluate $\int_{e^{-1}}^{e^2} |\frac{\log_e x}{x}| \, dx$.

(b) Calculate the sum of first 50 terms of

$$\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + \cdots$$

$[7 + 8 = 15]$

3. (a) Let a biased coin upon being tossed has probability of turning on head $\frac{2}{5}$ and probability of turning on tail $\frac{3}{5}$. The coin is tossed five times. Determine the probability of turning up exactly three heads, all of them consecutive.

(b) Show that $f(x) = \int_a^x |t| \, dt$ is differentiable at all $x > a$.

$[7 + 8 = 15]$

4. (a) The numbers $1, 2, 3, \ldots, 100$ are written down on each of the cards $A$, $B$ and $C$. One number is selected at random from each of the cards. Find the probability that the numbers so selected can be the measures (in cm) of three sides of a right-angled triangle.

(b) Find the area of the region bounded by the following lines $3x - y - 3 = 0$, $2x + y - 12 = 0$ and $x - 2y - 1 = 0$.

$[8 + 7 = 15]$

5. (a) If $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, then show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

(b) Consider the following functions $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$; $x \in \mathbb{R} - \{-1, 0, 1\}$. Show that $\frac{f(x)}{g(x)} \geq 2\sqrt{2}$.

$[8 + 7 = 15]$

6. (a) Shannon Foods is a business that produces and supplies chicken nuggets to selected restaurants in the city. The company has a fixed cost of 500 dollars per day, and its variable cost consists of a material cost of 5 dollars per packet produced, and a labour cost of 0.1 times the square of the number of nugget packets produced per day. The company sells each packet for 205 dollars. What is the optimal number of chicken nugget packets that Shannon Foods should produce per day to maximize its profit?

(b) Consider the square matrix $A = \begin{bmatrix} 1 & -2 \\ p & 2 \end{bmatrix}$. Find the coefficient of $p^3$ in determinant of $5A^3$.

$[9 + 6 = 15]$

7. (a) If $x \in \mathbb{R}$, find the real values of $a$ for which $x^2 - 2ax - 2a^2 + 1$ is always positive.

(b) There are three persons aged 60, 65 and 70 years. The probabilities of their living 5 more years are 0.8, 0.6 and 0.3 respectively. Find the probability that at least two of the three persons will remain alive 5 years hence.

$[6 + 9 = 15]$

8. (a) If the numbers appeared on two throws of a fair six-faced die are $\alpha$ and $\beta$, then find the probability that $x^2 + \alpha x + \beta > 0$, for all $x \in \mathbb{R}$.

(b) Four seeds are planted with each one having 80% chance of germinating. Find the probability that at least one seed will germinate.

(c) Show that $7^{2n} - 48n - 1$, where $n$ is positive integer, is divisible by 2304.

$[6 + 4 + 5 = 15]$