

10. Differentiation

Exercise 10A

1. Question

Differentiate each of the following w.r.t. x :

$$\sin 4x$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\sin x) = \cos x$$

$$\bullet \frac{d}{dx} (kx) = k$$

Let,

$$y = \sin 4x$$

$$\text{and } u = 4x$$

$$\text{therefore, } y = \sin u$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\sin u) \cdot \frac{d}{dx} (4x)$$

$$= \cos u \cdot 4 \dots\dots\dots \left(\because \frac{d}{dx} (\sin x) = \cos x \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= \cos 4x \cdot 4$$

$$= 4 \cos 4x$$

2. Question

Differentiate each of the following w.r.t. x :

$$\cos 5x$$

Answer

Formulae :

- $\frac{d}{dx} (\cos x) = -\sin x$

- $\frac{d}{dx} (kx) = k$

Let,

$$y = \cos 5x$$

$$\text{and } u = 5x$$

$$\text{therefore, } y = \cos u$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\cos u) \cdot \frac{d}{dx} (5x)$$

$$= -\sin u \cdot 5 \dots\dots\dots \left(\because \frac{d}{dx} (\cos x) = -\sin x \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= -\sin 5x \cdot 5$$

$$= -5 \sin 5x$$

3. Question

Differentiate each of the following w.r.t. x :

$$\tan 3x$$

Answer

Formulae :

- $\frac{d}{dx} (\tan x) = \sec^2 x$

- $\frac{d}{dx} (kx) = k$

Let,

$$y = \tan 3x$$

$$\text{and } u = 3x$$

$$\text{therefore, } y = \tan u$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\tan u) \cdot \frac{d}{dx} (3x)$$

$$= \sec^2 u \cdot 3 \dots\dots\dots \left(\because \frac{d}{dx} (\tan x) = \sec^2 x \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= \sec^2 3x \cdot 3$$

$$= 3 \sec^2 3x$$

4. Question

Differentiate each of the following w.r.t. x:

$$\cos x^3$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\cos x) = -\sin x$$

$$\bullet \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

Let,

$$y = \cos x^3$$

$$\text{and } u = x^3$$

$$\text{therefore, } y = \cos u$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\cos u) \cdot \frac{d}{dx} (x^3)$$

$$= -\sin u \cdot 3x^2 \dots\dots\dots \left(\because \frac{d}{dx} (\cos x) = -\sin x \text{ \& } \frac{d}{dx} (x^n) = n \cdot x^{n-1} \right)$$

$$= -\sin x^3 \cdot 3x^2$$

$$= -3x^2 \sin x^3$$

5. Question

Differentiate each of the following w.r.t. x:

$$\cot^2 x$$

Answer

Formulae :

- $\frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$

- $\frac{d}{dx} (x^n) = n \cdot x^{n-1}$

Let,

$$y = \cot^2 x$$

$$\text{and } u = \cot x$$

$$\text{therefore, } y = u^2$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (u^2) \cdot \frac{d}{dx} (\cot x)$$

$$= 2u \cdot (-\operatorname{cosec}^2 x) \dots\dots\dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x \right)$$

$$= 2 \cot x \cdot (-\operatorname{cosec}^2 x)$$

$$= -2 \cot x \cdot \operatorname{cosec}^2 x$$

6. Question

Differentiate each of the following w.r.t. x :

$$\tan^3 x$$

Answer

Formulae :

- $\frac{d}{dx} (\tan x) = \sec^2 x$

- $\frac{d}{dx} (x^n) = n \cdot x^{n-1}$

Let,

$$y = \tan^3 x$$

$$\text{and } u = \tan x$$

$$\text{therefore, } y = u^3$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= \frac{d}{du} (u^3) \cdot \frac{d}{dx} (\tan x) \\
 &= 3u^2 \cdot \sec^2 x \dots\dots\dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (\tan x) = \sec^2 x \right) \\
 &= 3 \tan^2 x \cdot (\sec^2 x) \\
 &= 3 \tan^2 x \cdot \sec^2 x
 \end{aligned}$$

7. Question

Differentiate each of the following w.r.t. x:

$$\square \cot \sqrt{x}$$

Answer

Formulae :

- $\frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$

Let,

$$y = \cot \sqrt{x}$$

$$\text{and } u = \sqrt{x}$$

therefore, $y = \cot u$

Differentiating above equation w.r.t. x,

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule} \\
 \therefore \frac{dy}{dx} &= \frac{d}{du} (\cot u) \cdot \frac{d}{dx} (\sqrt{x}) \\
 &= -\operatorname{cosec}^2 u \cdot \frac{1}{2\sqrt{x}} \dots\dots\dots \left(\because \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x \text{ \& } \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \right) \\
 &= -\operatorname{cosec}^2 \sqrt{x} \cdot \frac{1}{2\sqrt{x}} \\
 &= \frac{-1}{2\sqrt{x}} \operatorname{cosec}^2 \sqrt{x}
 \end{aligned}$$

8. Question

Differentiate each of the following w.r.t. x:

$$\sqrt{\tan x}$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\tan x) = \sec^2 x$$

$$\bullet \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

Let,

$$y = \sqrt{\tan x}$$

and $u = \tan x$

therefore, $y = \sqrt{u}$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\sqrt{u}) \cdot \frac{d}{dx} (\tan x)$$

$$= \frac{1}{2\sqrt{u}} \cdot \sec^2 x \dots\dots\dots \left(\because \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \text{ \& } \frac{d}{dx} (\tan x) = \sec^2 x \right)$$

$$= \frac{1}{2\sqrt{\tan x}} \cdot \sec^2 x$$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}}$$

9. Question

Differentiate each of the following w.r.t. x :

$$(5 + 7x)^6$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

$$\bullet \frac{d}{dx} (kx) = k$$

$$\bullet \frac{d}{dx} (k) = 0$$

- $\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$

Let,

$$y = (5+7x)^6$$

$$\text{and } u = (5+7x)$$

$$\text{therefore, } y = u^6$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (u^6) \cdot \frac{d}{dx} (5 + 7x)$$

$$= 6. (u)^5 \cdot \left(\frac{d}{dx} (5) + \frac{d}{dx} (7x) \right) \dots\dots\dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= 6. (5+7x)^5 \cdot (0+7) \dots\dots\dots \left(\because \frac{d}{dx} (k) = 0 \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= 42. (5+7x)^5$$

10. Question

Differentiate each of the following w.r.t. x:

$$(3 - 4x)^5$$

Answer

Formulae :

- $\frac{d}{dx} (x^n) = n \cdot x^{n-1}$

- $\frac{d}{dx} (kx) = k$

- $\frac{d}{dx} (k) = 0$

- $\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$

Let,

$$y = (3-4x)^5$$

$$\text{and } u = (3-4x)$$

$$\text{therefore, } y = u^5$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (u^5) \cdot \frac{d}{dx} (3 - 4x)$$

$$= 5 \cdot (u)^4 \cdot \left(\frac{d}{dx} (3) + \frac{d}{dx} (-4x) \right) \dots\dots\dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= 5 \cdot (3-4x)^4 \cdot (0-4) \dots\dots\dots \left(\because \frac{d}{dx} (k) = 0 \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= -20 (3-4x)^4$$

11. Question

Differentiate each of the following w.r.t. x:

$$(2x^2 - 3x + 4)^5$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

$$\bullet \frac{d}{dx} (kx) = k$$

$$\bullet \frac{d}{dx} (k) = 0$$

$$\bullet \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Let,

$$y = (2x^2 - 3x + 4)^5$$

$$\text{and } u = (2x^2 - 3x + 4)$$

$$\text{therefore, } y = u^5$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (u^5) \cdot \frac{d}{dx} (2x^2 - 3x + 4)$$

$$= 5 \cdot (u)^4 \cdot \left(\frac{d}{dx} (2x^2) + \frac{d}{dx} (-3x) + \frac{d}{dx} (4) \right) \dots\dots\dots$$

$$\left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= 5. (2x^2 - 3x + 4)^4. (4x-3+0) \dots\dots\dots \left(\because \frac{d}{dx} (kx) = k \text{ \& } \frac{d}{dx} (k) = 0 \right)$$

$$= 5. (2x^2 - 3x + 4)^4 (4x-3)$$

12. Question

Differentiate each of the following w.r.t. x:

$$(ax^2 + bx + c)^6$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (x^n) = n. x^{n-1}$$

$$\bullet \frac{d}{dx} (kx) = k$$

$$\bullet \frac{d}{dx} (k) = 0$$

$$\bullet \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Let,

$$y = (ax^2 + bx + c)^6$$

$$\text{and } u = (ax^2 + bx + c)$$

$$\text{therefore, } y = u^6$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (u^6) \cdot \frac{d}{dx} (ax^2 + bx + c)$$

$$= 6. (u)^5 \cdot \left(\frac{d}{dx} (ax^2) + \frac{d}{dx} (bx) + \frac{d}{dx} (c) \right)$$

$$= 6. (ax^2 + bx + c)^5 \cdot \frac{d}{dx} (ax^2 + bx + c) \dots\dots\dots \left(\because \frac{d}{dx} (x^n) = n. x^{n-1} \text{ \& } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= 6. (ax^2 + bx + c)^5. (2ax+b+0) \dots\dots\dots \left(\because \frac{d}{dx} (kx) = k \text{ \& } \frac{d}{dx} (k) = 0 \right)$$

13. Question

Differentiate each of the following w.r.t. x:

$$\frac{1}{(x^2 - 3x + 5)^3}$$

Answer

Formulae :

- $\frac{d}{dx} \left(\frac{1}{x} \right) = \frac{-1}{x^2}$
- $\frac{d}{dx} (x^n) = n \cdot x^{n-1}$
- $\frac{d}{dx} (kx) = k$
- $\frac{d}{dx} (k) = 0$
- $\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$

Let,

$$y = \frac{1}{(x^2 - 3x + 5)^3}$$

$$\text{Let, } u = (x^2 - 3x + 5)^3$$

$$\text{Therefore, } y = \frac{1}{u}$$

$$\text{For } u = (x^2 - 3x + 5)^3$$

$$\text{Let, } v = (x^2 - 3x + 5)$$

$$\text{Therefore, } u = (v)^3$$

$$\text{Therefore, } y = \frac{1}{v^3}$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} \left(\frac{1}{u} \right) \cdot \frac{d}{dv} (v)^3 \cdot \frac{d}{dx} (x^2 - 3x + 5)$$

$$= \frac{-1}{u^2} \cdot 3v^2 \cdot \left(\frac{d}{dx} (x^2) + \frac{d}{dx} (-3x) + \frac{d}{dx} (5) \right)$$

$$\dots\dots\dots \left(\because \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{-1}{x^2}, \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= \frac{-1}{(x^2 - 3x + 5)^6} \cdot 3(x^2 - 3x + 5)^2 \cdot (2x - 3 + 0) \dots \dots \dots \left(\because \frac{d}{dx} (kx) = k \text{ \& \& } \frac{d}{dx} (k) = 0 \right)$$

$$= \frac{-3}{(x^2 - 3x + 5)^4} \cdot (2x - 3)$$

$$= \frac{-3(2x - 3)}{(x^2 - 3x + 5)^4}$$

14. Question

Differentiate each of the following w.r.t. x:

$$\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$$

Answer

Formulae :

- $\frac{d}{dx} (x^n) = n \cdot x^{n-1}$
- $\frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx} (k) = 0$
- $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \cdot \frac{d}{dx}(u) - u \cdot \frac{d}{dx}(v)}{(v)^2}$

Let,

$$y = \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$$

$$\text{and } u = \frac{a^2 - x^2}{a^2 + x^2}$$

$$\therefore y = \sqrt{u}$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots \dots \dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\sqrt{u}) \cdot \frac{d}{dx} \left(\frac{a^2 - x^2}{a^2 + x^2} \right)$$

$$\begin{aligned}
&= \frac{1}{2\sqrt{u}} \left(\frac{(a^2+x^2) \cdot \frac{d}{dx}(a^2-x^2) - (a^2-x^2) \cdot \frac{d}{dx}(a^2+x^2)}{(a^2+x^2)^2} \right) \dots\dots\dots \left(\because \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \cdot \frac{d}{dx}(u) - u \cdot \frac{d}{dx}(v)}{(v)^2} \text{ \& } \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \right) \\
&= \frac{1}{2\sqrt{\frac{a^2-x^2}{a^2+x^2}}} \left(\frac{(a^2+x^2) \cdot (-2x) - (a^2-x^2) \cdot (2x)}{(a^2+x^2)^2} \right) \dots\dots\dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (k) = 0 \right) \\
&= \frac{\sqrt{a^2+x^2}}{2\sqrt{a^2-x^2}} \cdot (2x) \left(\frac{-a^2-x^2-a^2+x^2}{(a^2+x^2)^2} \right) \\
&= \frac{(a^2+x^2)^{1/2}}{2(a^2-x^2)^{1/2}} \cdot (2x) \cdot \frac{-2a^2}{(a^2+x^2)^2} \\
&= \frac{-2a^2x}{(a^2-x^2)^{1/2} (a^2+x^2)^{2-\frac{1}{2}}} \\
&= \frac{-2a^2x}{(a^2-x^2)^{1/2} \cdot (a^2+x^2)^{3/2}}
\end{aligned}$$

15. Question

Differentiate each of the following w.r.t. x:

$$\sqrt{\frac{1+\sin x}{1-\sin x}}$$

Answer

Formulae :

- $1 - \sin^2 x = \cos^2 x$
- $\frac{d}{dx} (\sec x) = \sec x \cdot \tan x$
- $\frac{d}{dx} (\tan x) = \sec^2 x$

Let,

$$y = \sqrt{\frac{1+\sin x}{1-\sin x}}$$

Multiplying numerator and denominator by $(1+\sin x)$,

$$\therefore y = \sqrt{\frac{1+\sin x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}}$$

$$= \sqrt{\frac{(1 + \sin x)^2}{1 - \sin^2 x}}$$

$$= \sqrt{\frac{(1 + \sin x)^2}{\cos^2 x}} \dots\dots\dots (1 - \sin^2 x = \cos^2 x)$$

$$= \frac{1 + \sin x}{\cos x}$$

$$= \frac{1}{\cos x} + \frac{\sin x}{\cos x}$$

$$y = \sec x + \tan x$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\sec x + \tan x)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\sec x) + \frac{d}{dx} (\tan x)$$

$$= \sec x \cdot \tan x + \sec^2 x \dots\dots\dots \left(\because \frac{d}{dx} (\sec x) = \sec x \cdot \tan x \text{ \& } \frac{d}{dx} (\tan x) = \sec^2 x \right)$$

$$= \sec x (\tan x + \sec x)$$

16. Question

Differentiate each of the following w.r.t. x:

$$\cos^2 x^3$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\cos x) = -\sin x$$

$$\bullet \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

$$\bullet 2 \sin x \cdot \cos x = \sin 2x$$

Let,

$$y = \cos^2 x^3$$

$$\text{and } u = x^3$$

$$\text{therefore, } y = \cos^2 u$$

$$\text{let, } v = \cos u$$

therefore, $y = v^2$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (v^2) \cdot \frac{d}{du} (\cos u) \cdot \frac{d}{dx} (x^3)$$

$$= 2v \cdot (-\sin u) \cdot 3x^2 \dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (\cos x) = -\sin x \right)$$

$$= -2 \cos u \cdot \sin u \cdot 3x^2$$

$$= -\sin 2u \cdot 3x^2 \dots (\because 2 \sin x \cdot \cos x = \sin 2x)$$

$$= -\sin 2x^3 \cdot 3x^2$$

17. Question

Differentiate each of the following w.r.t. x :

$$\sec^3 (x^2+1)$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\sec x) = \sec x \cdot \tan x$$

$$\bullet \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

Let,

$$y = \sec^3 (x^2+1)$$

$$\text{and } u = x^2+1$$

$$\text{therefore, } y = \sec^3 u$$

$$\text{let, } v = \sec u$$

$$\text{therefore, } y = v^3$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (v^3) \cdot \frac{d}{du} (\sec u) \cdot \frac{d}{dx} (x^2 + 1)$$

$$= 3v^2 \cdot (\sec u \cdot \tan u) \cdot 2x \dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (\sec x) = \sec x \cdot \tan x \right)$$

$$= 3 \sec^2 u \cdot (\sec u \cdot \tan u) \cdot 2x$$

$$= 6x \cdot \sec^3 u \cdot \tan u$$

$$= 6x \cdot \sec^3(x^2 + 1) \cdot \tan(x^2 + 1)$$

18. Question

Differentiate each of the following w.r.t. x:

$$\sqrt{\cos 3x}$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\cos x) = -\sin x$$

$$\bullet \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\bullet \frac{d}{dx} (kx) = k$$

Let,

$$y = \sqrt{\cos 3x}$$

$$\text{and } u = 3x$$

$$\text{therefore, } y = \sqrt{\cos u}$$

$$\text{let, } v = \cos u$$

$$\text{therefore, } y = \sqrt{v}$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (\sqrt{v}) \cdot \frac{d}{du} (\cos u) \cdot \frac{d}{dx} (3x)$$

$$= \frac{1}{2\sqrt{v}} \cdot (-\sin u) \cdot 3 \dots\dots\dots \left(\because \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}, \frac{d}{dx} (\cos x) = -\sin x \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= \frac{-3}{2} \cdot \frac{\sin u}{\sqrt{\cos u}}$$

$$= \frac{-3}{2} \cdot \frac{\sin 3x}{\sqrt{\cos 3x}}$$

19. Question

Differentiate each of the following w.r.t. x :

$$\sqrt[3]{\sin 2x}$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\sin x) = \cos x$$

$$\bullet \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

$$\bullet \frac{d}{dx} (kx) = k$$

Let,

$$y = \sqrt[3]{\sin 2x}$$

and $u = 2x$

$$\text{therefore, } y = \sqrt[3]{\sin u}$$

let, $v = \sin u$

$$\text{therefore, } y = \sqrt[3]{v} = v^{3/2}$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (v^{1/3}) \cdot \frac{d}{du} (\sin u) \cdot \frac{d}{dx} (2x)$$

$$= \frac{1}{3} v^{-2/3} \cdot (\cos u) \cdot 2 \dots\dots\dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1}, \frac{d}{dx} (\sin x) = \cos x \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= \frac{2}{3} \frac{\cos u}{v^{2/3}}$$

$$= \frac{2}{3} \frac{\cos u}{(\sin u)^{2/3}}$$

$$= \frac{2}{3} \frac{\cos 2x}{(\sin 2x)^{2/3}}$$

20. Question

Differentiate each of the following w.r.t. x :

$$\sqrt{1 + \cot x}$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$$

$$\bullet \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\bullet \frac{d}{dx} (k) = 0$$

$$\bullet \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Let,

$$y = \sqrt{1 + \cot x}$$

$$\text{and } u = 1 + \cot x$$

$$\text{therefore, } y = \sqrt{u}$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\sqrt{u}) \cdot \frac{d}{dx} (1 + \cot x)$$

$$= \frac{1}{2\sqrt{u}} \cdot \left(\frac{d}{dx} (1) + \frac{d}{dx} (\cot x) \right) \dots\dots\dots \left(\because \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \text{ \& } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= \frac{1}{2\sqrt{1 + \cot x}} \cdot (0 - \operatorname{cosec}^2 x)$$

$$= \frac{-1}{2} \frac{\operatorname{cosec}^2 x}{\sqrt{1 + \cot x}}$$

21. Question

Differentiate each of the following w.r.t. x :

$$\operatorname{cosec}^3 \frac{1}{x^2}$$

Answer

Formulae :

- $\frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

- $\frac{d}{dx} (x^n) = n \cdot x^{n-1}$

Let,

$$y = \operatorname{cosec}^3 \frac{1}{x^2}$$

and $u = \frac{1}{x^2}$

therefore, $y = \operatorname{cosec}^3 u$

let, $v = \operatorname{cosec} u$

therefore, $y = v^3$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots \dots \dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (v^3) \cdot \frac{d}{du} (\operatorname{cosec} u) \cdot \frac{d}{dx} \left(\frac{1}{x^2} \right)$$

$$= 3v^2 \cdot (-\operatorname{cosec} u \cdot \cot u) \cdot \frac{d}{dx} (x^{-2})$$

$$\dots \dots \dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x \right)$$

$$= 3 \operatorname{cosec}^2 u \cdot (-\operatorname{cosec} u \cdot \cot u) \cdot (-2x^{-3})$$

$$= 3 \operatorname{cosec}^3 u \cdot \cot u \left(2 \frac{1}{x^3} \right)$$

$$= \frac{6}{x^3} \cdot \operatorname{cosec}^3 \left(\frac{1}{x^2} \right) \cdot \cot \left(\frac{1}{x^2} \right)$$

22. Question

Differentiate each of the following w.r.t. x :

$$\sqrt{\sin x^3}$$

Answer

Formulae :

- $\frac{d}{dx} (\sin x) = \cos x$

- $\frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$

- $\frac{d}{dx} (x^n) = n \cdot x^{n-1}$

Let,

$$y = \sqrt{\sin x^3}$$

and $u = x^3$

therefore, $y = \sqrt{\sin u}$

let, $v = \sin u$

therefore, $y = \sqrt{v}$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (\sqrt{v}) \cdot \frac{d}{du} (\sin u) \cdot \frac{d}{dx} (x^3)$$

$$= \frac{1}{2\sqrt{v}} \cdot (\cos u) \cdot 3x^2$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}, \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (\sin x) = \cos x \right)$$

$$= \frac{1}{2\sqrt{\sin u}} \cdot (\cos u) \cdot 3x^2$$

$$= \frac{3}{2} x^2 \cdot \frac{\cos x^3}{\sqrt{\sin x^3}}$$

23. Question

Differentiate each of the following w.r.t. x :

$$\sqrt{x \sin x}$$

Answer

Formulae :

- $\frac{d}{dx} (\sin x) = \cos x$

- $\frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$

- $\frac{d}{dx} (kx) = k$

- $\frac{d}{dx} (u.v) = u.\frac{d}{dx} (v) + v.\frac{d}{dx} (u)$

Let,

$$y = \sqrt{x \cdot \sin x}$$

and $u = x \cdot \sin x$

therefore, $y = \sqrt{u}$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots \dots \dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\sqrt{u}) \cdot \frac{d}{dx} (x \cdot \sin x)$$

$$= \frac{1}{2\sqrt{u}} \left(x \cdot \frac{d}{dx} (\sin x) + \sin x \cdot \frac{d}{dx} (x) \right)$$

$$\dots \dots \dots \left(\because \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \text{ \& } \frac{d}{dx} (u.v) = u.\frac{d}{dx} (v) + v.\frac{d}{dx} (u) \right)$$

$$= \frac{1}{2\sqrt{x \cdot \sin x}} (x \cdot (\cos x) + \sin x \cdot (1)) \dots \dots \dots \left(\because \frac{d}{dx} (kx) = k \text{ \& } \frac{d}{dx} (\sin x) = \cos x \right)$$

$$= \frac{(x \cdot \cos x + \sin x)}{2\sqrt{x \cdot \sin x}}$$

24. Question

Differentiate each of the following w.r.t. x :

$$\sqrt{\cot \sqrt{x}}$$

Answer

Formulae :

- $\frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$

- $\frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$

Let,

$$y = \sqrt{\cot \sqrt{x}}$$

And $u = \sqrt{x}$

therefore, $y = \sqrt{\cot u}$

let, $v = \cot u$

therefore, $y = \sqrt{v}$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (\sqrt{v}) \cdot \frac{d}{du} (\cot u) \cdot \frac{d}{dx} (\sqrt{x})$$

$$= \frac{1}{2\sqrt{v}} \cdot (-\operatorname{cosec}^2 u) \cdot \frac{1}{2\sqrt{x}}$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \text{ \& } \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 \right)$$

$$= \frac{1}{2\sqrt{\cot u}} \cdot (-\operatorname{cosec}^2 u) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{\cot \sqrt{x}}} \cdot (-\operatorname{cosec}^2 \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{-\operatorname{cosec}^2 \sqrt{x}}{4 \sqrt{x} \sqrt{\cot \sqrt{x}}}$$

25. Question

Differentiate each of the following w.r.t. x :

$$\cot^3 x^2$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$$

$$\bullet \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

Let,

$$y = \cot^3 x^2$$

$$\text{and } u = x^2$$

therefore, $y = \cot^3 u$

let, $v = \cot u$

therefore, $y = v^3$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (v^3) \cdot \frac{d}{du} (\cot u) \cdot \frac{d}{dx} (x^2)$$

$$= 3v^2 \cdot (-\operatorname{cosec}^2 u) \cdot 2x \dots\dots\dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x \right)$$

$$= 3 \cot^2 u \cdot (-\operatorname{cosec}^2 u) \cdot 2x$$

$$= -6x \cdot \cot^2 u \cdot \operatorname{cosec}^2 u$$

$$= -6x \cdot \cot^2 (x^2) \cdot \operatorname{cosec}^2 (x^2)$$

26. Question

Differentiate each of the following w.r.t. x :

$$\cos(\sin \sqrt{ax+b})$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\cos x) = -\sin x$$

$$\bullet \frac{d}{dx} (\sin x) = \cos x$$

$$\bullet \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\bullet \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Let,

$$y = \cos(\sin \sqrt{ax+b})$$

and $u = ax + b$

therefore, $y = \cos(\sin \sqrt{u})$

let, $v = \sqrt{u}$

therefore, $y = \cos(\sin v)$

let, $w = \sin v$

therefore, $y = \cos w$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots \dots \dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dw} (\cos w) \cdot \frac{d}{dv} (\sin v) \cdot \frac{d}{du} (\sqrt{u}) \cdot \frac{d}{dx} (ax + b)$$

$$= (-\sin w) \cdot (\cos v) \cdot \left(\frac{1}{2\sqrt{u}}\right) \cdot \left(\frac{d}{dx}(ax) + \frac{d}{dx}(b)\right)$$

$$\dots \dots \dots \left(\because \frac{d}{dx}(\cos x) = -\sin x, \frac{d}{dx}(\sin x) = \cos x, \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \text{ \& } \frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}\right)$$

$$= (-\sin(\sin v)) \cdot (\cos \sqrt{u}) \cdot \left(\frac{1}{2\sqrt{ax + b}}\right) \cdot (a + 0)$$

$$= (-\sin(\sin \sqrt{u})) \cdot (\cos \sqrt{ax + b}) \cdot \left(\frac{1}{2\sqrt{ax + b}}\right) \cdot (a)$$

$$= \left(\frac{-a \cdot \cos \sqrt{ax + b}}{2\sqrt{ax + b}}\right) \cdot (\sin(\sin \sqrt{ax + b}))$$

27. Question

Differentiate each of the following w.r.t. x :

$$\sqrt{\operatorname{cosec}(x^3 + 1)}$$

Answer

Formulae :

$$\bullet \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$$

$$\bullet \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

$$\bullet \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\bullet \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Let,

$$y = \sqrt{\operatorname{cosec}(x^3 + 1)}$$

$$\text{and } u = x^3 + 1$$

$$\text{therefore, } y = \sqrt{\operatorname{cosec} u}$$

$$\text{let, } v = \operatorname{cosec} u$$

$$\text{therefore, } y = \sqrt{v}$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (\sqrt{v}) \cdot \frac{d}{du} (\operatorname{cosec} u) \cdot \frac{d}{dx} (x^3 + 1)$$

$$= \frac{1}{2\sqrt{v}} \cdot (-\operatorname{cosec} u \cdot \cot u) \cdot \left(\frac{d}{dx} (x^3) + \frac{d}{dx} (1) \right)$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x, \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \text{ \& } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= \frac{1}{2\sqrt{\operatorname{cosec} u}} \cdot (-\operatorname{cosec}(x^3 + 1) \cdot \cot(x^3 + 1)) \cdot (3x^2 + 0)$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (x^n) = n \cdot x^{n-1} \right)$$

$$= \frac{1}{2\sqrt{\operatorname{cosec}(x^3 + 1)}} \cdot (-\operatorname{cosec}(x^3 + 1) \cdot \cot(x^3 + 1)) \cdot (3x^2)$$

$$= \frac{-3x^2}{2} \cdot \sqrt{\operatorname{cosec}(x^3 + 1)} \cdot \cot(x^3 + 1)$$

28. Question

Differentiate each of the following w.r.t. x :

$$\sin 5x \cos 3x$$

Answer

Formulae :

$$\bullet (2\sin a \cdot \cos b) = \sin(a + b) + \sin(a - b)$$

$$\bullet \frac{d}{dx} (\sin x) = \cos x$$

- $\frac{d}{dx} (kx) = k$

- $\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$

Let,

$$y = \sin 5x \cdot \cos 3x$$

$$y = \frac{1}{2} (2 \sin 5x \cdot \cos 3x)$$

$$y = \frac{1}{2} (\sin(5x + 3x) + \sin(5x - 3x)) \dots\dots\dots (\because (2 \sin a \cdot \cos b) = \sin(a + b) + \sin(a - b))$$

$$y = \frac{1}{2} (\sin(8x) + \sin(2x))$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{2} (\sin(8x) + \sin(2x)) \right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \left(\frac{d}{dx} \sin 8x + \frac{d}{dx} \sin 2x \right) \dots\dots\dots \left(\because \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= \frac{1}{2} (8 \cos 8x + 2 \cos 2x) \dots\dots\dots \left(\because \frac{d}{dx} (\sin x) = \cos x \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= 4 \cos 8x + \cos 2x$$

29. Question

Differentiate each of the following w.r.t. x :

$$\sin 2x \sin x$$

Answer

Formulae :

- $(2 \sin a \cdot \sin b) = \cos(a - b) - \cos(a + b)$

- $\frac{d}{dx} (\cos x) = -\sin x$

- $\frac{d}{dx} (kx) = k$

- $\frac{d}{dx} (u - v) = \frac{du}{dx} - \frac{dv}{dx}$

Let,

$$y = \sin 2x \cdot \sin x$$

$$y = \frac{1}{2} (2 \sin 2x \cdot \sin x)$$

$$y = \frac{1}{2} (\cos(2x - x) - \cos(2x + x)) \dots\dots\dots (\because (2 \sin a \cdot \sin b) = \cos(a - b) - \cos(a + b))$$

$$y = \frac{1}{2} (\cos x - \cos 3x)$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{2} (\cos x - \cos 3x) \right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \left(\frac{d}{dx} \cos x - \frac{d}{dx} \cos 3x \right) \dots\dots\dots \left(\because \frac{d}{dx} (u - v) = \frac{du}{dx} - \frac{dv}{dx} \right)$$

$$= \frac{1}{2} (-\sin x + 3 \sin 3x) \dots\dots\dots \left(\because \frac{d}{dx} (\cos x) = -\sin x \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= \frac{3}{2} \sin 3x - \frac{1}{2} \sin x$$

30. Question

Differentiate each of the following w.r.t. x:

$$\cos 4x \cos 2x$$

Answer

Formulae :

$$\bullet (2 \cos a \cdot \cos b) = \cos(a + b) + \cos(a - b)$$

$$\bullet \frac{d}{dx} (\cos x) = -\sin x$$

$$\bullet \frac{d}{dx} (kx) = k$$

$$\bullet \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Let,

$$y = \cos 4x \cdot \cos 2x$$

$$y = \frac{1}{2} (2 \cos 4x \cdot \cos 2x)$$

$$y = \frac{1}{2} (\cos(4x + 2x) + \cos(4x - 2x)) \dots\dots\dots (\because (2 \cos a \cdot \cos b) = \cos(a + b) + \cos(a - b))$$

$$y = \frac{1}{2} (\cos 6x + \cos 2x)$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{2} (\cos 6x + \cos 2x) \right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \left(\frac{d}{dx} \cos 6x + \frac{d}{dx} \cos 2x \right) \dots\dots\dots \left(\because \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= \frac{1}{2} (-6 \sin 6x - 2 \sin 2x) \dots\dots\dots \left(\because \frac{d}{dx} (\cos x) = -\sin x \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= -3 \sin 6x - \sin 2x$$

$$= - (3 \sin 6x + \sin 2x)$$

31. Question

Find $\frac{dy}{dx}$, when:

$$y = \sin \left(\frac{1+x^2}{1-x^2} \right)$$

Answer

Formulae :

$$\bullet \frac{2 \tan x}{1 + \tan^2 x} = \sin 2x$$

$$\bullet \frac{1 + \tan^2 x}{1 - \tan^2 x} = \cos 2x$$

$$\bullet \frac{d}{dx} (\sin x) = \cos x$$

$$\bullet \frac{d}{dx} (\cos x) = -\sin x$$

$$\bullet 1 + \tan^2 x = \sec^2 x$$

Given,

$$y = \sin \left(\frac{1+x^2}{1-x^2} \right)$$

Put $x = \tan a$

Therefore, $\frac{dx}{da} = \sec^2 a \dots\dots\dots \text{eq (1)}$

$$y = \sin \left(\frac{1 + \tan^2 a}{1 - \tan^2 a} \right)$$

$$y = \sin(\cos 2a) \dots\dots\dots \left(\because \frac{1+\tan^2 x}{1-\tan^2 x} = \cos 2x \right)$$

Differentiating above equation w.r.t. a ,

$$\frac{dy}{da} = \frac{d}{da} (\sin(\cos 2a))$$

$$= (\cos(\cos 2a)) \frac{d}{da} (\cos 2a) \dots\dots\dots \left(\because \frac{d}{dx} (\sin x) = \cos x \right)$$

$$= (\cos(\cos 2a)) \cdot (-\sin 2a) \cdot \frac{d}{da} (2a) \dots\dots\dots \left(\because \frac{d}{dx} (\cos x) = -\sin x \right)$$

$$= (-2\sin 2a) \cdot (\cos(\cos 2a))$$

$$= -2 \left(\frac{2 \tan a}{1+\tan^2 a} \right) \cdot \left(\cos \left(\frac{1+\tan^2 a}{1-\tan^2 a} \right) \right) \dots\dots\dots \left(\because \frac{1+\tan^2 x}{1-\tan^2 x} = \cos 2x \text{ \& } \frac{2 \tan x}{1+\tan^2 x} = \sin 2x \right)$$

But, $x = \tan a$

$$\frac{dy}{da} = -2 \left(\frac{2x}{1+x^2} \right) \cdot \left(\cos \left(\frac{1+x^2}{1-x^2} \right) \right)$$

$$\frac{dy}{da} = \left(\frac{-4x}{1+x^2} \right) \cdot \left(\cos \left(\frac{1+x^2}{1-x^2} \right) \right) \dots\dots\dots \text{eq (2)}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{da} \cdot \frac{da}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \left(\frac{-4x}{1+x^2} \right) \cdot \left(\cos \left(\frac{1+x^2}{1-x^2} \right) \right) \cdot \frac{1}{\sec^2 a} \dots\dots\dots \text{from eq (1) \& eq (2)}$$

$$= \left(\frac{-4x}{1+x^2} \right) \cdot \left(\cos \left(\frac{1+x^2}{1-x^2} \right) \right) \cdot \frac{1}{1+\tan^2 a} \dots\dots\dots (\because 1 + \tan^2 x = \sec^2 x)$$

$$= \left(\frac{-4x}{1+x^2} \right) \cdot \left(\cos \left(\frac{1+x^2}{1-x^2} \right) \right) \cdot \frac{1}{1+x^2} \dots\dots\dots (\because x = \tan a)$$

$$\therefore \frac{dy}{dx} = \frac{-4x}{(1+x^2)^2} \cdot \left(\cos \left(\frac{1+x^2}{1-x^2} \right) \right)$$

32. Question

Find $\frac{dy}{dx}$, when:

$$y = \frac{(\sin x + x^2)}{\cot 2x}$$

Answer

Formulae :

- $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \cdot \frac{d}{dx}(u) - u \cdot \frac{d}{dx}(v)}{(v)^2}$
- $\frac{d}{dx} (\sin x) = \cos x$
- $\frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx} (x^n) = n \cdot x^{n-1}$

Given,

$$y = \frac{\sin x + x^2}{\cot 2x}$$

Differentiating above equation w.r.t. x ,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\sin x + x^2}{\cot 2x} \right) \\ &= \frac{\cot 2x \cdot \frac{d}{dx}(\sin x + x^2) - (\sin x + x^2) \cdot \frac{d}{dx}(\cot 2x)}{(\cot 2x)^2} \dots \dots \dots \left(\because \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \cdot \frac{d}{dx}(u) - u \cdot \frac{d}{dx}(v)}{(v)^2} \right) \\ &= \frac{\cot 2x \cdot (\cos 2x + 2x) - (\sin x + x^2) \cdot (-2 \operatorname{cosec}^2 2x)}{(\cot 2x)^2} \\ &\dots \dots \dots \left(\because \frac{d}{dx} (\sin x) = \cos x, \frac{d}{dx} (x^n) = n \cdot x^{n-1} \text{ \& } \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x \right) \\ &= \frac{(\cos 2x + 2x)}{\cot 2x} - \frac{(\sin x + x^2) \cdot (-2 \operatorname{cosec}^2 2x)}{(\cot 2x)^2} \\ &= \tan 2x \cdot (\cos 2x + 2x) + \frac{(\sin x + x^2) \cdot \left(\frac{2}{\sin^2 x} \right)}{\frac{\cos^2 x}{\sin^2 x}} \\ &= \tan 2x \cdot (\cos 2x + 2x) + \frac{2 (\sin x + x^2)}{\cos^2 x} \\ &= \tan 2x \cdot (\cos 2x + 2x) + 2 \sec^2 2x \cdot (\sin x + x^2) \\ \therefore \frac{dy}{dx} &= \tan 2x \cdot (\cos 2x + 2x) + 2 \sec^2 2x \cdot (\sin x + x^2) \end{aligned}$$

33. Question

If $y = \frac{(\cos x - \sin x)}{(\cos x + \sin x)}$, prove that $\frac{dy}{dx} + y^2 + 1 = 0$.

Answer

Formulae :

- $\frac{\sin x}{\cos x} = \tan x$
- $\frac{1 - \tan x}{1 + \tan x} = \tan\left(\frac{\pi}{4} - x\right)$
- $\frac{d}{dx} (\tan x) = \sec^2 x$
- $\frac{d}{dx} (u - v) = \frac{du}{dx} - \frac{dv}{dx}$
- $\frac{d}{dx} (kx) = k$
- $\frac{d}{dx} (k) = 0$
- $\tan^2 x + 1 = \sec^2 x$

Given,

$$y = \frac{(\cos x - \sin x)}{(\cos x + \sin x)}$$

Dividing numerator and denominator by $\cos x$,

$$y = \frac{\left(1 - \frac{\sin x}{\cos x}\right)}{\left(1 + \frac{\sin x}{\cos x}\right)}$$

$$y = \frac{1 - \tan x}{1 + \tan x} \dots\dots\dots \left(\because \frac{\sin x}{\cos x} = \tan x\right)$$

$$y = \tan\left(\frac{\pi}{4} - x\right) \dots\dots\dots \left(\because \frac{1 - \tan x}{1 + \tan x} = \tan\left(\frac{\pi}{4} - x\right)\right)$$

Differentiating above equation w.r.t. x ,

$$\frac{dy}{dx} = \frac{d}{dx} \tan\left(\frac{\pi}{4} - x\right)$$

$$= \sec^2\left(\frac{\pi}{4} - x\right) \cdot \frac{d}{dx}\left(\frac{\pi}{4} - x\right) \dots\dots\dots \left(\because \frac{d}{dx} (\tan x) = \sec^2 x\right)$$

$$= \sec^2\left(\frac{\pi}{4} - x\right) \cdot \left(\frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{d}{dx}(x)\right) \dots\dots\dots \left(\because \frac{d}{dx} (u - v) = \frac{du}{dx} - \frac{dv}{dx}\right)$$

$$= \sec^2\left(x + \frac{\pi}{4}\right) \cdot (0 - 1) \dots\dots\dots \left(\because \frac{d}{dx}(kx) = k \text{ \& } \frac{d}{dx}(k) = 0\right)$$

$$= -\sec^2\left(x + \frac{\pi}{4}\right)$$

$$\therefore \frac{dy}{dx} = -\sec^2\left(x + \frac{\pi}{4}\right)$$

Now,

$$\frac{dy}{dx} + y^2 + 1 = -\sec^2\left(x + \frac{\pi}{4}\right) + \left(\tan^2\left(x + \frac{\pi}{4}\right) + 1\right)$$

$$= -\sec^2\left(x + \frac{\pi}{4}\right) + \left(\sec^2\left(x + \frac{\pi}{4}\right)\right) \dots\dots\dots (\because \tan^2 x + 1 = \sec^2 x)$$

$$= 0$$

$$\therefore \frac{dy}{dx} + y^2 + 1 = 0$$

Hence Proved.

34. Question

If $y = \frac{(\cos x + \sin x)}{(\cos x - \sin x)}$, prove that $\frac{dy}{dx} = \sec^2\left(x + \frac{\pi}{4}\right)$.

Answer

Formulae :

- $\frac{\sin x}{\cos x} = \tan x$
- $\frac{1 + \tan x}{1 - \tan x} = \tan\left(x + \frac{\pi}{4}\right)$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$
- $\frac{d}{dx}(kx) = k$
- $\frac{d}{dx}(k) = 0$

Given,

$$y = \frac{(\cos x + \sin x)}{(\cos x - \sin x)}$$

Dividing numerator and denominator by $\cos x$,

$$y = \frac{\left(1 + \frac{\sin x}{\cos x}\right)}{\left(1 - \frac{\sin x}{\cos x}\right)}$$

$$y = \frac{1 + \tan x}{1 - \tan x} \dots\dots\dots \left(\because \frac{\sin x}{\cos x} = \tan x\right)$$

$$y = \tan\left(x + \frac{\pi}{4}\right) \dots\dots\dots \left(\because \frac{1 + \tan x}{1 - \tan x} = \tan\left(x + \frac{\pi}{4}\right)\right)$$

Differentiating above equation w.r.t. x ,

$$\frac{dy}{dx} = \frac{d}{dx} \tan\left(x + \frac{\pi}{4}\right)$$

$$= \sec^2\left(x + \frac{\pi}{4}\right) \cdot \frac{d}{dx}\left(x + \frac{\pi}{4}\right) \dots\dots\dots \left(\because \frac{d}{dx}(\tan x) = \sec^2 x\right)$$

$$= \sec^2\left(x + \frac{\pi}{4}\right) \cdot \left(\frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{\pi}{4}\right)\right) \dots\dots\dots \left(\because \frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}\right)$$

$$= \sec^2\left(x + \frac{\pi}{4}\right) \cdot (1 + 0) \dots\dots\dots \left(\because \frac{d}{dx}(kx) = k \text{ \& \& } \frac{d}{dx}(k) = 0\right)$$

$$= \sec^2\left(x + \frac{\pi}{4}\right)$$

$$\therefore \frac{dy}{dx} = \sec^2\left(x + \frac{\pi}{4}\right)$$

Hence Proved.

Exercise 10B

1. Question

Differentiate each of the following w.r.t. x :

(i) e^{4x}

(ii) e^{-5x}

(iii) $(e)^{x^3}$

Answer

(i) Let $y = e^{4x}$ $z = 4x$

Formula : $\frac{d(e^x)}{dx} = e^x$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= (e^{4x}) \times 4 \\ &= 4e^{4x}\end{aligned}$$

(ii) Let $y = e^{-5x}$ $z = -5x$

Formula : $\frac{d(e^x)}{dx} = e^x$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= (e^{-5x}) \times (-5) \\ &= -5e^{-5x}\end{aligned}$$

(iii) Let $y = (e)^{x^3}$ $z = x^3$

Formula : $\frac{d(e^x)}{dx} = e^x$, $\frac{d(x^n)}{dx} = n \times x^{n-1}$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= ((e)^{x^3}) \times 3x^2 \\ &= 3x^2(e)^{x^3}\end{aligned}$$

2. Question

Differentiate each of the following w.r.t. x :

(i) $e^{2/x}$

(ii) $e^{\sqrt{x}}$

(iii) $e^{-2\sqrt{x}}$

Answer

(i) Let $y = e^{2/x}$ $z = 2/x$

Formula : $\frac{d(e^x)}{dx} = e^x$, $\frac{d(x^n)}{dx} = n \times x^{n-1}$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= \left(e^{2/x}\right) \times \left(\frac{-2}{x^2}\right) \\ &= \frac{-2}{x^2} \times e^{\frac{2}{x}}\end{aligned}$$

(ii) Let $y = e^{\sqrt{x}} z = \sqrt{x}$

Formula : $\frac{d(e^x)}{dx} = e^x, \frac{d(x^n)}{dx} = n \times x^{n-1}$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= \left(e^{\sqrt{x}}\right) \times \left(\frac{1}{2} \times x^{-0.5}\right) = \left(e^{\sqrt{x}}\right) \times \left(\frac{1}{2 \times \sqrt{x}}\right) \\ &= \frac{e^{\sqrt{x}}}{2\sqrt{x}}\end{aligned}$$

(iii) Let $y = e^{-2\sqrt{x}} z = -2\sqrt{x}$

Formula : $\frac{d(e^x)}{dx} = e^x, \frac{d(x^n)}{dx} = n \times x^{n-1}$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= \left(e^{-2\sqrt{x}}\right) \times \left(-2 \times \frac{1}{2} \times x^{-0.5}\right) = \left(e^{-2\sqrt{x}}\right) \times \left(\frac{-1}{\sqrt{x}}\right) \\ &= \frac{-e^{-2\sqrt{x}}}{\sqrt{x}}\end{aligned}$$

3. Question

Differentiate each of the following w.r.t. x:

(i) $e^{\cot x}$

(ii) $e^{-\sin 2x}$

(iii) $e^{\sqrt{\sin x}}$

Answer

(i) Let $y = e^{\cot x} z = \cot x$

Formula : $\frac{d(e^x)}{dx} = e^x, \frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= (e^{\cot x}) \times (-\operatorname{cosec}^2 x) \\ &= -\operatorname{cosec}^2 x e^{\cot x}\end{aligned}$$

(ii) Let $y = e^{-\sin 2x} z = -\sin 2x$

Formula : $\frac{d(e^x)}{dx} = e^x, \frac{d(\sin x)}{dx} = \cos x$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= (e^{-\sin 2x}) \times (-\cos 2x \times 2) \\ &= (-2 \cos 2x) e^{-\sin 2x}\end{aligned}$$

(iii) Let $y = e^{\sqrt{\sin x}} z = \sqrt{\sin x}$

Formula : $\frac{d(e^x)}{dx} = e^x, \frac{d(\sin x)}{dx} = \cos x$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= (e^{\sqrt{\sin x}}) \times \left(\frac{1}{2} \times (\sin x)^{-0.5} \times \cos x \right) = (e^{\sqrt{\sin x}}) \times \left(\frac{1 \times \cos x}{2\sqrt{\sin x}} \right) \\ &= \frac{\cos x}{2\sqrt{\sin x}} e^{\sqrt{\sin x}}\end{aligned}$$

4. Question

Differentiate each of the following w.r.t. x :

(i) $\tan (\log x)$

(ii) $\log (\sec x)$

(iii) $\log (\sin (x/2))$

Answer

(i) Let $y = \tan(\log x)$ $z = \log x$

$$\text{Formula : } \frac{d(\tan x)}{dx} = \sec^2 x, \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= (\sec^2 \log x) \times \left(\frac{1}{x}\right) \\ &= \frac{\sec^2(\log x)}{x}\end{aligned}$$

(ii) Let $y = \log(\sec x)$ $z = \sec x$

$$\text{Formula : } \frac{d(\sec x)}{dx} = \sec x \times \tan x, \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= \left(\frac{1}{\sec x}\right)(\sec x \times \tan x) \\ &= \tan x\end{aligned}$$

(iii) Let $y = \log(\sin(x/2))$ $z = \sin(x/2)$

$$\text{Formula : } \frac{d(\sin x)}{dx} = \cos x, \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= \left(\frac{1}{\sin(x/2)}\right)\left(\cos(x/2) \times \frac{1}{2}\right) \\ &= \frac{1}{2} \times \cot(x/2)\end{aligned}$$

5. Question

Differentiate each of the following w.r.t. x :

(i) $\log_3 x$

(ii) 2^{-x}

(iii) 3^{x+2}

Answer

(i) Let $y = \log_3 x$

Formula : $\log_a b = \frac{\log b}{\log a}, \frac{d(\log x)}{dx} = \frac{1}{x}$

Therefore $y = \frac{\log x}{\log 3}$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx} \\ &= \left(\frac{1}{\log 3}\right) \left(\frac{1}{x}\right) \\ &= \frac{1}{x(\log 3)}\end{aligned}$$

(ii) Let $y = 2^{-x}$ $z = -x$

Formula : $\frac{d(a^x)}{dx} = a^x (\log a)$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= (2^{-x}) \times (\log 2)(-1) \\ &= -2^{-x}(\log 2)\end{aligned}$$

(iii) Let $y = 3^{x+2}$ $z = x$

Therefore $Y = 3^2 \times 3^x$

Formula : $\frac{d(a^x)}{dx} = a^x (\log a)$

According to chain rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= 9(3^x) \times (\log 3)\end{aligned}$$

6. Question

Differentiate each of the following w.r.t. x :

$$(i) \log\left(x + \frac{1}{x}\right)$$

$$(ii) \log(\sin(3x))$$

$$(iii) \log\left(x + \sqrt{1+x^2}\right)$$

Answer

$$(i) \text{ Let } y = \log\left(x + \frac{1}{x}\right) \quad z = x + \frac{1}{x}$$

$$\text{Formula : } \frac{d(\log x)}{dx} = \frac{1}{x}, \quad \frac{d(x^n)}{dx} = n \times x^{n-1}$$

According to chain rule of differentiation

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

$$= \left(\frac{1}{x + \frac{1}{x}}\right) \times \left(1 - \frac{1}{x^2}\right)$$

$$= \left(\frac{x}{x^2 + 1}\right) \times \left(\frac{x^2 - 1}{x^2}\right)$$

$$= \left(\frac{x^2 - 1}{x(x^2 + 1)}\right)$$

$$(ii) \text{ Let } y = \log(\sin(3x)) \quad z = \sin(3x)$$

$$\text{Formula : } \frac{d(\sin x)}{dx} = \cos x, \quad \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to chain rule of differentiation

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

$$= \left(\frac{1}{\sin(3x)}\right) (\cos(3x) \times 3)$$

$$= 3 \times \cot(3x)$$

$$(iii) \text{ Let } y = \log\left(x + \sqrt{1+x^2}\right) \quad z = x + \sqrt{1+x^2}$$

$$\text{Formula : } \frac{d(\log x)}{dx} = \frac{1}{x}, \quad \frac{d(x^n)}{dx} = n \times x^{n-1}$$

According to chain rule of differentiation

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\
&= \left(\frac{1}{x + \sqrt{1+x^2}} \right) \times \left(1 + \frac{1}{2} (1+x^2)^{-0.5} 2x \right) \\
&= \left(\frac{1}{x + \sqrt{1+x^2}} \right) \times \left(1 + \frac{x}{1} (1+x^2)^{-0.5} \right) \\
&= \left(\frac{1}{x + \sqrt{1+x^2}} \right) \times \left(1 + \frac{x}{\sqrt{1+x^2}} \right) \\
&= \left(\frac{1}{x + \sqrt{1+x^2}} \right) \times \left(\frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} \right) \\
&= \left(\frac{1}{\sqrt{1+x^2}} \right)
\end{aligned}$$

7. Question

Differentiate each of the following w.r.t. x :

$$e^{\sqrt{x}} \log x$$

Answer

Let $y = e^{\sqrt{x}} \log x$, $z = e^{\sqrt{x}}$ and $w = \log (x)$

$$\text{Formula : } \frac{d(e^x)}{dx} = e^x, \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to product rule of differentiation

$$\begin{aligned}
\frac{dy}{dx} &= w \times \frac{dz}{dx} + z \times \frac{dw}{dx} \\
&= [\log (x) \times (e^{\sqrt{x}}) \times \frac{1}{2\sqrt{x}}] + [e^{\sqrt{x}} \times \frac{1}{x}] \\
&= e^{\sqrt{x}} \times \left[\frac{\log (x)}{2\sqrt{x}} + \frac{1}{x} \right] \\
&= e^{\sqrt{x}} \times \left[\frac{\sqrt{x} \log (x)}{2x} + \frac{2}{2x} \right] \\
&= e^{\sqrt{x}} \times \left[\frac{2 + \sqrt{x} \log (x)}{2x} \right]
\end{aligned}$$

8. Question

Differentiate each of the following w.r.t. x :

$$\log \sin \sqrt{x^2 + 1}$$

Answer

$$\text{Let } y = \log \sin \sqrt{1 + x^2}, z = \sin \sqrt{1 + x^2}$$

$$\text{Formula : } \frac{d(\sin x)}{dx} = \cos x, \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to chain rule of differentiation

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= \left[\frac{1}{\sin \sqrt{1 + x^2}} \right] \times [\cos \sqrt{1 + x^2}] \times \left[\frac{1}{2} \times \frac{1}{\sqrt{1 + x^2}} \times 2x \right] \\ &= [\cot \sqrt{1 + x^2}] \times \left[\frac{1}{1} \times \frac{1}{\sqrt{1 + x^2}} \times x \right] \\ &= \frac{x}{\sqrt{x^2 + 1}} \cot \sqrt{x^2 + 1} \end{aligned}$$

9. Question

Differentiate each of the following w.r.t. x:

$$e^{2x} \sin 3x$$

Answer

$$\text{Let } y = e^{2x} \sin 3x, z = e^{2x} \text{ and } w = \sin 3x$$

$$\text{Formula : } \frac{d(e^x)}{dx} = e^x \text{ and } \frac{d(\sin x)}{dx} = \cos x$$

According to product rule of differentiation

$$\begin{aligned} \frac{dy}{dx} &= w \times \frac{dz}{dx} + z \times \frac{dw}{dx} \\ &= [\sin 3x \times (2 \times e^{2x})] + [e^{2x} \times 3 \cos 3x] \\ &= e^{2x} \times [2 \sin 3x + 3 \cos 3x] \end{aligned}$$

10. Question

Differentiate each of the following w.r.t. x:

$$e^{3x} \cos 2x$$

Answer

$$\text{Let } y = e^{3x} \cos 2x, z = e^{3x} \text{ and } w = \cos 2x$$

Formula : $\frac{d(e^x)}{dx} = e^x$ and $\frac{d(\cos x)}{dx} = -\sin x$

According to product rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= w \times \frac{dz}{dx} + z \times \frac{dw}{dx} \\ &= [\cos 2x \times (3 \times e^{3x})] + [e^{3x} \times (-2 \sin 2x)] \\ &= e^{3x} \times [3 \cos 2x - 2 \sin 2x]\end{aligned}$$

11. Question

Differentiate each of the following w.r.t. x:

$$e^{-5x} \cot 4x$$

Answer

Let $y = e^{-5x} \cot 4x$, $z = e^{-5x}$ and $w = \cot 4x$

Formula : $\frac{d(e^x)}{dx} = e^x$ and $\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

According to product rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= w \times \frac{dz}{dx} + z \times \frac{dw}{dx} \\ &= [\cot 4x \times (-5e^{-5x})] + [e^{-5x} \times (-4 \operatorname{cosec}^2 4x)] \\ &= -e^{-5x} \times [5 \cot 4x + 4 \operatorname{cosec}^2 4x]\end{aligned}$$

12. Question

Differentiate each of the following w.r.t. x:

$$e^x \log (\sin 2x)$$

Answer

Let $y = e^x \log (\sin 2x)$, $z = e^x$ and $w = \log (\sin 2x)$

Formula : $\frac{d(e^x)}{dx} = e^x$, $\frac{d(\log x)}{dx} = \frac{1}{x}$ and $\frac{d(\sin x)}{dx} = \cos x$

According to product rule of differentiation

$$\begin{aligned}\frac{dy}{dx} &= w \times \frac{dz}{dx} + z \times \frac{dw}{dx} \\ &= [\log (\sin 2x) \times (e^x)] + [e^x \times \frac{1}{\sin 2x} \times 2 \cos 2x]\end{aligned}$$

$$= e^x \times \left[\log(\sin 2x) + \frac{2 \cos 2x}{\sin 2x} \right]$$

$$= e^x \times [\log(\sin 2x) + 2 \cot 2x]$$

13. Question

Differentiate each of the following w.r.t. x:

$$\log(\operatorname{cosec} x - \cot x)$$

Answer

$$\text{Let } y = \log(\operatorname{cosec} x - \cot x), z = (\operatorname{cosec} x - \cot x)$$

Formula :

$$\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x, \frac{d(\log x)}{dx} = \frac{1}{x} \text{ and } \frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$$

According to chain rule of differentiation

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

$$= \left[\frac{1}{(\operatorname{cosec} x - \cot x)} \right] \times [-\operatorname{cosec} x \cot x - (-\operatorname{cosec}^2 x)]$$

$$= \left[\frac{1}{(\operatorname{cosec} x - \cot x)} \right] \times [-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x]$$

$$= \left[\frac{1}{(\operatorname{cosec} x - \cot x)} \right] \times [\operatorname{cosec} x (\operatorname{cosec} x - \cot x)]$$

$$= \operatorname{cosec} x$$

14. Question

Differentiate each of the following w.r.t. x:

$$\log \left(\sec \frac{x}{2} + \tan \frac{x}{2} \right)$$

Answer

$$\text{Let } y = \log \left(\sec \frac{x}{2} + \tan \frac{x}{2} \right), z = \left(\sec \frac{x}{2} + \tan \frac{x}{2} \right)$$

Formula :

$$\frac{d(\sec x)}{dx} = \sec x \tan x, \frac{d(\log x)}{dx} = \frac{1}{x} \text{ and } \frac{d(\tan x)}{dx} = \sec^2 x$$

According to chain rule of differentiation

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\
 &= \left[\frac{1}{\left(\sec \frac{x}{2} + \tan \frac{x}{2}\right)} \right] \times \left[\left(\sec \frac{x}{2} \tan \frac{x}{2} \times \frac{1}{2} \right) + \left(\sec^2 \frac{x}{2} \times \frac{1}{2} \right) \right] \\
 &= \left[\frac{1}{\left(\sec \frac{x}{2} + \tan \frac{x}{2}\right)} \right] \times \left[\frac{1}{2} \sec \frac{x}{2} \left(\sec \frac{x}{2} + \tan \frac{x}{2} \right) \right] \\
 &= \frac{1}{2} \sec \frac{x}{2}
 \end{aligned}$$

15. Question

Differentiate each of the following w.r.t. x:

$$\sqrt{\frac{1+e^x}{1-e^x}}$$

Answer

$$\text{Let } y = \sqrt{\frac{1+e^x}{1-e^x}}, \quad u = 1 + e^x, \quad v = 1 - e^x, \quad z = \frac{1+e^x}{1-e^x}$$

$$\text{Formula : } \frac{d(e^x)}{dx} = e^x$$

According to quotient rule of differentiation

$$\text{If } z = \frac{u}{v}$$

$$\begin{aligned}
 \frac{dz}{dx} &= \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2} \\
 &= \frac{(1 - e^x) \times (e^x) - (1 + e^x) \times (-e^x)}{(1 - e^x)^2} \\
 &= \frac{e^x - e^{2x} + e^x + e^{2x}}{(1 - e^x)^2} \\
 &= \frac{2e^x}{(1 - e^x)^2}
 \end{aligned}$$

According to chain rule of differentiation

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

$$\begin{aligned}
&= \left[\frac{1}{2} \times \left(\frac{1+e^x}{1-e^x} \right)^{\frac{1}{2}-1} \right] \times \left[\frac{2e^x}{(1-e^x)^2} \right] \\
&= \left[\frac{e^x}{1} \times \left(\frac{1+e^x}{1} \right)^{-\frac{1}{2}} \right] \times \left[\frac{1}{(1-e^x)^{2-\frac{1}{2}}} \right] \\
&= \left[\frac{e^x}{(1+e^x)^{\frac{1}{2}} \times (1-e^x)^{2-\frac{1}{2}}} \right] \\
&= \left[\frac{e^x}{(1+e^x)^{\frac{1}{2}} \times (1-e^x)^{\frac{1}{2}} \times (1-e^x)^1} \right] \\
&= \left[\frac{e^x}{((1+e^x)(1-e^x))^{\frac{1}{2}} \times (1-e^x)^1} \right] \\
&= \frac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}
\end{aligned}$$

16. Question

Differentiate each of the following w.r.t. x :

$$\frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Answer

$$\text{Let } y = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad u = e^x + e^{-x}, \quad v = e^x - e^{-x}$$

$$\text{Formula : } \frac{d(e^x)}{dx} = e^x$$

According to quotient rule of differentiation

$$\text{If } y = \frac{u}{v}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2} \\
&= \frac{(e^x - e^{-x}) \times (e^x - e^{-x}) - (e^x + e^{-x}) \times (e^x + e^{-x})}{(e^x - e^{-x})^2} \\
&= \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2}
\end{aligned}$$

$$= \frac{(e^x - e^{-x} + e^x + e^{-x})(e^x - e^{-x} - e^x - e^{-x})}{(e^x - e^{-x})^2}$$

$$(a^2 - b^2 = (a - b)(a + b))$$

$$= \frac{(2e^x)(-2e^{-x})}{(e^x - e^{-x})^2}$$

$$= \frac{-4}{(e^x - e^{-x})^2}$$

17. Question

Differentiate each of the following w.r.t. x:

$$xe^{\sqrt{\sin x}}$$

Answer

$$\text{Let } y = xe^{\sqrt{\sin x}}, z = x \text{ and } w = e^{\sqrt{\sin x}}$$

$$\text{Formula : } \frac{d(e^x)}{dx} = e^x, \frac{d(\sin x)}{dx} = \cos x$$

According to product rule of differentiation

$$\begin{aligned} \frac{dy}{dx} &= w \times \frac{dz}{dx} + z \times \frac{dw}{dx} \\ &= [e^{\sqrt{\sin x}} \times (1)] + [x \times e^{\sqrt{\sin x}} \times \frac{1}{2} \times \frac{1}{\sqrt{\sin x}} \times \cos x] \\ &= e^{\sqrt{\sin x}} \times \left[1 + \frac{x \cos x}{2\sqrt{\sin x}}\right] \end{aligned}$$

18. Question

Differentiate each of the following w.r.t. x:

$$e^{\sin x} \sin(e^x)$$

Answer

$$\text{Let } y = e^{\sin x} \sin e^x, z = e^{\sin x} \text{ and } w = \sin e^x$$

$$\text{Formula : } \frac{d(e^x)}{dx} = e^x, \frac{d(\sin x)}{dx} = \cos x$$

According to product rule of differentiation

$$\frac{dy}{dx} = w \times \frac{dz}{dx} + z \times \frac{dw}{dx}$$

$$\begin{aligned}
&= [\sin e^x \times (e^{\sin x} \times \cos x)] + [e^{\sin x} \times \cos e^x \times e^x] \\
&= e^{\sin x} [(\sin e^x \times \cos x) + (\cos e^x \times e^x)] \\
&= e^{\sin x} (e^x \cos e^x + \cos x \sin e^x)
\end{aligned}$$

19. Question

Differentiate each of the following w.r.t. x :

$$e^{\sqrt{1-x^2}} \tan x$$

Answer

$$\text{Let } y = e^{\sqrt{1-x^2}} \tan x, \quad z = e^{\sqrt{1-x^2}} \text{ and } w = \tan x$$

$$\text{Formula : } \frac{d(e^x)}{dx} = e^x, \quad \frac{d(\tan x)}{dx} = \sec^2 x$$

According to product rule of differentiation

$$\begin{aligned}
\frac{dy}{dx} &= w \times \frac{dz}{dx} + z \times \frac{dw}{dx} \\
&= [\tan x \times (e^{\sqrt{1-x^2}} \times \frac{1}{2} \times \frac{1}{\sqrt{1-x^2}} \times (-2x))] + [e^{\sqrt{1-x^2}} \times \sec^2 x] \\
&= e^{\sqrt{1-x^2}} \times \left[\sec^2 x - \frac{x \tan x}{\sqrt{1-x^2}} \right]
\end{aligned}$$

20. Question

Differentiate each of the following w.r.t. x :

$$\frac{e^x}{1 + \cos x}$$

Answer

$$\text{Let } y = \frac{e^x}{1 + \cos x}, \quad u = e^x, \quad v = 1 + \cos x$$

$$\text{Formula: } \frac{d(e^x)}{dx} = e^x, \quad \frac{d(\cos x)}{dx} = -\sin x$$

According to quotient rule of differentiation

$$\text{If } y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$

$$= \frac{(1 + \cos x) \times (e^x) - (e^x) \times (-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{e^x(1 + \cos x + \sin x)}{(1 + \cos x)^2}$$

21. Question

Differentiate each of the following w.r.t. x:

$$x^3 e^x \cos x$$

Answer

Let $y = x^3 e^x \cos x$, $z = x^3$ and $w = e^x \cos x$

Formula : $\frac{d(e^x)}{dx} = e^x$ and $\frac{d(\cos x)}{dx} = -\sin x$

$$\frac{dw}{dx} = [\cos x \times (e^x)] + [e^x \times (-\sin x)] = e^x[\cos x - \sin x]$$

According to product rule of differentiation

$$\begin{aligned} \frac{dy}{dx} &= w \times \frac{dz}{dx} + z \times \frac{dw}{dx} \\ &= [e^x \cos x \times (3x^2)] + [x^3 \times (e^x[\cos x - \sin x])] \\ &= e^x x^2 \times [3 \cos x + x \cos x - x \sin x] \\ &= e^x x^2 (x \cos x - x \sin x + 3 \cos x) \end{aligned}$$

22. Question

Differentiate each of the following w.r.t. x:

$$e^{x \cos x}$$

Answer

Let $y = e^{x \cos x}$, $z = x \cos x$

Formula : $\frac{d(e^x)}{dx} = e^x$ and $\frac{d(\cos x)}{dx} = -\sin x$

$$\frac{dz}{dx} = [\cos x \times (1)] + [x \times (-\sin x)] = [\cos x - x \sin x] \text{ (Using product rule)}$$

According to chain rule of differentiation

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

$$= [e^{x \cos x}] \times [\cos x - x \sin x]$$

$$= e^{x \cos x} (\cos x - x \sin x)$$

Exercise 10C

1. Question

Differentiate each of the following w.r.t. x:

$$\cos^{-1} 2x$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\text{ii) } \frac{d}{dx} (kx) = k$$

Answer :

Let,

$$y = \cos^{-1} 2x$$

$$\text{and } u = 2x$$

$$\text{therefore, } y = \cos^{-1} u$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\cos^{-1} u) \cdot \frac{d}{dx} (2x)$$

$$= \frac{-1}{\sqrt{1-u^2}} \cdot 2$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= \frac{-2}{\sqrt{1-(2x)^2}}$$

$$= \frac{-2}{\sqrt{1-4x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{-2}{\sqrt{1-4x^2}}$$

2. Question

Differentiate each of the following w.r.t. x:

$$\tan^{-1} x^2$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\text{ii) } \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

Answer :

Let,

$$y = \tan^{-1} x^2$$

$$\text{and } u = x^2$$

$$\text{therefore, } y = \tan^{-1} u$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\tan^{-1} u) \cdot \frac{d}{dx} (x^2)$$

$$= \frac{1}{1+u^2} \cdot 2x$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \text{ \& } \frac{d}{dx} (x^n) = n \cdot x^{n-1} \right)$$

$$= \frac{2x}{1+(x^2)^2}$$

$$= \frac{2x}{1+x^4}$$

$$\therefore \frac{dy}{dx} = \frac{2x}{1+x^4}$$

3. Question

Differentiate each of the following w.r.t. x:

$$\sec^{-1} \sqrt{x}$$

Answer

Formulae :

$$i) \frac{d}{dx} (\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$ii) \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

Answer :

Let,

$$y = \sec^{-1} \sqrt{x}$$

$$\text{and } u = \sqrt{x}$$

$$\text{therefore, } y = \sec^{-1} u$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\sec^{-1} u) \cdot \frac{d}{dx} (\sqrt{x})$$

$$= \frac{1}{u\sqrt{u^2-1}} \cdot \frac{1}{2\sqrt{x}}$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \text{ \& } \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \right)$$

$$= \frac{1}{\sqrt{x} \sqrt{(\sqrt{x})^2-1}} \cdot \left(\frac{1}{2\sqrt{x}} \right)$$

$$= \frac{1}{2\sqrt{x} \cdot \sqrt{x\sqrt{x}-1}}$$

$$= \frac{1}{2x\sqrt{x-1}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2x\sqrt{x-1}}$$

4. Question

Differentiate each of the following w.r.t. x:

$$\sin^{-1} \frac{x}{a}$$

Answer

Formulae :

$$i) \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$ii) \frac{d}{dx} (kx) = k$$

Answer :

Let,

$$y = \sin^{-1} \left(\frac{x}{a} \right)$$

$$\text{and } u = \frac{x}{a}$$

$$\text{therefore, } y = \sin^{-1} u$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\sin^{-1} u) \cdot \frac{d}{dx} \left(\frac{x}{a} \right)$$

$$= \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{a}$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{\frac{a^2 - x^2}{a^2}}} \cdot \frac{1}{a}$$

$$= \frac{a}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{a^2 - x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

5. Question

Differentiate each of the following w.r.t. x:

$$\tan^{-1}(\log x)$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\text{ii) } \frac{d}{dx} (\log x) = \frac{1}{x}$$

Answer :

Let,

$$y = \tan^{-1} (\log x)$$

and $u = \log x$

therefore, $y = \tan^{-1} u$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\tan^{-1} u) \cdot \frac{d}{dx} (\log x)$$

$$= \frac{1}{1+u^2} \cdot \frac{1}{x}$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \text{ \& } \frac{d}{dx} (\log x) = \frac{1}{x} \right)$$

$$= \frac{1}{1+(\log x)^2} \cdot \frac{1}{x}$$

$$= \frac{1}{x \{1 + (\log x)^2\}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x \{1 + (\log x)^2\}}$$

6. Question

Differentiate each of the following w.r.t. x:

$$\cot^{-1}(e^x)$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\cot^{-1}x) = \frac{-1}{1+x^2}$$

$$\text{ii) } \frac{d}{dx} (e^x) = e^x$$

Answer :

Let,

$$y = \cot^{-1}(e^x)$$

$$\text{and } u = e^x$$

$$\text{therefore, } y = \cot^{-1}u$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\cot^{-1}u) \cdot \frac{d}{dx} (e^x)$$

$$= \frac{-1}{1+u^2} \cdot e^x$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\cot^{-1}x) = \frac{-1}{1+x^2} \text{ \& } \frac{d}{dx} (e^x) = e^x \right)$$

$$= \frac{-1}{1+(e^x)^2} \cdot e^x$$

$$= \frac{-e^x}{1+e^{2x}}$$

$$\therefore \frac{dy}{dx} = \frac{-e^x}{1+e^{2x}}$$

7. Question

Differentiate each of the following w.r.t. x:

$$\log(\tan^{-1}x)$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\log x) = \frac{1}{x}$$

$$\text{ii) } \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

Answer :

Let,

$$y = \log(\tan^{-1})$$

$$\text{and } u = \tan^{-1} x$$

$$\text{therefore, } y = \log u$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\log u) \cdot \frac{d}{dx} (\tan^{-1} x)$$

$$= \frac{1}{u} \cdot \frac{1}{1+x^2}$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\log x) = \frac{1}{x} \text{ \& } \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \right)$$

$$= \frac{1}{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

$$= \frac{1}{(1+x^2) \cdot \tan^{-1} x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(1+x^2) \cdot \tan^{-1} x}$$

8. Question

Differentiate each of the following w.r.t. x :

$$\cot^{-1} x^3$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\text{ii) } \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

Answer :

Let,

$$y = \cot^{-1} (x^3)$$

$$\text{and } u = x^3$$

$$\text{therefore, } y = \cot^{-1} u$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\cot^{-1} u) \cdot \frac{d}{dx} (x^3)$$

$$= \frac{-1}{1+u^2} \cdot 3x^2$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{1+x^2} \text{ \& } \frac{d}{dx} (x^n) = n \cdot x^{n-1} \right)$$

$$= \frac{-1}{1+(x^3)^2} \cdot 3x^2$$

$$= \frac{-3x^2}{1+x^6}$$

$$\therefore \frac{dy}{dx} = \frac{-3x^2}{1+x^6}$$

9. Question

Differentiate each of the following w.r.t. x :

$$\sin^{-1}(\cos x)$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\text{ii) } \frac{d}{dx} (\cos x) = -\sin x$$

$$\text{iii) } \sin^2 x + \cos^2 x = 1$$

Answer :

Let,

$$y = \sin^{-1}(\cos x)$$

$$\text{and } u = \cos x$$

$$\text{therefore, } y = \sin^{-1}u$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\sin^{-1}u) \cdot \frac{d}{dx} (\cos x)$$

$$= \frac{1}{\sqrt{1-u^2}} \cdot (-\sin x)$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \text{ \& } \frac{d}{dx} (\cos x) = -\sin x \right)$$

$$= \frac{1}{\sqrt{1-(\cos x)^2}} \cdot (-\sin x)$$

$$= \frac{1}{\sqrt{\sin^2 x}} \cdot (-\sin x) \dots\dots\dots (\because \sin^2 x + \cos^2 x = 1)$$

$$= \frac{1}{\sin x} \cdot (-\sin x)$$

$$= -1$$

$$\therefore \frac{dy}{dx} = -1$$

10. Question

Differentiate each of the following w.r.t. x :

$$(1+x^2) \tan^{-1} x$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\text{ii) } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$\text{iii) } \frac{d}{dx} (\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\text{iv) } \frac{d}{dx} (k) = 0$$

$$\text{v) } \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

Answer :

Let,

$$y = (1 + x^2)\tan^{-1}x$$

$$\text{Let, } u = (1+x^2) \text{ and } v = \tan^{-1}x$$

therefore, $y = u \cdot v$

$$\therefore \frac{dy}{dx} = (1 + x^2) \cdot \frac{d}{dx} (\tan^{-1}x) + (\tan^{-1}u) \cdot \frac{d}{dx} (1 + x^2)$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx} \right)$$

$$= (1 + x^2) \cdot \frac{1}{1 + x^2} + (\tan^{-1}x) \left\{ \frac{d}{dx} (1) + \frac{d}{dx} (x^2) \right\}$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\tan^{-1}x) = \frac{1}{1+x^2} \text{ \& } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= 1 + (\tan^{-1}x)(0 + 2x)$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (k) = 0 \text{ \& } \frac{d}{dx} (x^n) = n \cdot x^{n-1} \right)$$

$$= 1 + 2x \tan^{-1}x$$

$$\therefore \frac{dy}{dx} = 1 + 2x \tan^{-1}x$$

11. Question

Differentiate each of the following w.r.t. x :

$$\tan^{-1}(\cot x)$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\text{ii) } \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$$

$$\text{iii) } 1 + \cot^2 x = \operatorname{cosec}^2 x$$

Answer :

Let,

$$y = \tan^{-1}(\cot x)$$

and $u = \cot x$

therefore, $y = \tan^{-1} u$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\tan^{-1} u) \cdot \frac{d}{dx} (\cot x)$$

$$= \frac{1}{1 + u^2} \cdot (-\operatorname{cosec}^2 x)$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \text{ \& } \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x \right)$$

$$= \frac{-\operatorname{cosec}^2 x}{1 + (\cot x)^2}$$

$$= \frac{-\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} \dots\dots\dots (\because 1 + \cot^2 x = \operatorname{cosec}^2 x)$$

$$= -1$$

$$\therefore \frac{dy}{dx} = -1$$

12. Question

Differentiate each of the following w.r.t. x :

$$\log(\sin^{-1} x^4)$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\log x) = \frac{1}{x}$$

$$\text{ii) } \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\text{iii) } \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

Answer :

Let,

$$y = \log(\sin^{-1}x^4)$$

$$\text{and } u = x^4$$

$$\text{therefore, } y = \log(\sin^{-1}u)$$

$$\text{let, } v = \sin^{-1}u$$

$$\text{therefore, } y = \log v$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (\log v) \cdot \frac{d}{du} (\sin^{-1}u) \cdot \frac{d}{dx} (x^4)$$

$$= \frac{1}{v} \cdot \left(\frac{1}{\sqrt{1-u^2}} \right) \cdot 4x^3$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\log x) = \frac{1}{x}, \frac{d}{dx} (\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \text{ \& } \frac{d}{dx} (x^n) = n \cdot x^{n-1} \right)$$

$$= \frac{1}{\sin^{-1}u} \cdot \left(\frac{1}{\sqrt{1-(x^4)^2}} \right) \cdot 4x^3$$

$$= \frac{1}{\sin^{-1}x^4} \cdot \left(\frac{1}{\sqrt{1-x^8}} \right) \cdot 4x^3$$

$$= \frac{4x^3}{\sin^{-1}x^4 \cdot \sqrt{1-x^8}}$$

$$\therefore \frac{dy}{dx} = \frac{4x^3}{\sin^{-1}x^4 \cdot \sqrt{1-x^8}}$$

13. Question

Differentiate each of the following w.r.t. x :

$$(\cot^{-1}x^2)^3$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\cot^{-1}x) = \frac{-1}{1+x^2}$$

$$\text{ii) } \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

Answer :

Let,

$$y = (\cot^{-1} x^2)^3$$

$$\text{and } u = x^2$$

$$\text{therefore, } y = (\cot^{-1} u)^3$$

$$\text{let, } v = \cot^{-1} u$$

$$\text{therefore, } y = v^3$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (v^3) \cdot \frac{d}{du} (\cot^{-1} u) \cdot \frac{d}{dx} (x^2)$$

$$= 3v^2 \cdot \left(\frac{-1}{1+u^2} \right) \cdot 2x$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{1+x^2} \text{ \& } \frac{d}{dx} (x^n) = n \cdot x^{n-1} \right)$$

$$= 3(\cot^{-1} u)^2 \cdot \left(\frac{-1}{1+(x^2)^2} \right) \cdot 2x$$

$$= (\cot^{-1}(x^2))^2 \cdot \frac{-6x}{1+(x^2)^2}$$

$$= \frac{-6x (\cot^{-1}(x^2))^2}{1+x^4}$$

$$\therefore \frac{dy}{dx} = \frac{-6x (\cot^{-1}(x^2))^2}{1+x^4}$$

14. Question

Differentiate each of the following w.r.t. x:

$$\tan^{-1}(\cos \sqrt{x})$$

Answer

Formulae :

$$i) \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$ii) \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$iii) \frac{d}{dx} (\cos x) = -\sin x$$

Answer :

Let,

$$y = \tan^{-1}(\cos \sqrt{x})$$

$$\text{and } u = \sqrt{x}$$

$$\text{therefore, } y = \tan^{-1}(\cos u)$$

$$\text{let, } v = \cos u$$

$$\text{therefore, } y = \tan^{-1} v$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (\tan^{-1} v) \cdot \frac{d}{du} (\cos u) \cdot \frac{d}{dx} (\sqrt{x})$$

$$= \frac{1}{1+v^2} \cdot (-\sin u) \cdot \frac{1}{2\sqrt{x}}$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}, \frac{d}{dx} (\cos x) = -\sin x \text{ \& } \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \right)$$

$$= \frac{1}{1+(\cos u)^2} \cdot (-\sin \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{1+(\cos \sqrt{x})^2} \cdot (-\sin \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{-\sin \sqrt{x}}{2\sqrt{x} (1+(\cos \sqrt{x})^2)}$$

$$\therefore \frac{dy}{dx} = \frac{-\sin \sqrt{x}}{2\sqrt{x} (1+(\cos \sqrt{x})^2)}$$

15. Question

Differentiate each of the following w.r.t. x:

$$\tan(\sin^{-1} x)$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\text{ii) } \frac{d}{dx} (\tan x) = \sec^2 x$$

Answer :

Let,

$$y = \tan(\sin^{-1} x)$$

$$\text{and } u = \sin^{-1} x$$

$$\text{therefore, } y = \tan u$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{du} (\tan u) \cdot \frac{d}{dx} (\sin^{-1} x)$$

$$= \sec^2 u \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\tan x) = \sec^2 x \text{ \& } \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \right)$$

$$= \sec^2 (\sin^{-1} x) \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{\sec^2 (\sin^{-1} x)}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 (\sin^{-1} x)}{\sqrt{1-x^2}}$$

16. Question

Differentiate each of the following w.r.t. x:

$$e^{\tan^{-1} \sqrt{x}}$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\text{ii) } \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\text{iii) } \frac{d}{dx} (e^x) = e^x$$

Answer :

Let,

$$y = e^{\tan^{-1} \sqrt{x}}$$

$$\text{and } u = \sqrt{x}$$

$$\text{therefore, } y = e^{\tan^{-1} u}$$

$$\text{let, } v = \tan^{-1} u$$

$$\text{therefore, } y = e^v$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (e^v) \cdot \frac{d}{du} (\tan^{-1} u) \cdot \frac{d}{dx} (\sqrt{x})$$

$$= e^v \cdot \left(\frac{1}{1+u^2} \right) \cdot \frac{1}{2\sqrt{x}}$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}, \frac{d}{dx} (e^x) = e^x \text{ \& } \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}} \right)$$

$$= e^{\tan^{-1} u} \cdot \left(\frac{1}{1+(\sqrt{x})^2} \right) \cdot \frac{1}{2\sqrt{x}}$$

$$= e^{\tan^{-1} \sqrt{x}} \cdot \left(\frac{1}{1+x} \right) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{e^{\tan^{-1} \sqrt{x}}}{2\sqrt{x}(1+x)}$$

$$\therefore \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}$$

17. Question

Differentiate each of the following w.r.t. x:

$$\sqrt{\sin^{-1} x^2}$$

Answer

Formulae :

$$\text{i) } \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\text{ii) } \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\text{iii) } \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

Answer :

Let,

$$y = \sqrt{\sin^{-1} x^2}$$

$$\text{and } u = x^2$$

$$\text{therefore, } y = \sqrt{\sin^{-1} u}$$

$$\text{let, } v = \sin^{-1} u$$

$$\text{therefore, } y = \sqrt{v}$$

Differentiating above equation w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dv} (\sqrt{v}) \cdot \frac{d}{du} (\sin^{-1} u) \cdot \frac{d}{dx} (x^2)$$

$$= \frac{1}{2\sqrt{v}} \cdot \left(\frac{1}{\sqrt{1-u^2}} \right) \cdot 2x$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}, \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \text{ \& } \frac{d}{dx} (x^n) = n \cdot x^{n-1} \right)$$

$$= \frac{1}{2\sqrt{\sin^{-1} u}} \cdot \left(\frac{1}{\sqrt{1-(x^2)^2}} \right) \cdot 2x$$

$$= \frac{1}{\sqrt{\sin^{-1}(x^2)}} \cdot \left(\frac{1}{\sqrt{1-x^4}} \right) \cdot x$$

$$= \frac{x}{\sqrt{\sin^{-1}(x^2)} (\sqrt{1-x^4})}$$

$$\therefore \frac{dy}{dx} = \frac{x}{\sqrt{\sin^{-1}(x^2)} (\sqrt{1-x^4})}$$

18. Question

If $y = \sin^{-1}(\cos x) + \cos^{-1}(\sin x)$, prove that $\frac{dy}{dx} = -2$.

Answer

Given : $y = \sin^{-1}(\cos x) + \cos^{-1}(\sin x)$

To Prove : $\frac{dy}{dx} = -2$

Formulae :

$$\text{i) } \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\text{ii) } \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\text{iii) } \frac{d}{dx} (\cos x) = -\sin x$$

$$\text{iv) } \frac{d}{dx} (\sin x) = \cos x$$

$$\text{v) } \sin^2 x + \cos^2 x = 1$$

$$\text{vi) } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Answer :

Given equation,

$$y = \sin^{-1}(\cos x) + \cos^{-1}(\sin x)$$

$$\text{Let } s = \sin^{-1}(\cos x) \text{ \& } t = \cos^{-1}(\sin x)$$

Therefore, $y = s + t$ eq(1)

I. For $\sin^{-1}(\cos x)$

let $u = \cos x$

therefore, $s = \sin^{-1}u$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{ds}{dx} = \frac{ds}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{ds}{dx} = \frac{d}{du} (\sin^{-1}u) \cdot \frac{d}{dx} (\cos x)$$

$$= \frac{1}{\sqrt{1-u^2}} \cdot (-\sin x)$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \text{ \& } \frac{d}{dx} (\cos x) = -\sin x \right)$$

$$= \frac{1}{\sqrt{1-(\cos x)^2}} \cdot (-\sin x)$$

$$= \frac{1}{\sqrt{\sin^2 x}} \cdot (-\sin x) \dots\dots\dots (\because \sin^2 x + \cos^2 x = 1)$$

$$= \frac{1}{\sin x} \cdot (-\sin x)$$

$$= -1$$

$$\therefore \frac{ds}{dx} = -1 \dots\dots\dots \text{eq(2)}$$

II. For $\cos^{-1}(\sin x)$

let $u = \sin x$

therefore, $t = \cos^{-1}u$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dt}{dx} = \frac{dt}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dt}{dx} = \frac{d}{du} (\cos^{-1}u) \cdot \frac{d}{dx} (\sin x)$$

$$= \frac{-1}{\sqrt{1-u^2}} \cdot (\cos x)$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}} \text{ \& } \frac{d}{dx} (\sin x) = \cos x \right)$$

$$= \frac{-1}{\sqrt{1 - (\sin x)^2}} \cdot (\cos x)$$

$$= \frac{-1}{\sqrt{\cos^2 x}} \cdot (\cos x) \dots\dots\dots (\because \sin^2 x + \cos^2 x = 1)$$

$$= \frac{-1}{\cos x} \cdot (\cos x)$$

$$= -1$$

$$\therefore \frac{dt}{dx} = -1 \dots\dots\dots \text{eq(2)}$$

Differentiating eq(1) w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (s + t)$$

$$= \frac{ds}{dx} + \frac{dt}{dx} \dots\dots\dots \left(\because \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= -1 -1 \dots\dots\dots \text{from eq(2) and eq(3)}$$

$$= -2$$

$$\therefore \frac{dy}{dx} = -2$$

Hence proved !!!

19. Question

Prove that $\frac{d}{dx} \{2x \tan^{-1} x - \log(1 + x^2)\} = 2 \tan^{-1} x$.

Answer

To Prove $\therefore \frac{d}{dx} \{2x \tan^{-1} x - \log(1 + x^2)\} = 2 \tan^{-1} x$

Formulae :

$$\text{i) } \frac{d}{dx} (u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\text{ii) } \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\text{iii) } \frac{d}{dx} (kx) = k$$

$$\text{iv) } \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$v) \frac{d}{dx} (kx) = 0$$

$$vi) \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

$$vii) \frac{d}{dx} (u - v) = \frac{du}{dx} - \frac{dv}{dx}$$

Answer :

Let,

$$y = 2x \tan^{-1}x - \log(1 + x^2)$$

$$\text{Let } s = 2x \tan^{-1}x \text{ \& } t = \log(1 + x^2)$$

Therefore, $y = s - t$ eq(1)

I. For $2x \tan^{-1}x$

$$\text{let } u = 2x \text{ \& } v = \tan^{-1}x$$

$$\text{therefore, } s = u \cdot v$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{ds}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \dots\dots\dots \left(\because \frac{d}{dx} (u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx} \right)$$

$$\therefore \frac{ds}{dx} = 2x \frac{d}{dx} (\tan^{-1}x) + \tan^{-1}x \frac{d}{dx} (2x)$$

$$= 2x \cdot \frac{1}{1+x^2} + \tan^{-1}x \cdot 2$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (\tan^{-1}x) = \frac{1}{1+x^2} \text{ \& } \frac{d}{dx} (kx) = k \right)$$

$$= \frac{2x}{1+x^2} + 2 \tan^{-1}x$$

$$\therefore \frac{ds}{dx} = \frac{2x}{1+x^2} + 2 \tan^{-1}x \dots\dots\dots \text{eq(2)}$$

II. For $\log(1 + x^2)$

$$\text{let } u = (1 + x^2)$$

$$\text{therefore, } t = \log u$$

Differentiating above equation w.r.t. x ,

$$\therefore \frac{dt}{dx} = \frac{dt}{du} \cdot \frac{du}{dx} \dots\dots\dots \text{By chain rule}$$

$$\therefore \frac{dt}{dx} = \frac{d}{du} (\log u) \cdot \frac{d}{dx} (1 + x^2)$$

$$= \frac{1}{u} \cdot \left(\frac{d}{dx} (1) + \frac{d}{dx} (x^2) \right) \dots\dots\dots \left(\because \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \right)$$

$$= \frac{1}{(1 + x^2)} \cdot (0 + 2x)$$

$$\dots\dots\dots \left(\because \frac{d}{dx} (k) = 0 \text{ \& } \frac{d}{dx} (x^n) = n \cdot x^{n-1} \right)$$

$$= \frac{2x}{1 + x^2}$$

$$\therefore \frac{dt}{dx} = \frac{2x}{1+x^2} \dots\dots\dots \text{eq(3)}$$

Differentiating eq(1) w.r.t. x,

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (s - t)$$

$$= \frac{ds}{dx} - \frac{dt}{dx} \dots\dots\dots \left(\because \frac{d}{dx} (u - v) = \frac{du}{dx} - \frac{dv}{dx} \right)$$

$$= \frac{2x}{1+x^2} + 2 \tan^{-1} x - \frac{2x}{1+x^2} \dots\dots\dots \text{from eq(2) and eq(3)}$$

$$= 2 \tan^{-1} x$$

$$\therefore \frac{dy}{dx} = 2 \tan^{-1} x$$

Hence proved !!!

Exercise 10D

1. Question

Differentiate each of the following w.r.t x:

$$\square \sin^{-1} \left\{ \sqrt{\frac{1 - \cos x}{2}} \right\}$$

Answer

$$\text{To find: Value of } \sin^{-1} \left\{ \sqrt{\frac{1 - \cos x}{2}} \right\}$$

Formula used: (i) $\cos \theta = 2 \sin^2 \frac{\theta}{2}$

We have, $\sin^{-1} \left\{ \sqrt{\frac{1 - \cos x}{2}} \right\}$

$$\Rightarrow \sin^{-1} \left\{ \sqrt{\frac{2 \sin^2 \frac{x}{2}}{2}} \right\}$$

$$\Rightarrow \sin^{-1} \left\{ \sqrt{\sin^2 \frac{x}{2}} \right\}$$

$$\Rightarrow \sin^{-1} \left\{ \sin \frac{x}{2} \right\}$$

$$\Rightarrow \frac{x}{2}$$

Now, we can see that $\sin^{-1} \left\{ \sqrt{\frac{1 - \cos x}{2}} \right\} = \frac{x}{2}$

Now differentiating ,

$$\Rightarrow \frac{d\left(\frac{x}{2}\right)}{dx}$$

$$\Rightarrow \frac{1}{2} \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{2}$$

Ans) $\frac{1}{2}$

2. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left(\frac{\sin x}{1 + \cos x} \right)$$

Answer

To find: Value of $\tan^{-1} \left(\frac{\sin x}{1 + \cos x} \right)$

Formula used: (i) $\sin 2\theta = 2 \sin \theta \cos \theta$

(ii) $1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$

We have, $\tan^{-1} \left(\frac{\sin x}{1 + \cos x} \right)$

$$\Rightarrow \tan^{-1} \left(\frac{\sin x}{2 \cos^2 \frac{x}{2}} \right)$$

$$\Rightarrow \tan^{-1} \left(\frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right)$$

$$\Rightarrow \tan^{-1} \left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right)$$

$$\Rightarrow \tan^{-1} \left(\tan \frac{x}{2} \right)$$

$$\Rightarrow \frac{x}{2}$$

Now, we can see that $\tan^{-1} \left(\frac{\sin x}{1 + \cos x} \right) = \frac{x}{2}$

Now differentiating ,

$$\Rightarrow \frac{d \left(\frac{x}{2} \right)}{dx}$$

$$\Rightarrow \frac{1}{2} \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{2}$$

$$\text{Ans) } \frac{1}{2}$$

3. Question

Differentiate each of the following w.r.t x:

$$\cot^{-1} \left(\frac{1 + \cos x}{\sin x} \right)$$

Answer

To find: Value of $\cot^{-1} \left(\frac{1 + \cos x}{\sin x} \right)$

Formula used: (i) $\sin 2\theta = 2 \sin \theta \cos \theta$

$$(ii) 1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$$

We have, $\cot^{-1} \left(\frac{1+\cos x}{\sin x} \right)$

$$\Rightarrow \cot^{-1} \left(\frac{2\cos^2 \frac{x}{2}}{\sin x} \right)$$

$$\Rightarrow \cot^{-1} \left(\frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} \right)$$

$$\Rightarrow \cot^{-1} \left(\frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} \right)$$

$$\Rightarrow \cot^{-1} \left(\cot \frac{x}{2} \right)$$

$$\Rightarrow \frac{x}{2}$$

Now, we can see that $\cot^{-1} \left(\frac{1+\cos x}{\sin x} \right) = \frac{x}{2}$

Now differentiating ,

$$\Rightarrow \frac{d\left(\frac{x}{2}\right)}{dx}$$

$$\Rightarrow \frac{1}{2} \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{2}$$

$$\text{Ans) } \frac{1}{2}$$

4. Question

Differentiate each of the following w.r.t x:

$$\cot^{-1} \left(\sqrt{\frac{1+\cos x}{1-\cos x}} \right)$$

Answer

$$\text{To find: Value of } \cot^{-1} \left(\sqrt{\frac{1+\cos x}{1-\cos x}} \right)$$

Formula used: (i) $\sin 2\theta = 2\sin \theta \cos \theta$

$$(ii) 1 + \cos \theta = 2\cos^2 \frac{\theta}{2}$$

We have, $\cot^{-1} \left(\sqrt{\frac{1+\cos x}{1-\cos x}} \right)$

$$\Rightarrow \cot^{-1} \left(\sqrt{\frac{1+\cos x}{1-\cos x}} \sqrt{\frac{1+\cos x}{1+\cos x}} \right)$$

$$\Rightarrow \cot^{-1} \left(\sqrt{\frac{(1+\cos x)^2}{1-\cos^2 x}} \right)$$

$$\Rightarrow \cot^{-1} \left(\sqrt{\frac{(1+\cos x)^2}{\sin^2 x}} \right)$$

$$\Rightarrow \cot^{-1} \left(\frac{1+\cos x}{\sin x} \right)$$

$$\Rightarrow \cot^{-1} \left(\frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} \right)$$

$$\Rightarrow \cot^{-1} \left(\frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} \right)$$

$$\Rightarrow \cot^{-1} \left(\cot \frac{x}{2} \right)$$

$$\Rightarrow \frac{x}{2}$$

Now, we can see that $\cot^{-1} \left(\sqrt{\frac{1+\cos x}{1-\cos x}} \right) = \frac{x}{2}$

Now differentiating ,

$$\Rightarrow \frac{d\left(\frac{x}{2}\right)}{dx}$$

$$\Rightarrow \frac{1}{2} \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{2}$$

$$\text{Ans) } \frac{1}{2}$$

5. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)$$

Answer

To find: Value of $\tan^{-1}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)$

Formula used: (i) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

We have, $\tan^{-1}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)$

Dividing numerator and denominator by $\cos x$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{\cos x + \sin x}{\cos x}}{\frac{\cos x - \sin x}{\cos x}}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{1 + \tan x}{1 - \tan x}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\tan \frac{\pi}{4} + \tan x}{1 - \tan x \tan \frac{\pi}{4}}\right)$$

$$\Rightarrow \tan^{-1}\left(\tan\left(\frac{\pi}{4} + x\right)\right)$$

$$\Rightarrow \frac{\pi}{4} + x$$

Now, we can see that $\tan^{-1}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) = \frac{\pi}{4} + x$

Now differentiating ,

$$\Rightarrow \frac{d\left(\frac{\pi}{4} + x\right)}{dx}$$

$$\Rightarrow \frac{d\left(\frac{\pi}{4}\right)}{dx} + \frac{dx}{dx}$$

$$\Rightarrow 0 + 1$$

$$\Rightarrow 1$$

Ans) 1

6. Question

Differentiate each of the following w.r.t x:

$$\cot^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

Answer

To find: Value of $\cot^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

Formula used: (i) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

We have, $\cot^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

Dividing numerator and denominator by $\cos x$

$$\Rightarrow \cot^{-1}\left(\frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan x \tan \frac{\pi}{4}}\right)$$

$$\Rightarrow \cot^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right)$$

$$\Rightarrow \cot^{-1}\left(\cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - x\right)\right)\right)$$

$$\Rightarrow \cot^{-1}\left(\cot\left(\frac{\pi}{4} + x\right)\right)$$

$$\Rightarrow \frac{\pi}{4} + x$$

Now, we can see that $\cot^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \frac{\pi}{4} + x$

Now differentiating ,

$$\Rightarrow \frac{d\left(\frac{\pi}{4} + x\right)}{dx}$$

$$\Rightarrow \frac{d\left(\frac{\pi}{4}\right)}{dx} + \frac{dx}{dx}$$

$$\Rightarrow 0 + 1$$

$$\Rightarrow 1$$

Ans) 1

7. Question

Differentiate each of the following w.r.t x:

$$\cot^{-1}\left(\sqrt{\frac{1+\cos 3x}{1-\cos 3x}}\right)$$

Answer

To find: Value of $\cot^{-1}\left(\sqrt{\frac{1+\cos 3x}{1-\cos 3x}}\right)$

Formula used: (i) $1 - \cos \theta = 2\sin^2 \frac{\theta}{2}$

(ii) $1 + \cos \theta = 2\cos^2 \frac{\theta}{2}$

We have, $\cot^{-1}\left(\sqrt{\frac{1+\cos 3x}{1-\cos 3x}}\right)$

$$\Rightarrow \cot^{-1}\left(\sqrt{\frac{1+\cos 3x}{2\sin^2 \frac{3x}{2}}}\right)$$

$$\Rightarrow \cot^{-1}\left(\sqrt{\frac{2\cos^2 \frac{3x}{2}}{2\sin^2 \frac{3x}{2}}}\right)$$

$$\Rightarrow \cot^{-1}\left(\sqrt{\cot^2\left(\frac{3x}{2}\right)}\right)$$

$$\Rightarrow \cot^{-1}\left(\cot\left(\frac{3x}{2}\right)\right)$$

$$\Rightarrow \frac{3x}{2}$$

Now, we can see that $\cot^{-1} \left(\sqrt{\frac{1+\cos 3x}{1-\cos 3x}} \right) = \frac{3x}{2}$

Now differentiating ,

$$\Rightarrow \frac{d\left(\frac{3x}{2}\right)}{dx}$$

$$\Rightarrow \frac{3}{2} \frac{dx}{dx}$$

$$\Rightarrow \frac{3}{2}$$

Ans) $\frac{3}{2}$

8. Question

Differentiate each of the following w.r.t x:

$$\sec^{-1} \left(\frac{1+\tan^2 x}{1-\tan^2 x} \right)$$

Answer

To find: Value of $\sec^{-1} \left(\frac{1+\tan^2 x}{1-\tan^2 x} \right)$

Formula used: (i) $\cos 2\theta = \frac{1-\tan^2 \theta}{1+\tan^2 \theta}$

We have, $\sec^{-1} \left(\frac{1+\tan^2 x}{1-\tan^2 x} \right)$

Dividing numerator and denominator by $1+\tan^2 x$

$$\Rightarrow \sec^{-1} \left(\frac{\left(\frac{1+\tan^2 x}{1+\tan^2 x} \right)}{\left(\frac{1-\tan^2 x}{1+\tan^2 x} \right)} \right)$$

$$\Rightarrow \sec^{-1} \left(\frac{1}{\left(\frac{1-\tan^2 x}{1+\tan^2 x} \right)} \right)$$

$$\Rightarrow \sec^{-1}\left(\frac{1}{\cos 2x}\right)$$

$$\Rightarrow \sec^{-1}(\sec 2x)$$

$$\Rightarrow 2x$$

$$\text{Now, we can see that } \sec^{-1}\left(\frac{1+\tan^2 x}{1-\tan^2 x}\right) = 2x$$

Now differentiating ,

$$\Rightarrow \frac{d(2x)}{dx}$$

$$\Rightarrow 2 \frac{dx}{dx}$$

$$\Rightarrow 2$$

Ans) 2

9. Question

Differentiate each of the following w.r.t x:

$$\sin^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)$$

Answer

$$\text{To find: Value of } \sin^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)$$

$$\text{Formula used: (i) } \cos 2\theta = \frac{1-\tan^2 \theta}{1+\tan^2 \theta}$$

$$\text{We have, } \sin^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)$$

$$\Rightarrow \sin^{-1}(\cos 2x)$$

$$\Rightarrow \sin^{-1}\left(\sin\left(\frac{\pi}{2}-2x\right)\right)$$

$$\Rightarrow \frac{\pi}{2}-2x$$

$$\text{Now, we can see that } \sin^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right) = \frac{\pi}{2}-2x$$

Now differentiating ,

$$\Rightarrow \frac{d\left(\frac{\pi}{2} - 2x\right)}{dx}$$

$$\Rightarrow \frac{d\left(\frac{\pi}{2}\right)}{dx} - 2 \frac{dx}{dx}$$

$$\Rightarrow 0 - 2$$

$$\Rightarrow -2$$

Ans) -2

10. Question

Differentiate each of the following w.r.t x:

$$\operatorname{cosec}^{-1}\left(\frac{1+\tan^2 x}{2 \tan x}\right)$$

Answer

To find: Value of $\operatorname{cosec}^{-1}\left(\frac{1+\tan^2 x}{2 \tan x}\right)$

Formula used: (i) $\sin 2\theta = \frac{2 \tan \theta}{1+\tan^2 \theta}$

We have, $\operatorname{cosec}^{-1}\left(\frac{1+\tan^2 x}{2 \tan x}\right)$

Dividing Numerator and Denominator with $1+\tan^2 x$

$$\Rightarrow \operatorname{cosec}^{-1}\left(\frac{\left(\frac{1+\tan^2 x}{1+\tan^2 x}\right)}{\left(\frac{2 \tan x}{1+\tan^2 x}\right)}\right)$$

$$\Rightarrow \operatorname{cosec}^{-1}\left(\frac{(1)}{\left(\frac{2 \tan x}{1+\tan^2 x}\right)}\right)$$

$$\Rightarrow \operatorname{cosec}^{-1}\left(\frac{1}{\sin 2x}\right)$$

$$\Rightarrow \operatorname{cosec}^{-1}(\operatorname{cosec} 2x)$$

$$\Rightarrow 2x$$

Now, we can see that $\operatorname{cosec}^{-1}\left(\frac{1+\tan^2 x}{2\tan x}\right) = 2x$

Now differentiating ,

$$\Rightarrow \frac{d(2x)}{dx}$$

$$\Rightarrow 2 \frac{dx}{dx}$$

$$\Rightarrow 2$$

Ans) 2

11. Question

Differentiate each of the following w.r.t x:

$$\cot^{-1}(\operatorname{cosec} x + \cot x)$$

Answer

To find: Value of $\cot^{-1}(\operatorname{cosec} x + \cot x)$

Formula used: (i) $\sin 2\theta = 2\sin \theta \cos \theta$

$$(ii) 1 + \cos \theta = 2\cos^2 \frac{\theta}{2}$$

We have, $\cot^{-1}(\operatorname{cosec} x + \cot x)$

$$\Rightarrow \cot^{-1}\left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{2\cos^2 \frac{x}{2}}{\sin x}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{\cos \frac{x}{2}}{\sin \frac{x}{2}}\right)$$

$$\Rightarrow \cot^{-1}\left(\cot \frac{x}{2}\right)$$

$$\Rightarrow \frac{x}{2}$$

Now, we can see that $\cot^{-1}(\operatorname{cosec} x + \cot x) = \frac{x}{2}$

Now differentiating ,

$$\Rightarrow \frac{d\left(\frac{x}{2}\right)}{dx}$$

$$\Rightarrow \frac{1}{2} \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{2}$$

$$\text{Ans) } \frac{1}{2}$$

12. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1}(\cot x) + \cot^{-1}(\tan x)$$

Answer

To find: Value of $\tan^{-1}(\cot x) + \cot^{-1}(\tan x)$

The formula used: (i) $\tan \theta = \cot \left(\frac{\pi}{2} - \theta\right)$

(ii) $\cot \theta = \tan \left(\frac{\pi}{2} - \theta\right)$

We have, $\tan^{-1}(\cot x) + \cot^{-1}(\tan x)$

$$\Rightarrow \tan^{-1}\left[\tan \left(\frac{\pi}{2} - x\right)\right] + \cot^{-1}\left[\cot \left(\frac{\pi}{2} - x\right)\right]$$

$$\Rightarrow \left(\frac{\pi}{2} - x\right) + \left(\frac{\pi}{2} - x\right)$$

$$\Rightarrow \pi - 2x$$

Now, we can see that $\tan^{-1}(\cot x) + \cot^{-1}(\tan x) = \pi - 2x$

Now differentiating ,

$$\Rightarrow \frac{d(\pi - 2x)}{dx}$$

$$\Rightarrow \frac{d\pi}{dx} - \frac{d2x}{dx}$$

$$\Rightarrow -2$$

Ans) -2

13. Question

Differentiate each of the following w.r.t x:

$$\sin^{-1}\left\{\sqrt{1-x^2}\right\}$$

Answer

To find: Value of $\sin^{-1}\{\sqrt{1-x^2}\}$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\sin^{-1}\{\sqrt{1-x^2}\}$

$\Rightarrow$ Putting $x = \cos \theta$

$\theta = \cos^{-1}x \dots (i)$

Putting $x = \cos \theta$ in the equation

$$\Rightarrow \sin^{-1}\{\sqrt{1-\cos^2 \theta}\}$$

$$\Rightarrow \sin^{-1}(\sqrt{\sin^2 \theta})$$

$$\Rightarrow \sin^{-1}(\sin \theta)$$

$$\Rightarrow \theta$$

$$\Rightarrow \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d(\cos^{-1}x)}{dx} \text{ [From (i)]}$$

$$\Rightarrow -\frac{1}{\sqrt{1-x^2}}$$

$$\text{Ans) } -\frac{1}{\sqrt{1-x^2}}$$

14. Question

Differentiate each of the following w.r.t x:

$$\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)$$

Answer

To find: Value of $\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

(ii) $\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

We have, $\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)$

$\Rightarrow$ Putting $x = \cos \theta$

$\theta = \cos^{-1}x \dots (i)$

Putting $x = \cos \theta$ in the equation

$\Rightarrow \sin^{-1}\left(\sqrt{\frac{1-\cos \theta}{2}}\right)$

$\Rightarrow \sin^{-1}\left(\sqrt{\sin^2 \frac{\theta}{2}}\right)$

$\Rightarrow \sin^{-1}\left(\sin \frac{\theta}{2}\right)$

$\Rightarrow \frac{\theta}{2}$

Now, we can see that $\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right) = \frac{\theta}{2}$

$\Rightarrow \theta = \cos^{-1}x$

$\Rightarrow \frac{d\left(\frac{\theta}{2}\right)}{dx}$

$\Rightarrow \frac{d\left(\frac{\cos^{-1}x}{2}\right)}{dx}$

$\Rightarrow -\frac{1}{2\sqrt{1-x^2}}$

$$\text{Ans) } -\frac{1}{2\sqrt{1-x^2}}$$

15. Question

Differentiate each of the following w.r.t x:

$$\cos^{-1} \left\{ \sqrt{\frac{1+x}{2}} \right\}$$

Answer

$$\text{To find: Value of } \cos^{-1} \left(\sqrt{\frac{1+x}{2}} \right)$$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \cos^{-1} \left(\sqrt{\frac{1+x}{2}} \right)$$

$$\Rightarrow \text{Putting } x = \cos \theta$$

$$\theta = \cos^{-1} x \dots (i)$$

Putting $x = \cos \theta$ in the equation

$$\Rightarrow \cos^{-1} \left(\sqrt{\frac{1+\cos \theta}{2}} \right)$$

$$\Rightarrow \cos^{-1} \left(\sqrt{\cos^2 \frac{\theta}{2}} \right)$$

$$\Rightarrow \cos^{-1} \left(\cos \frac{\theta}{2} \right)$$

$$\Rightarrow \frac{\theta}{2}$$

$$\text{Now, we can see that } \cos^{-1} \left(\sqrt{\frac{1+x}{2}} \right) = \frac{\theta}{2}$$

$$\Rightarrow \theta = \cos^{-1} x$$

$$\Rightarrow \frac{d\left(\frac{\theta}{2}\right)}{dx}$$

$$\Rightarrow \frac{d\left(\frac{\cos^{-1}x}{2}\right)}{dx}$$

$$\Rightarrow -\frac{1}{2\sqrt{1-x^2}}$$

$$\text{Ans) } -\frac{1}{2\sqrt{1-x^2}}$$

16. Question

Differentiate each of the following w.r.t x:

$$\cos^{-1}\left\{\sqrt{1-x^2}\right\}$$

Answer

To find: Value of $\cos^{-1}(\sqrt{1-x^2})$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\cos^{-1}(\sqrt{1-x^2})$

$\Rightarrow$ Putting $x = \sin\theta$

$\theta = \sin^{-1}x \dots (i)$

Putting $x = \sin\theta$ in the equation

$$\Rightarrow \cos^{-1}\left(\sqrt{1-(\sin\theta)^2}\right)$$

$$\Rightarrow \cos^{-1}(\sqrt{1-\sin^2\theta})$$

$$\Rightarrow \cos^{-1}(\cos\theta)$$

$$\Rightarrow \theta$$

Now, we can see that $\cos^{-1}(\sqrt{1-x^2}) = \theta$

$$\Rightarrow \theta = \sin^{-1}x$$

$$\Rightarrow \frac{d(\theta)}{dx}$$

$$\Rightarrow \frac{d(\sin^{-1}x)}{dx}$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}}$$

$$\text{Ans) } \frac{1}{\sqrt{1-x^2}}$$

17. Question

Differentiate each of the following w.r.t x:

$$\sin^{-1}\left\{2x\sqrt{1-x^2}\right\}$$

Answer

To find: Value of $\sin^{-1}(2x\sqrt{1-x^2})$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow$ Putting $x = \sin\theta$

$\theta = \sin^{-1}x \dots (i)$

Putting $x = \sin\theta$ in the equation

$$\Rightarrow \sin^{-1}\left(2\sin\theta\sqrt{1-(\sin\theta)^2}\right)$$

$$\Rightarrow \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$\Rightarrow \sin^{-1}(2\sin\theta\cos\theta)$$

$$\Rightarrow \sin^{-1}(\sin 2\theta)$$

$$\Rightarrow 2\theta$$

$$\Rightarrow 2\sin^{-1}x$$

Now, we can see that $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$

Now Differentiating

$$\Rightarrow \frac{d2\theta}{dx} = \frac{d(2\sin^{-1}x)}{dx}$$

$$\Rightarrow 2 \frac{d(\theta)}{dx}$$

$$\Rightarrow 2 \frac{d(\sin^{-1} x)}{dx}$$

$$\Rightarrow 2 \frac{1}{\sqrt{1-x^2}}$$

$$\text{Ans) } \frac{2}{\sqrt{1-x^2}}$$

18. Question

Differentiate each of the following w.r.t x:

$$\sin^{-1}(3x - 4x^3)$$

Answer

To find: Value of $\sin^{-1}(3x - 4x^3)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\sin^{-1}(3x - 4x^3)$

$\Rightarrow$ Putting $x = \sin \theta$

$\theta = \sin^{-1} x \dots (i)$

Putting $x = \sin \theta$ in the equation

$$\Rightarrow \sin^{-1}(3\sin \theta - 4(\sin \theta)^3)$$

$$\Rightarrow \sin^{-1}(3\sin \theta - 4 \sin^3 \theta)$$

$$\Rightarrow \sin^{-1}(\sin 3\theta)$$

$$\Rightarrow 3\theta$$

Now, we can see that $\sin^{-1}(3x - 4x^3) = 3\theta$

Now Differentiating

$$\Rightarrow \frac{d3\theta}{dx} = \frac{d(3\sin^{-1} x)}{dx}$$

$$\Rightarrow 3 \frac{d(\sin^{-1} x)}{dx}$$

$$\Rightarrow 3 \frac{1}{\sqrt{1-x^2}}$$

$$\text{Ans) } \frac{3}{\sqrt{1-x^2}}$$

19. Question

Differentiate each of the following w.r.t x:

$$\sin^{-1}(1-2x^2)$$

Answer

To find: Value of $\sin^{-1}(1-2x^2)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\sin^{-1}(1-2x^2)$

$\Rightarrow$ Putting $x = \sin \theta$

$\theta = \sin^{-1}x \dots (i)$

Putting $x = \sin \theta$ in the equation

$$\Rightarrow \sin^{-1}(1-2(\sin \theta)^2)$$

$$\Rightarrow \sin^{-1}(1-2\sin^2 \theta)$$

$$\Rightarrow \sin^{-1}(\cos 2\theta)$$

$$\Rightarrow \sin^{-1}\left(\sin\left(\frac{\pi}{2}-2\theta\right)\right)$$

$$\Rightarrow \frac{\pi}{2}-2\theta$$

Now, we can see that $\sin^{-1}(1-2x^2) = \frac{\pi}{2}-2\theta$

Now Differentiating

$$\Rightarrow \frac{d\left(\frac{\pi}{2}-2\theta\right)}{dx} = \frac{d\left(\frac{\pi}{2}\right)}{dx} - \frac{d2\theta}{dx}$$

$$\Rightarrow 0 - \frac{d2\theta}{dx}$$

$$\Rightarrow -2 \frac{d\sin^{-1}x}{dx}$$

$$\Rightarrow \frac{-2}{\sqrt{1-x^2}}$$

$$\text{Ans) } \frac{-2}{\sqrt{1-x^2}}$$

20. Question

Differentiate each of the following w.r.t x:

$$\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

Answer

To find: Value of $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

$\Rightarrow$ Putting $x = \sin\theta$

$\theta = \sin^{-1}x \dots (i)$

Putting $x = \sin\theta$ in the equation

$$\Rightarrow \sec^{-1}\left(\frac{1}{\sqrt{1-(\sin\theta)^2}}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{1}{\sqrt{\cos^2\theta}}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{1}{\cos\theta}\right)$$

$$\Rightarrow \sec^{-1}(\sec\theta)$$

$$\Rightarrow \theta$$

$$\text{Now, we can see that } \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \theta$$

Now Differentiating

$$\Rightarrow \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d(\sin^{-1}x)}{dx}$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}}$$

$$\text{Ans) } \frac{1}{\sqrt{1-x^2}}$$

21. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

Answer

$$\text{To find: Value of } \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\text{The formula used: (i) } \cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$$

$$\text{(ii) } \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \text{Putting } x = \sin\theta$$

$$\theta = \sin^{-1}x \dots \text{(i)}$$

Putting $x = \sin\theta$ in the equation

$$\Rightarrow \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-(\sin\theta)^2}}\right)$$

$$\Rightarrow \tan^{-1} \left(\frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} \right)$$

$$\Rightarrow \tan^{-1} \left(\frac{\sin \theta}{\sqrt{\cos^2 \theta}} \right)$$

$$\Rightarrow \tan^{-1} \left(\frac{\sin \theta}{\cos \theta} \right)$$

$$\Rightarrow \tan^{-1}(\tan \theta)$$

$$\Rightarrow \theta$$

Now, we can see that $\tan^{-1} \left(\frac{x}{\sqrt{1-x^2}} \right) = \theta$

Now Differentiating

$$\Rightarrow \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d(\sin^{-1} x)}{dx}$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}}$$

Ans) $\frac{1}{\sqrt{1-x^2}}$

22. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left(\frac{x}{1 + \sqrt{1-x^2}} \right)$$

Answer

To find: Value of $\tan^{-1} \left(\frac{x}{1 + \sqrt{1-x^2}} \right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

(ii) $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

We have, $\tan^{-1} \left(\frac{x}{1 + \sqrt{1-x^2}} \right)$

⇒ Putting $x = \sin\theta$

$$\theta = \sin^{-1}x \dots (i)$$

Putting $x = \sin\theta$ in the equation

$$\Rightarrow \tan^{-1}\left(\frac{\sin\theta}{1+\sqrt{1-(\sin\theta)^2}}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\sin\theta}{1+\sqrt{1-\sin^2\theta}}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\sin\theta}{1+\sqrt{\cos^2\theta}}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)$$

$$\Rightarrow \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\Rightarrow \frac{\theta}{2}$$

Now, we can see that $\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right) = \frac{\theta}{2}$

Now Differentiating

$$\Rightarrow \frac{d\left(\frac{\theta}{2}\right)}{dx}$$

$$\Rightarrow \frac{1}{2} \frac{d(\theta)}{dx}$$

$$\Rightarrow \frac{1}{2} \frac{d\sin^{-1}x}{dx}$$

$$\Rightarrow \frac{1}{2\sqrt{1-x^2}}$$

Ans) $\frac{1}{2\sqrt{1-x^2}}$

23. Question

Differentiate each of the following w.r.t x:

$$\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

Answer

To find: Value of $\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

(ii) $\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

We have, $\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$\Rightarrow$ Putting $x = \sin\theta$

$\theta = \sin^{-1}x \dots$ (i)

Putting $x = \sin\theta$ in the equation

$$\Rightarrow \cot^{-1}\left(\frac{\sqrt{1-(\sin\theta)^2}}{\sin\theta}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{\sqrt{\cos^2\theta}}{\sin\theta}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \cot^{-1}(\cot\theta)$$

$$\Rightarrow \theta$$

Now, we can see that $\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \theta$

Now Differentiating

$$\Rightarrow \frac{d(\theta)}{dx}$$

$$\Rightarrow \frac{d(\sin^{-1}x)}{dx}$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}}$$

$$\text{Ans) } \frac{1}{\sqrt{1-x^2}}$$

24. Question

Differentiate each of the following w.r.t x:

$$\sec^{-1}\left(\frac{1}{1-2x^2}\right)$$

Answer

To find: Value of $\sec^{-1}\left(\frac{1}{1-2x^2}\right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\sec^{-1}\left(\frac{1}{1-2x^2}\right)$

$\Rightarrow$ Putting $x = \sin \theta$

$\theta = \sin^{-1}x \dots (i)$

Putting $x = \sin \theta$ in the equation

$$\Rightarrow \sec^{-1}\left(\frac{1}{1-2(\sin \theta)^2}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{1}{1-2\sin^2 \theta}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$\Rightarrow \sec^{-1}(\sec 2\theta)$$

$$\Rightarrow 2\theta$$

Now, we can see that $\sec^{-1}\left(\frac{1}{1-2x^2}\right) = 2\theta$

Now Differentiating

$$\Rightarrow \frac{d(2\theta)}{dx}$$

$$\Rightarrow 2 \frac{d(\sin^{-1}x)}{dx}$$

$$\Rightarrow \frac{2}{\sqrt{1-x^2}}$$

$$\text{Ans) } \frac{2}{\sqrt{1-x^2}}$$

25. Question

Differentiate each of the following w.r.t x:

$$\sin^{-1} \left\{ \frac{1}{\sqrt{1+x^2}} \right\}$$

Answer

$$\text{To find: Value of } \sin^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right)$$

$$\text{The formula used: (i) } \cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$$

$$\text{(ii) } \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \sin^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right)$$

$$\Rightarrow \text{Putting } x = \cot \theta$$

$$\theta = \cot^{-1}x \dots \text{(i)}$$

Putting $x = \cot \theta$ in the equation

$$\Rightarrow \sin^{-1} \left(\frac{1}{\sqrt{1+(\cot \theta)^2}} \right)$$

$$\Rightarrow \sin^{-1} \left(\frac{1}{\sqrt{1+\cot^2 \theta}} \right)$$

$$\Rightarrow \sin^{-1} \left(\frac{1}{\sqrt{\operatorname{cosec}^2 \theta}} \right)$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{\operatorname{cosec}\theta}\right)$$

$$\Rightarrow \sin^{-1}(\sin\theta)$$

$$\Rightarrow \theta$$

$$\text{Now, we can see that } \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \theta$$

Now Differentiating

$$\Rightarrow \frac{d(\theta)}{dx}$$

$$\Rightarrow \frac{d(\cot^{-1}x)}{dx}$$

$$\Rightarrow -\frac{1}{1+x^2}$$

$$\text{Ans) } -\frac{1}{1+x^2}$$

26. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1}\left(\frac{1+x}{1-x}\right)$$

Answer

$$\text{To find: Value of } \tan^{-1}\left(\frac{1+x}{1-x}\right)$$

$$\text{The formula used: (i) } \cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\text{(ii) } \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \tan^{-1}\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \text{Putting } x = \tan\theta$$

$$\theta = \tan^{-1}x \dots \text{(i)}$$

Putting $x = \tan\theta$ in the equation

$$\Rightarrow \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$\Rightarrow \tan^{-1} \left(\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \right)$$

$$\Rightarrow \tan^{-1} \left(\tan \frac{\pi}{4} + \theta \right)$$

$$\Rightarrow \frac{\pi}{4} + \theta$$

Now, we can see that $\tan^{-1} \left(\frac{1+x}{1-x} \right) = \frac{\pi}{4} + \theta$

Now Differentiating

$$\Rightarrow \frac{d \left(\frac{\pi}{4} + \theta \right)}{dx}$$

$$\Rightarrow \frac{d \left(\frac{\pi}{4} \right)}{dx} + \frac{d(\theta)}{dx}$$

$$\Rightarrow 0 + \frac{d(\theta)}{dx}$$

$$\Rightarrow \frac{d(\tan^{-1} x)}{dx}$$

$$\Rightarrow \frac{1}{1+x^2}$$

Ans) $\frac{1}{1+x^2}$

27. Question

Differentiate each of the following w.r.t x:

$$\cot^{-1} \left(\frac{1+x}{1-x} \right)$$

Answer

To find: Value of $\cot^{-1} \left(\frac{1+x}{1-x} \right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

(ii) $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

We have, $\cot^{-1} \left(\frac{1+x}{1-x} \right)$

⇒ Putting $x = \tan\theta$

$$\theta = \tan^{-1}x \dots (i)$$

Putting $x = \tan\theta$ in the equation

$$\Rightarrow \cot^{-1} \left(\frac{1 + \tan\theta}{1 - \tan\theta} \right)$$

$$\Rightarrow \cot^{-1} \left(\frac{\tan \frac{\pi}{4} + \tan\theta}{1 - \tan \frac{\pi}{4} \tan\theta} \right)$$

$$\Rightarrow \cot^{-1} \left(\tan \frac{\pi}{4} + \theta \right)$$

$$\Rightarrow \cot^{-1} \left(\cot \left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \theta \right) \right) \right)$$

$$\Rightarrow \cot^{-1} \left(\cot \left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \theta \right) \right) \right)$$

$$\Rightarrow \cot^{-1} \left(\cot \left(\frac{\pi}{4} - \theta \right) \right)$$

$$\Rightarrow \frac{\pi}{4} - \theta$$

$$\text{Now, we can see that } \cot^{-1} \left(\frac{1+x}{1-x} \right) = \frac{\pi}{4} - \theta$$

Now Differentiating

$$\Rightarrow \frac{d \left(\frac{\pi}{4} - \theta \right)}{dx}$$

$$\Rightarrow \frac{d \left(\frac{\pi}{4} \right)}{dx} - \frac{d(\theta)}{dx}$$

$$\Rightarrow 0 - \frac{d(\theta)}{dx}$$

$$\Rightarrow - \frac{d(\tan^{-1}x)}{dx}$$

$$\Rightarrow - \frac{1}{1+x^2}$$

$$\text{Ans) } - \frac{1}{1+x^2}$$

28. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

Answer

To find: Value of $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$\Rightarrow$ Putting $x = \tan\theta$

$\theta = \tan^{-1}x \dots (i)$

Putting $x = \tan\theta$ in the equation

$$\Rightarrow \tan^{-1}\left(\frac{3\tan\theta - (\tan\theta)^3}{1 - 3(\tan\theta)^2}\right)$$

$$\Rightarrow \tan^{-1}(\tan 3\theta)$$

$$\Rightarrow \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$$\Rightarrow \tan^{-1}(\tan 3\theta)$$

$$\Rightarrow 3\theta$$

Now, we can see that $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = 3\theta$

Now Differentiating

$$\Rightarrow \frac{d(3\theta)}{dx}$$

$$\Rightarrow 3 \frac{d(\tan^{-1}x)}{dx}$$

$$\Rightarrow \frac{3}{1+x^2}$$

$$\text{Ans) } \frac{3}{1+x^2}$$

29. Question

Differentiate each of the following w.r.t x:

$$\operatorname{cosec}^{-1}\left(\frac{1+x^2}{2x}\right)$$

Answer

To find: Value of $\operatorname{cosec}^{-1}\left(\frac{1+x^2}{2x}\right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\operatorname{cosec}^{-1}\left(\frac{1+x^2}{2x}\right)$

$\Rightarrow$ Putting $x = \tan \theta$

$\theta = \tan^{-1}x \dots (i)$

Putting $x = \tan \theta$ in the equation

$$\Rightarrow \operatorname{cosec}^{-1}\left(\frac{1+(\tan \theta)^2}{2 \tan \theta}\right)$$

$$\Rightarrow \operatorname{cosec}^{-1}\left(\frac{1+\tan^2 \theta}{2 \tan \theta}\right)$$

$$\Rightarrow \operatorname{cosec}^{-1}\left(\frac{1}{\sin 2\theta}\right)$$

$$\Rightarrow \operatorname{cosec}^{-1}(\operatorname{cosec} 2\theta)$$

$$\Rightarrow 2\theta$$

Now, we can see that $\operatorname{cosec}^{-1}\left(\frac{1+x^2}{2x}\right) = 2\theta$

Now Differentiating

$$\Rightarrow \frac{d(2\theta)}{dx}$$

$$\Rightarrow 2 \frac{d(\tan^{-1}x)}{dx}$$

$$\Rightarrow \frac{2}{1+x^2}$$

Ans) $\frac{2}{1+x^2}$

30. Question

Differentiate each of the following w.r.t x:

$$\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$$

Answer

To find: Value of $\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

$\Rightarrow$ Putting $x = \tan \theta$

$\theta = \tan^{-1}x \dots (i)$

Putting $x = \tan \theta$ in the equation

$$\Rightarrow \sec^{-1}\left(\frac{1+(\tan \theta)^2}{1-(\tan \theta)^2}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{1+\tan^2 \theta}{1-\tan^2 \theta}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$\Rightarrow \sec^{-1}(\sec 2\theta)$$

$$\Rightarrow 2\theta$$

Now, we can see that $\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right) = 2\theta$

Now Differentiating

$$\Rightarrow \frac{d(2\theta)}{dx}$$

$$\Rightarrow 2 \frac{d(\tan^{-1}x)}{dx}$$

$$\Rightarrow \frac{2}{1+x^2}$$

$$\text{Ans) } \frac{2}{1+x^2}$$

31. Question

Differentiate each of the following w.r.t x:

$$\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

Answer

$$\text{To find: Value of } \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \text{Putting } x = \tan \theta$$

$$\theta = \tan^{-1}x \dots (i)$$

Putting $x = \tan \theta$ in the equation

$$\Rightarrow \sin^{-1}\left(\frac{1}{\sqrt{1+(\tan \theta)^2}}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{\sqrt{1+\tan^2 \theta}}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{\sqrt{\sec^2 \theta}}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{\sec \theta}\right)$$

$$\Rightarrow \sin^{-1}(\cos \theta)$$

$$\Rightarrow \sin^{-1}\left(\sin\left(\frac{\pi}{2} - \theta\right)\right)$$

$$\Rightarrow \frac{\pi}{2} - \theta$$

Now, we can see that $\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \frac{\pi}{2} - \theta$

Now Differentiating

$$\Rightarrow \frac{d\left(\frac{\pi}{2} - \theta\right)}{dx}$$

$$\Rightarrow \frac{d\left(\frac{\pi}{2}\right)}{dx} - \frac{d(\theta)}{dx}$$

$$\Rightarrow 0 - \frac{d(\tan^{-1}x)}{dx}$$

$$\Rightarrow -\frac{1}{1+x^2}$$

$$\text{Ans) } -\frac{1}{1+x^2}$$

32. Question

Differentiate each of the following w.r.t x:

$$\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$$

Answer

To find: Value of $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

The formula used: (i) $\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

$\Rightarrow$ Putting $x = \tan\theta$

$\theta = \tan^{-1}x \dots (i)$

Putting $x = \tan\theta$ in the equation

$$\Rightarrow \sec^{-1}\left(\frac{(\tan\theta)^2+1}{(\tan\theta)^2-1}\right)$$

$$\Rightarrow \sec^{-1} \left(\frac{\tan^2 \theta + 1}{\tan^2 \theta - 1} \right)$$

$$\Rightarrow \sec^{-1} \left[- \left(\frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} \right) \right]$$

$$\Rightarrow \pi - \sec^{-1} \left(\frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} \right)$$

$$\Rightarrow \pi - \sec^{-1} \left(\frac{1}{\cos 2\theta} \right)$$

$$\Rightarrow \pi - \sec^{-1}(\sec 2\theta)$$

$$\Rightarrow \pi - 2\theta$$

$$\Rightarrow \pi - 2\tan^{-1}x$$

Now, we can see that $\sec^{-1} \left(\frac{x^2 + 1}{x^2 - 1} \right) = \pi - 2\tan^{-1}x$

Now Differentiating

$$\Rightarrow \frac{d(\pi - 2\tan^{-1}x)}{dx}$$

$$\Rightarrow \frac{d(\pi)}{dx} - \frac{d(2\tan^{-1}x)}{dx}$$

$$\Rightarrow 0 - 2 \frac{d(\tan^{-1}x)}{dx}$$

$$\Rightarrow -\frac{2}{1+x^2}$$

$$\text{Ans) } -\frac{1}{1+x^2}$$

33. Question

Differentiate each of the following w.r.t x:

$$\cos^{-1} \left(\frac{1 - x^{2n}}{1 + x^{2n}} \right)$$

Answer

To find: Value of $\cos^{-1} \left(\frac{1 - x^{2n}}{1 + x^{2n}} \right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \cos^{-1} \left(\frac{1 - x^{2n}}{1 + x^{2n}} \right)$$

$$\Rightarrow \cos^{-1} \left(\frac{1 - (x^n)^2}{1 + (x^n)^2} \right)$$

$$\Rightarrow \text{Putting } x^n = \tan \theta$$

$$\theta = \tan^{-1}(x^n) \dots (i)$$

Putting $x^n = \tan \theta$ in the equation

$$\Rightarrow \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right)$$

$$\Rightarrow \cos^{-1}(\cos 2\theta)$$

$$\Rightarrow 2\theta$$

$$\Rightarrow 2\tan^{-1}(x^n)$$

$$\text{Now, we can see that } \cos^{-1} \left(\frac{1 - x^{2n}}{1 + x^{2n}} \right) = 2\tan^{-1}(x^n)$$

Now Differentiating

$$\Rightarrow \frac{d(2\tan^{-1}(x^n))}{dx}$$

$$\Rightarrow 2 \frac{d(\tan^{-1}(x^n))}{dx^n} \frac{dx^n}{dx}$$

$$\Rightarrow 2 \frac{1}{1+(x^n)^2} nx^{n-1}$$

$$\Rightarrow \frac{2nx^{n-1}}{1+x^{2n}}$$

$$\text{Ans) } \frac{2nx^{n-1}}{1+x^{2n}}$$

34. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left\{ \frac{x}{\sqrt{a^2 - x^2}} \right\}$$

Answer

To find: Value of $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$

$\Rightarrow$ Putting $x = a\sin\theta$

$$\sin\theta = \frac{x}{a}$$

$$\theta = \sin^{-1}\left(\frac{x}{a}\right) \dots (i)$$

Putting $x = a\sin\theta$ in the equation

$$\Rightarrow \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-(a\sin\theta)^2}}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2(1-\sin^2\theta)}}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$\Rightarrow \tan^{-1}(\tan\theta)$$

$$\Rightarrow \theta$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{a}\right)$$

$$\text{Now, we can see that } \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \sin^{-1}\left(\frac{x}{a}\right)$$

Now Differentiating

$$\Rightarrow \frac{d\left(\sin^{-1}\left(\frac{x}{a}\right)\right)}{dx}$$

$$\Rightarrow \frac{d\left(\sin^{-1}\left(\frac{x}{a}\right)\right)}{d\left(\frac{x}{a}\right)} \frac{d\left(\frac{x}{a}\right)}{dx}$$

$$\Rightarrow \left(\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right) \frac{1}{a}$$

$$\Rightarrow \left(\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\right) \frac{1}{a}$$

$$\Rightarrow \left(\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}\right) \frac{1}{a}$$

$$\Rightarrow \left(\frac{a}{\sqrt{a^2-x^2}}\right) \frac{1}{a}$$

$$\Rightarrow \frac{1}{\sqrt{a^2-x^2}}$$

$$\text{Ans) } \frac{1}{\sqrt{a^2-x^2}}$$

35. Question

Differentiate each of the following w.r.t x:

$$\sin^{-1}\left\{2ax\sqrt{1-a^2x^2}\right\}$$

Answer

To find: Value of $\sin^{-1} \left\{ 2ax\sqrt{1-a^2x^2} \right\}$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\sin^{-1} \left\{ 2ax\sqrt{1-a^2x^2} \right\}$

$\Rightarrow$ Putting $ax = \sin \theta$

$$\theta = \sin^{-1}(ax) \dots (i)$$

Putting $ax = \sin \theta$ in the equation

$$\Rightarrow \sin^{-1} \left\{ 2\sin \theta \sqrt{1-(\sin \theta)^2} \right\}$$

$$\Rightarrow \sin^{-1} \left\{ 2\sin \theta \sqrt{1-\sin^2 \theta} \right\}$$

$$\Rightarrow \sin^{-1} \{ 2\sin \theta \cos \theta \}$$

$$\Rightarrow \sin^{-1} \{ \sin 2\theta \}$$

$$\Rightarrow 2\theta$$

$$\Rightarrow 2 \sin^{-1}(ax)$$

$$\text{Now, we can see that } \sin^{-1} \left\{ 2ax\sqrt{1-a^2x^2} \right\} = 2 \sin^{-1}(ax)$$

Now Differentiating

$$\Rightarrow \frac{d(2 \sin^{-1}(ax))}{dx}$$

$$\Rightarrow 2 \frac{d(\sin^{-1}(ax))}{dax} \frac{dax}{dx}$$

$$\Rightarrow \left(2 \frac{1}{\sqrt{1-(ax)^2}} \right) a$$

$$\Rightarrow \left(\frac{2a}{\sqrt{1-a^2x^2}} \right)$$

Ans) $\frac{2a}{\sqrt{1-a^2x^2}}$

36. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left\{ \frac{\sqrt{1+a^2x^2}-1}{ax} \right\}$$

Answer

To find: Value of $\tan^{-1} \left\{ \frac{\sqrt{1+a^2x^2}-1}{ax} \right\}$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

(ii) $\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

We have, $\tan^{-1} \left\{ \frac{\sqrt{1+a^2x^2}-1}{ax} \right\}$

$\Rightarrow$ Putting $ax = \tan \theta$

$\theta = \tan^{-1}(ax) \dots (i)$

Putting $ax = \tan \theta$ in the equation

$$\Rightarrow \tan^{-1} \left\{ \frac{\sqrt{1+(\tan \theta)^2}-1}{\tan \theta} \right\}$$

$$\Rightarrow \tan^{-1} \left\{ \frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta} \right\}$$

$$\Rightarrow \tan^{-1} \left\{ \frac{\sec \theta - 1}{\tan \theta} \right\}$$

$$\Rightarrow \tan^{-1} \left\{ \frac{\frac{1}{\cos \theta} - 1}{\frac{\sin \theta}{\cos \theta}} \right\}$$

$$\Rightarrow \tan^{-1} \left\{ \frac{1 - \cos \theta}{\frac{\sin \theta}{\cos \theta}} \right\}$$

$$\Rightarrow \tan^{-1} \left\{ \frac{1 - \cos \theta}{\sin \theta} \right\}$$

$$\Rightarrow \tan^{-1} \left\{ \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right\}$$

$$\Rightarrow \tan^{-1} \left\{ \tan \frac{\theta}{2} \right\}$$

$$\Rightarrow \frac{\theta}{2}$$

$$\Rightarrow \frac{\tan^{-1}(ax)}{2}$$

Now, we can see that $\tan^{-1} \left\{ \frac{\sqrt{1+a^2x^2}-1}{ax} \right\} = \frac{\tan^{-1}(ax)}{2}$

Now Differentiating

$$\Rightarrow \frac{d \left(\frac{\tan^{-1}(ax)}{2} \right)}{dx}$$

$$\Rightarrow \frac{1}{2} \frac{d(\tan^{-1}(ax))}{dax} \frac{dax}{dx}$$

$$\Rightarrow \frac{1}{2} \left(\frac{1}{1+(ax)^2} \right) a$$

$$\Rightarrow \frac{a}{2(1+a^2x^2)}$$

Ans) $\frac{a}{2(1+a^2x^2)}$

37. Question

Differentiate each of the following w.r.t x:

$$\sin^{-1} \left\{ \frac{x^2}{\sqrt{x^4 + a^4}} \right\}$$

Answer

To find: Value of $\sin^{-1} \left\{ \frac{x^2}{\sqrt{x^4 + a^4}} \right\}$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \sin^{-1} \left\{ \frac{x^2}{\sqrt{x^4 + a^4}} \right\}$$

$$\Rightarrow \text{Putting } x^2 = a^2 \cot \theta$$

$$\theta = \cot^{-1} \left(\frac{x^2}{a^2} \right) \dots (i)$$

Putting $x^2 = a^2 \cot \theta$ in the equation

$$\Rightarrow \sin^{-1} \left\{ \frac{a^2 \cot \theta}{\sqrt{(a^2 \cot \theta)^2 + a^4}} \right\}$$

$$\Rightarrow \sin^{-1} \left\{ \frac{a^2 \cot \theta}{\sqrt{a^4 \cot^2 \theta + a^4}} \right\}$$

$$\Rightarrow \sin^{-1} \left\{ \frac{a^2 \cot \theta}{\sqrt{a^4 (\cot^2 \theta + 1)}} \right\}$$

$$\Rightarrow \sin^{-1} \left\{ \frac{a^2 \cot \theta}{a^2 \operatorname{cosec} \theta} \right\}$$

$$\Rightarrow \sin^{-1} \{ \cos \theta \}$$

$$\Rightarrow \sin^{-1} \left\{ \sin \left(\frac{\pi}{2} - \theta \right) \right\}$$

$$\Rightarrow \frac{\pi}{2} - \theta$$

$$\Rightarrow \frac{\pi}{2} - \cot^{-1} \left(\frac{x^2}{a^2} \right)$$

$$\text{Now, we can see that } \sin^{-1} \left\{ \frac{x^2}{\sqrt{x^4 + a^4}} \right\} = \frac{\pi}{2} - \cot^{-1} \left(\frac{x^2}{a^2} \right)$$

Now Differentiating

$$\Rightarrow \frac{d \left(\frac{\pi}{2} - \cot^{-1} \left(\frac{x^2}{a^2} \right) \right)}{dx}$$

$$\Rightarrow \frac{d \left(\frac{\pi}{2} \right)}{dx} - \frac{d \left(\cot^{-1} \left(\frac{x^2}{a^2} \right) \right)}{dx}$$

$$\Rightarrow 0 - \frac{d\left(\cot^{-1}\left(\frac{x^2}{a^2}\right)\right)}{d\frac{x^2}{a^2}} \frac{d\frac{x^2}{a^2}}{dx}$$

$$\Rightarrow \left(\frac{1}{1 + \left(\frac{x^2}{a^2}\right)^2}\right) \frac{1}{a^2} 2x$$

$$\Rightarrow \left(\frac{a^4}{a^4 + x^4}\right) \frac{1}{a^2} 2x$$

$$\Rightarrow \left(\frac{2a^2x}{a^4 + x^4}\right)$$

$$\text{Ans) } \frac{2a^2x}{a^4 + x^4}$$

38. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left\{ \frac{e^{2x} + 1}{e^{2x} - 1} \right\}$$

Answer

To find: Value of $\tan^{-1} \left\{ \frac{e^{2x} + 1}{e^{2x} - 1} \right\}$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\tan^{-1} \left\{ \frac{e^{2x} + 1}{e^{2x} - 1} \right\}$

$$\Rightarrow \tan^{-1} \left\{ \frac{1 + e^{2x}}{-(1 - e^{2x})} \right\}$$

$$-\tan^{-1} \left\{ \frac{1 + e^{2x}}{1 - e^{2x}} \right\}$$

Putting $e^{2x} = \tan \theta$

$$\theta = \tan^{-1}(e^{2x}) \dots (i)$$

Putting $e^{2x} = \tan \theta$ in the equation

$$\Rightarrow -\tan^{-1} \left\{ \frac{1 + \tan \theta}{1 - \tan \theta} \right\}$$

$$\Rightarrow -\tan^{-1} \left\{ \frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \right\}$$

$$\Rightarrow -\tan^{-1} \left\{ \tan \left(\frac{\pi}{4} + \theta \right) \right\}$$

$$\Rightarrow -\left(\frac{\pi}{4} + \theta \right)$$

$$\Rightarrow -\frac{\pi}{4} - \theta$$

$$\Rightarrow -\frac{\pi}{4} - \tan^{-1}(e^{2x})$$

Now, we can see that $\tan^{-1} \left\{ \frac{e^{2x} + 1}{e^{2x} - 1} \right\} = -\frac{\pi}{4} - \tan^{-1}(e^{2x})$

Now Differentiating

$$\Rightarrow \frac{d \left(-\frac{\pi}{4} - \tan^{-1}(e^{2x}) \right)}{dx}$$

$$\Rightarrow \frac{d \left(-\frac{\pi}{4} \right)}{dx} - \frac{d(\tan^{-1}(e^{2x}))}{dx}$$

$$\Rightarrow 0 - \frac{d(\tan^{-1}(e^{2x}))}{de^{2x}} \frac{de^{2x}}{d2x} \frac{d2x}{dx}$$

$$\Rightarrow - \left(\frac{1}{1 + (e^{2x})^2} \right) e^{2x} 2$$

$$\Rightarrow - \left(\frac{2e^{2x}}{1 + e^{4x}} \right)$$

$$\Rightarrow \frac{-2e^{2x}}{1 + e^{4x}}$$

Ans) $\frac{-2e^{2x}}{1 + e^{4x}}$

39. Question

Differentiate each of the following w.r.t x:

$$\cos^{-1}(2x) + 2 \cos^{-1} \sqrt{1 - 4x^2}$$

Answer

To find: Value of $\cos^{-1}(2x) + 2 \cos^{-1} \sqrt{1-4x^2}$

The formula used: (i) $\sin \theta = \cos \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

We have, $\cos^{-1}(2x) + 2 \cos^{-1} \sqrt{1-4x^2}$

Putting $2x = \cos \theta$

$$\theta = \cos^{-1}(2x) \dots (i)$$

Putting $e^{2x} = \tan \theta$ in the equation

$$\Rightarrow \cos^{-1}(\cos \theta) + 2 \cos^{-1} \sqrt{1 - (\cos \theta)^2}$$

$$\Rightarrow \cos^{-1}(\cos \theta) + 2 \cos^{-1} \sqrt{1 - \cos^2 \theta}$$

$$\Rightarrow \theta + 2 \cos^{-1} \sqrt{\sin^2 \theta}$$

$$\Rightarrow \theta + 2 \cos^{-1}(\sin \theta)$$

$$\Rightarrow \theta + 2 \cos^{-1} \left(\cos \left(\frac{\pi}{2} - \theta \right) \right)$$

$$\Rightarrow \theta + 2 \left(\frac{\pi}{2} - \theta \right)$$

$$\Rightarrow \pi - \theta$$

$$\Rightarrow \pi - \cos^{-1}(2x)$$

Now, we can see that $\cos^{-1}(2x) + 2 \cos^{-1} \sqrt{1-4x^2} = \pi - \cos^{-1}(2x)$

Now Differentiating

$$\Rightarrow \frac{d(\pi - \cos^{-1}(2x))}{dx}$$

$$\Rightarrow \frac{d(\pi)}{dx} - \frac{d(\cos^{-1}(2x))}{dx}$$

$$\Rightarrow 0 - \frac{d(\cos^{-1}(2x))}{d2x} \frac{d2x}{dx}$$

$$\Rightarrow \left(\frac{1}{\sqrt{1 - (2x)^2}} \right)^2$$

$$\Rightarrow \left(\frac{2}{\sqrt{1 - 4x^2}} \right)$$

$$\text{Ans) } \frac{2}{\sqrt{1 - 4x^2}}$$

40. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left(\frac{a - x}{1 + ax} \right)$$

Answer

To find: Value of $\tan^{-1} \left\{ \frac{a-x}{1+ax} \right\}$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

We have, $\tan^{-1} \left\{ \frac{a-x}{1+ax} \right\}$

$$\Rightarrow \tan^{-1} a - \tan^{-1} x$$

Now Differentiating

$$\Rightarrow \frac{d(\tan^{-1} a - \tan^{-1} x)}{dx}$$

$$\Rightarrow \frac{d(\tan^{-1} a)}{dx} - \frac{d(\tan^{-1} x)}{dx}$$

$$\Rightarrow 0 - \frac{1}{1+x^2}$$

$$\text{Ans) } -\frac{1}{1+x^2}$$

41. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left\{ \frac{\sqrt{x} - x}{1 + x^{3/2}} \right\}$$

Answer

To find: Value of $\tan^{-1} \left(\frac{\sqrt{x} - x}{1 + x^{3/2}} \right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\tan^{-1} \left(\frac{\sqrt{x} - x}{1 + x^{3/2}} \right)$

$$\Rightarrow \tan^{-1} \left(\frac{\sqrt{x} - x}{1 + x\sqrt{x}} \right)$$

$$\Rightarrow \tan^{-1} \sqrt{x} - \tan^{-1} x$$

Now Differentiating

$$\Rightarrow \frac{d(\tan^{-1} \sqrt{x} - \tan^{-1} x)}{dx}$$

$$\Rightarrow \frac{d(\tan^{-1} \sqrt{x})}{dx} - \frac{d(\tan^{-1} x)}{dx}$$

$$\Rightarrow \frac{d(\tan^{-1} \sqrt{x})}{d\sqrt{x}} \cdot \frac{d\sqrt{x}}{dx} - \frac{d(\tan^{-1} x)}{dx}$$

$$\Rightarrow \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{1+x^2}$$

$$\Rightarrow \frac{1}{2\sqrt{x}(1+x)} - \frac{1}{1+x^2}$$

$$\text{Ans) } \frac{1}{2\sqrt{x}(1+x)} - \frac{1}{1+x^2}$$

42. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left(\frac{\sqrt{a} + \sqrt{x}}{1 - \sqrt{ax}} \right)$$

Answer

To find: Value of $\tan^{-1} \left(\frac{\sqrt{a} + \sqrt{x}}{1 - \sqrt{ax}} \right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\tan^{-1} \left(\frac{\sqrt{a} + \sqrt{x}}{1 - \sqrt{ax}} \right)$

$$\Rightarrow \tan^{-1} \left(\frac{\sqrt{a} + \sqrt{x}}{1 - \sqrt{x}\sqrt{a}} \right)$$

$$\Rightarrow \tan^{-1} \sqrt{a} + \tan^{-1} \sqrt{x}$$

Now Differentiating

$$\Rightarrow \frac{d(\tan^{-1} \sqrt{a} + \tan^{-1} \sqrt{x})}{dx}$$

$$\Rightarrow \frac{d(\tan^{-1} \sqrt{a})}{dx} - \frac{d(\tan^{-1} \sqrt{x})}{dx}$$

$$\Rightarrow 0 - \frac{d(\tan^{-1} \sqrt{x})}{d\sqrt{x}} \cdot \frac{d\sqrt{x}}{dx}$$

$$\Rightarrow - \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow - \frac{1}{2\sqrt{x}(1+x)}$$

$$\text{Ans) } - \frac{1}{2\sqrt{x}(1+x)}$$

43. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left(\frac{3 - 2x}{1 + 6x} \right)$$

Answer

Given: Value of $\tan^{-1} \left(\frac{3-2x}{1+6x} \right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\tan^{-1} \left(\frac{3-2x}{1+6x} \right)$

$$\Rightarrow \tan^{-1} \left(\frac{3-2x}{1+3 \times 2x} \right)$$

$$\Rightarrow \tan^{-1} 3 - \tan^{-1} 2x$$

Now Differentiating

$$\Rightarrow \frac{d(\tan^{-1} 3 - \tan^{-1} 2x)}{dx}$$

$$\Rightarrow 0 - \frac{d(\tan^{-1} 2x)}{d2x} \frac{d2x}{dx}$$

$$\Rightarrow -\frac{1}{1+(2x)^2} \cdot 2$$

$$\Rightarrow -\frac{2}{1+4x^2}$$

$$\text{Ans) } -\frac{2}{1+4x^2}$$

44. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left(\frac{5x}{1-6x^2} \right)$$

Answer

Given: Value of $\tan^{-1} \left(\frac{5x}{1-6x^2} \right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\tan^{-1} \left(\frac{5x}{1-6x^2} \right)$

$$\Rightarrow \tan^{-1} \left(\frac{3x + 2x}{1 - 3x \times 2x} \right)$$

$$\Rightarrow \tan^{-1} 3x + \tan^{-1} 2x$$

Now Differentiating

$$\Rightarrow \frac{d(\tan^{-1} 3x + \tan^{-1} 2x)}{dx}$$

$$\Rightarrow \frac{d(\tan^{-1} 3x)}{d3x} \frac{d3x}{dx} + \frac{d(\tan^{-1} 2x)}{d2x} \frac{d2x}{dx}$$

$$\Rightarrow \frac{1}{1 + (3x)^2} 3 + \frac{1}{1 + (2x)^2} 2$$

$$\Rightarrow \frac{3}{1 + 9x^2} + \frac{2}{1 + 4x^2}$$

$$\text{Ans) } \frac{3}{1 + 9x^2} + \frac{2}{1 + 4x^2}$$

45. Question

Differentiate each of the following w.r.t x:

$$\tan^{-1} \left(\frac{2x}{1+15x^2} \right)$$

Answer

Given: Value of $\tan^{-1} \left(\frac{2x}{1+15x^2} \right)$

The formula used: (i) $\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$

$$(ii) \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have, $\tan^{-1} \left(\frac{2x}{1+15x^2} \right)$

$$\Rightarrow \tan^{-1} \left(\frac{5x - 3x}{1 + 5x \times 3x} \right)$$

$$\Rightarrow \tan^{-1} 5x - \tan^{-1} 3x$$

Now Differentiating

$$\Rightarrow \frac{d(\tan^{-1} 5x - \tan^{-1} 3x)}{dx}$$

$$\Rightarrow \frac{d(\tan^{-1} 5x)}{d5x} \frac{d5x}{dx} - \frac{d(\tan^{-1} 3x)}{d3x} \frac{d3x}{dx}$$

$$\Rightarrow \frac{1}{1 + (5x)^2} 5 + \frac{1}{1 + (3x)^2} 3$$

$$\Rightarrow \frac{5}{1+25x^2} + \frac{3}{1+9x^2}$$

$$\text{Ans) } \frac{5}{1+25x^2} + \frac{3}{1+9x^2}$$

46. Question

Differentiate each of the following w.r.t x:

$$\text{If } t = \tan^{-1} \left(\frac{ax - b}{bx + a} \right), \text{ prove that } \frac{dy}{dx} = \frac{1}{(1 + x^2)}.$$

Answer

$$\text{Given: Value of } \tan^{-1} \left(\frac{ax-b}{bx+a} \right)$$

$$\text{To Prove: } \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\text{The formula used: (i) } \cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$$

$$\text{(ii) } \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \tan^{-1} \left(\frac{ax-b}{bx+a} \right)$$

Dividing numerator and denominator with a

$$\Rightarrow \tan^{-1} \left(\frac{\frac{ax-b}{a}}{\frac{bx+a}{a}} \right)$$

$$\Rightarrow \tan^{-1} \left(\frac{x - \frac{b}{a}}{1 + \frac{b}{a}x} \right)$$

$$\Rightarrow \tan^{-1} x - \tan^{-1} \left(\frac{b}{a} \right)$$

Now Differentiating

$$\Rightarrow \frac{d\left(\tan^{-1} x - \tan^{-1}\left(\frac{b}{a}\right)\right)}{dx}$$

$$\Rightarrow \frac{d(\tan^{-1} x)}{dx} - \frac{d\left(\tan^{-1}\left(\frac{b}{a}\right)\right)}{dx}$$

$$\Rightarrow \frac{1}{1+x^2} + 0$$

$$\text{Ans) } \frac{1}{1+x^2}$$

47. Question

Differentiate each of the following w.r.t x:

$$\text{If } y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right), \text{ show that } \frac{dy}{dx} = \frac{4}{(1+x^2)}.$$

Answer

$$\text{Given: Value of } y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$$

$$\text{To Prove: } \frac{dy}{dx} = \frac{4}{(1+x^2)}$$

$$\text{The formula used: (i) } \cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$$

$$\text{(ii) } \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$$

$$\text{Putting } x = \tan \theta$$

$$\theta = \tan^{-1} x$$

Dividing numerator and denominator with a

$$\Rightarrow \sin^{-1}\left(\frac{2\tan\theta}{1+(\tan\theta)^2}\right) + \sec^{-1}\left(\frac{1+(\tan\theta)^2}{1-(\tan\theta)^2}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$$

$$\Rightarrow \sin^{-1}(\sin 2\theta) + \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$\Rightarrow \sin^{-1}(\sin 2\theta) + \sec^{-1}(\sec 2\theta)$$

$$\Rightarrow 2\theta + 2\theta$$

$$\Rightarrow 4\theta$$

$$\Rightarrow 4\tan^{-1}x$$

Now Differentiating

$$\Rightarrow \frac{d(4\tan^{-1}x)}{dx}$$

$$\Rightarrow 4 \frac{1}{1+x^2}$$

$$\text{Ans) } \frac{4}{1+x^2}$$

48. Question

Differentiate each of the following w.r.t x:

$$\text{If } y = \sec^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right), \text{ show that } \frac{dy}{dx} = 0.$$

Answer

$$\text{Given: Value of } y = \sec^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$$

$$\text{To Prove: } \frac{dy}{dx} = 0$$

$$\text{Formula used: (i) } \cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\text{(ii) } \frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{We have, } \sec^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{x-1}{x+1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$$

$$\Rightarrow \frac{\pi}{2}$$

Now Differentiating

$$\Rightarrow \frac{d\left(\frac{\pi}{2}\right)}{dx}$$

$$\Rightarrow 0$$

$$\text{Ans) } \frac{4}{1+x^2}$$

49. Question

Differentiate each of the following w.r.t x:

$$\text{If } y = \sin \left\{ 2 \tan^{-1} \left(\sqrt{\frac{1-x}{1+x}} \right) \right\}, \text{ show that } \frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}.$$

Answer

$$\text{Given: Value of } y = \sin \left\{ 2 \tan^{-1} \left(\sqrt{\frac{1-x}{1+x}} \right) \right\}$$

$$\text{To Prove: } \frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$$

$$\text{Formula used: (i) } \frac{d(\cos^{-1}x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\text{Let } x = \cos \theta$$

$$\theta = \cos^{-1}x$$

Putting $x = \cos \theta$ in equation

$$\Rightarrow \sin \left\{ 2 \tan^{-1} \left(\sqrt{\frac{1-\cos \theta}{1+\cos \theta}} \right) \right\}$$

$$\Rightarrow \sin \left\{ 2 \tan^{-1} \left(\sqrt{\frac{2 \sin^2 \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}} \right) \right\}$$

$$\Rightarrow \sin \left\{ 2 \tan^{-1} \left(\sqrt{\tan^2 \frac{\theta}{2}} \right) \right\}$$

$$\Rightarrow \sin \left\{ 2 \tan^{-1} \left(\tan \frac{\theta}{2} \right) \right\}$$

$$\Rightarrow \sin \left\{ 2 \frac{\theta}{2} \right\}$$

$$\Rightarrow \sin \theta$$

$$\Rightarrow \sin(\cos^{-1}x)$$

Now Differentiating

$$\Rightarrow \frac{d(\sin(\cos^{-1}x))}{dx}$$

$$\Rightarrow \frac{d(\sin(\cos^{-1}x))}{d\cos^{-1}x} \frac{d\cos^{-1}x}{dx}$$

$$\Rightarrow -\cos(\cos^{-1}x) \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow -\frac{x}{\sqrt{1-x^2}}$$

$$\text{Ans) } \frac{4}{1+x^2}$$

50. Question

Differentiate each of the following w.r.t x:

$$\text{If } y = \tan^{-1} \left\{ \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right\}. \text{ Prove that } \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}.$$

Answer

$$\text{Given: Value of } y = \tan^{-1} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

$$\text{To Prove: } \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$$

$$\text{The formula used: (i) } \cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$$

$$\text{(ii) } \frac{d(\cos^{-1}x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\text{Let } x = \cos 2\theta$$

$$2\theta = \cos^{-1}x$$

$$\theta = \frac{1}{2}\cos^{-1}x$$

Putting $x = \cos 2\theta$

$$y = \tan^{-1} \frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}$$

$$y = \tan^{-1} \frac{\sqrt{2 \cos^2 \theta} - \sqrt{2 \sin^2 \theta}}{\sqrt{2 \cos^2 \theta} + \sqrt{2 \sin^2 \theta}}$$

$$y = \tan^{-1} \frac{\sqrt{2} \cos \theta - \sqrt{2} \sin \theta}{\sqrt{2} \cos \theta + \sqrt{2} \sin \theta}$$

$$y = \tan^{-1} \frac{\sqrt{2}(\cos \theta - \sin \theta)}{\sqrt{2}(\cos \theta + \sin \theta)}$$

Dividing by $\cos \theta$ in the numerator and denominator

$$y = \tan^{-1} \frac{\frac{\cos \theta - \sin \theta}{\cos \theta}}{\frac{\cos \theta + \sin \theta}{\cos \theta}}$$

$$y = \tan^{-1} \frac{1 - \tan \theta}{1 + \tan \theta}$$

$$y = \tan^{-1} \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta}$$

$$y = \tan^{-1} \tan \left(\frac{\pi}{4} - \theta \right)$$

$$y = \frac{\pi}{4} - \theta$$

$$y = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x$$

Now Differentiating

$$\Rightarrow \frac{d \left(\frac{\pi}{4} - \frac{1}{2} \cos^{-1} x \right)}{dx}$$

$$\Rightarrow \frac{d \left(\frac{\pi}{4} \right)}{dx} - \frac{1}{2} \frac{d \cos^{-1} x}{dx}$$

$$\Rightarrow \frac{1}{2} \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{1}{2\sqrt{1-x^2}}$$

$$\text{Ans) } \frac{1}{2\sqrt{1-x^2}}$$

51. Question

Differentiate each of the following w.r.t x:

$$\text{Differentiate } \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right) \text{ w. r. t. } x$$

Answer

$$\text{Given: Value of } y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$$

$$\text{To find: } \frac{dy}{dx}$$

$$\text{The formula used: (i) } \cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$$

$$\text{(ii) } \frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$$

$$y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$$

$$y = \sin^{-1}\left(\frac{2^x \cdot 2}{1+(2^2)^x}\right)$$

$$y = \sin^{-1}\left(\frac{2^x \cdot 2}{1+(2^x)^2}\right)$$

$$\text{Let } 2^x = \tan \theta$$

$$\theta = \tan^{-1}(2^x)$$

$$\text{Putting } 2^x = \tan \theta$$

$$y = \sin^{-1}\left(\frac{\tan \theta \cdot 2}{1+(\tan \theta)^2}\right)$$

$$y = \sin^{-1}\left(\frac{2 \tan \theta}{1+\tan^2 \theta}\right)$$

$$y = \sin^{-1}(\sin 2\theta)$$

$$y = 2\theta$$

$$y = 2\tan^{-1}(2^x)$$

Now Differentiating

$$\Rightarrow \frac{d(2\tan^{-1}(2^x))}{dx}$$

$$\Rightarrow 2 \frac{d(\tan^{-1}(2^x))}{d2^x} \frac{d2^x}{dx}$$

$$\Rightarrow 2 \frac{1}{1+(2^x)^2} \cdot 2^x \log 2$$

$$\Rightarrow \frac{2^{1+x} \log 2}{1+4^x}.$$

$$\text{Ans) } \frac{2^{1+x} \log 2}{1+4^x}$$

Exercise 10E

1. Question

Find , when:

$$x^2 + y^2 = 4$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(x^n)}{dx} = n \times x^{(n-1)}$$

According to the chain rule of differentiation

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$$

Therefore ,

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = \frac{d(4)}{dx}$$

$$2x + 2y \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

2. Question

Find , when:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(x^n)}{dx} = n \times x^{(n-1)}$$

According to the chain rule of differentiation

$$\frac{d(y^2/b^2)}{dx} = \frac{d(y^2/b^2)}{dy} \times \frac{dy}{dx} = \frac{2y}{b^2} \times \frac{dy}{dx}$$

Therefore ,

$$\frac{d(x^2/a^2)}{dx} + \frac{d(y^2/b^2)}{dx} = \frac{d(1)}{dx}$$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{2x}{a^2}}{\frac{2y}{b^2}}$$

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$$

3. Question

Find , when:

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

Answer

Let us differentiate the whole equation w.r.t x

Formula : $\frac{d(x^n)}{dx} = n \times x^{(n-1)}$

According to the chain rule of differentiation

$$\frac{d(\sqrt{y})}{dx} = \frac{d(\sqrt{y})}{dy} \times \frac{dy}{dx} = \frac{1}{2\sqrt{y}} \times \frac{dy}{dx}$$

Therefore ,

$$\frac{d(\sqrt{x})}{dx} + \frac{d(\sqrt{y})}{dx} = \frac{d(\sqrt{a})}{dx}$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}}$$

$$\frac{dy}{dx} = \frac{-2\sqrt{y}}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}$$

4. Question

Find , when:

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Answer

Let us differentiate the whole equation w.r.t x

Formula : $\frac{d(x^n)}{dx} = n \times x^{(n-1)}$

According to the chain rule of differentiation

$$\frac{d(y^{2/3})}{dx} = \frac{d(y^{2/3})}{dy} \times \frac{dy}{dx} = \frac{2}{3y^{1/3}} \times \frac{dy}{dx}$$

Therefore ,

$$\frac{d(x^{2/3})}{dx} + \frac{d(y^{2/3})}{dx} = \frac{d(a^{2/3})}{dx}$$

$$\frac{2}{3x^{1/3}} + \frac{2}{3y^{1/3}} \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{2}{3x^{1/3}}}{\frac{2}{3y^{1/3}}}$$

$$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}$$

5. Question

Find , when:

$$xy = c^2$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(x^n)}{dx} = n \times x^{(n-1)}$$

According to product rule of differentiation

$$\frac{d(xy)}{dx} = \frac{xd(y)}{dx} + \frac{yd(x)}{dx} = x \times \frac{dy}{dx} + y$$

Therefore ,

$$\frac{d(xy)}{dx} = \frac{d(c^2)}{dx}$$

$$x \times \frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-xy}{x^2}$$

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

6. Question

Find , when:

$$x^2 + y^2 - 3xy = 1$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(x^n)}{dx} = n \times x^{(n-1)}$$

According to chain rule of differentiation

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$$

According to product rule of differentiation

$$\frac{d(xy)}{dx} = \frac{xd(y)}{dx} + \frac{yd(x)}{dx} = x \times \frac{dy}{dx} + y$$

Therefore ,

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} - 3 \frac{d(xy)}{dx} = \frac{d(1)}{dx}$$

$$2x + 2y \times \frac{dy}{dx} - 3(x \times \frac{dy}{dx} + y) = 0$$

$$(2y - 3x) \frac{dy}{dx} + 2x - 3y = 0$$

$$\frac{dy}{dx} = \frac{-(2x - 3y)}{2y - 3x}$$

$$\frac{dy}{dx} = \frac{2x - 3y}{3x - 2y}$$

7. Question

Find , when:

$$xy^2 - x^2y - 5 = 0$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(x^n)}{dx} = n \times x^{(n-1)}$$

According to chain rule of differentiation

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$$

According to product rule of differentiation

$$\frac{d(xy)}{dx} = \frac{xd(y)}{dx} + \frac{yd(x)}{dx} = x \times \frac{dy}{dx} + y$$

Therefore ,

$$\frac{d(xy^2)}{dx} - \frac{d(x^2y)}{dx} = \frac{d(5)}{dx}$$

$$x \times \frac{d(y^2)}{dx} + y^2 - [x^2 \times \frac{d(y)}{dx} + y \times 2x] = 0$$

$$x \times (2y \times \frac{dy}{dx}) + y^2 - [x^2 \times \frac{d(y)}{dx} + y \times 2x] = 0$$

$$2xy \frac{dy}{dx} - x^2 \frac{dy}{dx} + y^2 - 2xy = 0$$

$$\frac{dy}{dx} = \frac{2xy - y^2 dy}{2xy - x^2 dx} = \frac{y^2 - 2xy}{x^2 - 2xy}$$

8. Question

Find , when:

$$(x^2 + y^2)^2 = xy$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(x^n)}{dx} = n \times x^{(n-1)}$$

According to chain rule of differentiation

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$$

According to product rule of differentiation

$$\frac{d(xy)}{dx} = \frac{xd(y)}{dx} + \frac{yd(x)}{dx} = x \times \frac{dy}{dx} + y$$

Therefore ,

$$\frac{d((x^2 + y^2)^2)}{dx} = \frac{d(xy)}{dx}$$

$$2(x^2 + y^2) \times \frac{d(x^2 + y^2)}{dx} = [x \times \frac{d(y)}{dx} + y]$$

$$2(x^2 + y^2) \times [2x + 2y \times \frac{dy}{dx}] = [x \times \frac{d(y)}{dx} + y]$$

$$4x(x^2 + y^2) + 4y(x^2 + y^2) \frac{dy}{dx} = x \frac{dy}{dx} + y$$

$$\frac{dy}{dx} [4y(x^2 + y^2) - x] = y - 4x(x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{y - 4x(x^2 + y^2)}{[4y(x^2 + y^2) - x]}$$

$$\frac{dy}{dx} = \frac{y - 4x^3 - 4xy^2}{4y^3 + 4x^2y - x}$$

9. Question

Find , when:

$$x^2 + y^2 = \log (xy)$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(x^n)}{dx} = n \times x^{(n-1)}, \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to chain rule of differentiation

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$$

According to product rule of differentiation

$$\frac{d(xy)}{dx} = \frac{xd(y)}{dx} + \frac{yd(x)}{dx} = x \times \frac{dy}{dx} + y$$

Therefore ,

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = \frac{d(\log xy)}{dx}$$

$$2x + 2y \frac{d(y)}{dx} = \left[\frac{1}{xy} \frac{d(xy)}{dx} \right]$$

$$2x + 2y \frac{d(y)}{dx} = \frac{1}{xy} (x \frac{dy}{dx} + y)$$

$$2x + 2y \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx} + \frac{1}{x}$$

$$\frac{dy}{dx} \left[2y - \frac{1}{y} \right] = \frac{1}{x} - 2x$$

$$\frac{dy}{dx} \left(\frac{2y^2 - 1}{y} \right) = \frac{1 - 2x^2}{x}$$

$$\frac{dy}{dx} = \frac{y(1 - 2x^2)}{x(2y^2 - 1)}$$

10. Question

Find , when:

$$x^n + y^n = a^n$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(x^n)}{dx} = n \times x^{(n-1)}$$

According to the chain rule of differentiation

$$\frac{d(y^n)}{dx} = \frac{d(y^n)}{dy} \times \frac{dy}{dx} = ny^{n-1} \times \frac{dy}{dx}$$

Therefore ,

$$\frac{d(x^n)}{dx} + \frac{d(y^n)}{dx} = \frac{d(a^n)}{dx}$$

$$nx^{n-1} + ny^{n-1} \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-nx^{n-1}}{ny^{n-1}}$$

$$\frac{dy}{dx} = \frac{-x^{n-1}}{y^{n-1}}$$

11. Question

Find , when:

$$x \sin 2y = y \cos 2x$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\sin x)}{dx} = \cos x, \frac{d(\cos x)}{dx} = -\sin x$$

According to the chain rule of differentiation

$$\frac{d(\sin 2y)}{dx} = \frac{d(\sin 2y)}{dy} \times \frac{dy}{dx} = 2 \cos 2y \times \frac{dy}{dx}$$

According to the product rule of differentiation

$$\frac{d(x \sin 2y)}{dx} = \frac{x d(\sin 2y)}{dx} + \frac{\sin 2y d(x)}{dx} = x \times \frac{d(\sin 2y)}{dx} + \sin 2y$$

Therefore ,

$$\frac{d(x \sin 2y)}{dx} = \frac{d(y \cos 2x)}{dx}$$

$$x \times \frac{d(\sin 2y)}{dx} + \sin 2y = \cos 2x \times \frac{d(y)}{dx} + y(-2 \sin 2x)$$

$$x \times 2 \cos 2y \times \frac{dy}{dx} + \sin 2y = \cos 2x \times \frac{d(y)}{dx} + y(-2 \sin 2x)$$

$$\frac{dy}{dx} [2x \cos 2y - \cos 2x] = -2y \sin 2x - \sin 2y$$

$$\frac{dy}{dx} = \frac{-(2y \sin 2x + \sin 2y)}{2x \cos 2y - \cos 2x}$$

$$\frac{dy}{dx} = \frac{(2y \sin 2x + \sin 2y)}{\cos 2x - 2x \cos 2y}$$

12. Question

Find , when:

$$\sin^2 x + 2 \cos y + xy$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\sin x)}{dx} = \cos x, \frac{d(\cos x)}{dx} = -\sin x$$

According to chain rule of differentiation

$$\frac{d(\cos y)}{dx} = \frac{d(\cos y)}{dy} \times \frac{dy}{dx} = -\sin y \times \frac{dy}{dx}$$

Therefore ,

$$\frac{d(\sin^2 x)}{dx} + \frac{d(2 \cos y)}{dx} + \frac{d(xy)}{dx} = 0$$

$$2 \sin x \times \frac{d(\sin x)}{dx} + 2 \left(-\sin y \times \frac{dy}{dx} \right) + x \times \frac{dy}{dx} + y = 0$$

$$2 \sin x \times \cos x + y = 2 \left(\sin y \times \frac{dy}{dx} \right) - x \times \frac{dy}{dx}$$

$$\frac{dy}{dx} [2 \sin y - x] = \sin 2x + y$$

$$\frac{dy}{dx} = \frac{\sin 2x + y}{2x \cos 2y - \cos 2x}$$

$$\frac{dy}{dx} = \frac{\sin 2x + y}{2 \sin y - x}$$

13. Question

Find , when:

$$y \sec x + \tan x + x^2 y = 0$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\sec x)}{dx} = \sec x \tan x, \frac{d(\tan x)}{dx} = \sec^2 x$$

According to product rule of differentiation

$$\frac{d(x^2 y)}{dx} = \frac{x^2 dy}{dx} + \frac{y d(x^2)}{dx} = x^2 \frac{dy}{dx} + 2xy$$

Therefore ,

$$\frac{d(y \sec x)}{dx} + \frac{d(\tan x)}{dx} + \frac{d(x^2 y)}{dx} = 0$$

$$\sec x \times \frac{d(y)}{dx} + y \sec x \tan x + \sec^2 x + x^2 \frac{dy}{dx} + 2xy = 0$$

$$\frac{dy}{dx} [x^2 + \sec x] = -(y \sec x \tan x + \sec^2 x + 2xy)$$

$$\frac{dy}{dx} = \frac{-(y \sec x \tan x + \sec^2 x + 2xy)}{x^2 + \sec x}$$

14. Question

Find , when:

$$\cot (xy) + xy = y$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$$

According to chain rule of differentiation

$$\frac{d(\cot xy)}{dx} = -\operatorname{cosec}^2 xy \times \frac{d(xy)}{dx}$$

According to product rule of differentiation

$$\frac{d(xy)}{dx} = \frac{xdy}{dx} + \frac{yd(x)}{dx} = x \times \frac{dy}{dx} + y$$

Therefore ,

$$\frac{d(\cot xy)}{dx} + \frac{d(xy)}{dx} = \frac{dy}{dx}$$

$$-\operatorname{cosec}^2 xy \times \frac{d(xy)}{dx} + \frac{d(xy)}{dx} = \frac{dy}{dx}$$

$$\frac{d(xy)}{dx} [-\operatorname{cosec}^2 xy + 1] = \frac{dy}{dx}$$

$$\left[x \frac{dy}{dx} + y \right] [-\cot^2 xy] = \frac{dy}{dx} \text{ (Since, } 1 - \operatorname{cosec}^2 xy = -\cot^2 xy \text{)}$$

$$x \frac{dy}{dx} (-\cot^2 xy) - y \cot^2 xy = \frac{dy}{dx}$$

$$\frac{dy}{dx} [-x \cot^2 xy - 1] = y \cot^2 xy$$

$$\frac{dy}{dx} = \frac{-y \cot^2 xy}{x \cot^2 xy + 1}$$

15. Question

Find , when:

$$y \tan x - y^2 \cos x + 2x = 0$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\tan x)}{dx} = \sec^2 x, \frac{d(\cos x)}{dx} = -\sin x$$

According to chain rule of differentiation

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$$

According to product rule of differentiation

$$\frac{d(y \tan x)}{dx} = y \sec^2 x + \tan x \times \frac{dy}{dx}$$

Therefore ,

$$\frac{d(y \tan x)}{dx} - \frac{d(y^2 \cos x)}{dx} + \frac{d(2x)}{dx} = 0$$

$$y \sec^2 x + \tan x \times \frac{dy}{dx} - \cos x \frac{d(y^2)}{dx} - y^2(-\sin x) + 2 = 0$$

$$y \sec^2 x + \tan x \times \frac{dy}{dx} - \cos x \left(2y \frac{dy}{dx} \right) + y^2(\sin x) + 2 = 0$$

$$y \sec^2 x + \frac{dy}{dx} [\tan x - 2y \cos x] + y^2(\sin x) + 2 = 0$$

$$y \sec^2 x + y^2(\sin x) + 2 = \frac{dy}{dx} [2y \cos x - \tan x]$$

$$\frac{dy}{dx} = \frac{y \sec^2 x + y^2 \sin x + 2}{2y \cos x - \tan x}$$

16. Question

Find , when:

$$e^x \log y = \sin^{-1} x + \sin^{-1} y$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}, \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to chain rule of differentiation

$$\frac{d(\sin^{-1} y)}{dx} = \frac{d(\sin^{-1} y)}{dy} \times \frac{dy}{dx} = \frac{1}{\sqrt{1-y^2}} \times \frac{dy}{dx}$$

According to product rule of differentiation

$$\frac{d(e^x \log y)}{dx} = e^x \log y + e^x \times \frac{d(\log y)}{dx} = e^x \log y + e^x \times \frac{1}{y} \times \frac{dy}{dx}$$

Therefore ,

$$\frac{d(e^x \log y)}{dx} = \frac{d(\sin^{-1} x)}{dx} + \frac{d(\sin^{-1} y)}{dx}$$

$$e^x \log y + e^x \frac{1}{y} \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \times \frac{dy}{dx}$$

$$\frac{dy}{dx} \left[e^x \frac{1}{y} - \frac{1}{\sqrt{1-y^2}} \right] = \frac{1}{\sqrt{1-x^2}} - e^x \log y$$

$$\frac{dy}{dx} \left[\frac{e^x \sqrt{1-y^2} - y}{y \sqrt{1-y^2}} \right] = \frac{1 - (e^x \log y \sqrt{1-x^2})}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{y \sqrt{1-y^2}}{e^x \sqrt{1-y^2} - y} \times \frac{1 - (e^x \log y \sqrt{1-x^2})}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = y \times \sqrt{\frac{1-y^2}{1-x^2}} \times \frac{1 - (e^x \log y \sqrt{1-x^2})}{(e^x \sqrt{1-y^2}) - y}$$

17. Question

Find , when:

$$xy \log (x + y) = 1$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(x^n)}{dx} = n \times x^{(n-1)}, \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to product rule of differentiation

$$\frac{d(xy)}{dx} = \frac{xd(y)}{dx} + \frac{yd(x)}{dx} = x \times \frac{dy}{dx} + y$$

Therefore ,

$$\frac{d(xy \times \log x + y)}{dx} = \frac{d(1)}{dx}$$

$$\log x + y \times \frac{d(xy)}{dx} + xy \times \frac{d(\log x + y)}{dx} = \frac{d(1)}{dx}$$

$$\log x + y \left[x \frac{dy}{dx} + y \right] + xy \left[\frac{1}{x+y} \times \left(1 + \frac{dy}{dx} \right) \right] = 0$$

$$\frac{dy}{dx} [x \times \log x + y] + y \times \log(x + y) + \frac{xy}{x + y} \left(1 + \frac{dy}{dx} \right) = 0$$

$$\frac{dy}{dx} \left(x \log(x+y) + \frac{xy}{x+y} \right) = - \left(y \log(x+y) + \frac{xy}{x+y} \right)$$

$$\frac{dy}{dx} [(x^2 + xy) \log(x+y) + xy] = -[(y^2 + xy) \log(x+y) + xy]$$

$$\frac{dy}{dx} = \frac{-y^2 \log(x+y) - xy \log(x+y) - xy}{x[(x+y) \log(x+y) + y]} \times \frac{x}{x} \text{ (Multiply and divide by } x)$$

$$\frac{dy}{dx} = \frac{-y xy \log(x+y) - x xy \log(x+y) - x^2 y}{x^2 [(x+y) \log(x+y) + y]}$$

$$\frac{dy}{dx} = \frac{-y(1) - x(1) - x^2 y}{x^2 [(x+y) \log(x+y) + y]}$$

$$\frac{dy}{dx} = \frac{-(x+y+x^2 y)}{x^2 \{y + (x+y) \log(x+y)\}}$$

18. Question

Find , when:

$$\tan(x+y) + \tan(x-y) = 1$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\tan x)}{dx} = \sec^2 x$$

Therefore ,

$$\frac{d(\tan(x+y))}{dx} + \frac{d(\tan(x-y))}{dx} = \frac{d(1)}{dx}$$

$$\sec^2(x+y) \left[1 + \frac{dy}{dx} \right] + \sec^2(x-y) \left[1 - \frac{dy}{dx} \right] = 0$$

$$\sec^2(x+y) + \sec^2(x+y) \frac{dy}{dx} + \sec^2(x-y) - \sec^2(x-y) \frac{dy}{dx} = 0$$

$$\sec^2(x+y) + \sec^2(x-y) = \frac{dy}{dx} [\sec^2(x-y) - \sec^2(x+y)]$$

$$\frac{dy}{dx} = \frac{\sec^2(x+y) + \sec^2(x-y)}{\sec^2(x+y) - \sec^2(x-y)}$$

19. Question

Find , when:

$$\log \sqrt{x^2 + y^2} = \tan^{-1} \frac{y}{x}$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}, \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to quotient rule of differentiation

$$\frac{d(y/x)}{dx} = \frac{\frac{xd(y)}{dx} - yd(x)}{x^2} = \frac{x \frac{dy}{dx} - y}{x^2}$$

Therefore ,

$$\frac{d(\log \sqrt{x^2 + y^2})}{dx} = \frac{d(\tan^{-1} \frac{y}{x})}{dx}$$

$$\frac{1}{\sqrt{x^2 + y^2}} \times \frac{d(\sqrt{x^2 + y^2})}{dx} = \frac{1}{1 + (\frac{y}{x})^2} \times \frac{d(\frac{y}{x})}{dx}$$

$$\frac{1}{\sqrt{x^2 + y^2}} \times \frac{1}{2\sqrt{x^2 + y^2}} \times [2x + 2y \frac{d(y)}{dx}] = \frac{1}{1 + (\frac{y}{x})^2} \times \frac{x \frac{dy}{dx} - y}{x^2}$$

$$\frac{1}{x^2 + y^2} \times [x + y \frac{dy}{dx}] = \frac{x^2}{x^2 + y^2} \times \frac{x \frac{dy}{dx} - y}{x^2}$$

$$x + y \frac{dy}{dx} = x \frac{dy}{dx} - y$$

$$\frac{dy}{dx} [x - y] = x + y$$

$$\frac{dy}{dx} = \frac{x + y}{x - y}$$

20. Question

Find , when:

$$\text{If } y = x \sin y, \text{ prove that } \left(x \cdot \frac{dy}{dx} \right) = \frac{y}{(1 - x \cos y)}.$$

There is correction in question Prove that should be $\frac{dy}{dx} = \frac{\sin y}{1 - x \cos y}$ instead of

$$\left(x \cdot \frac{dy}{dx} \right) = \frac{y}{(1 - x \cos y)} \text{ to get the required answer.}$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\sin x)}{dx} = \cos x$$

According to chain rule of differentiation

$$\frac{d(\sin y)}{dx} = \frac{d(\sin y)}{dy} \times \frac{dy}{dx} = \cos y \times \frac{dy}{dx}$$

Therefore ,

$$\frac{d(y)}{dx} = \frac{d(x \sin y)}{dx}$$

$$\frac{dy}{dx} = x \frac{d(\sin y)}{dx} + \sin y$$

$$\frac{dy}{dx} = x \cos y \frac{dy}{dx} + \sin y$$

$$\frac{dy}{dx} [1 - x \cos y] = \sin y$$

$$\frac{dy}{dx} = \frac{\sin y}{1 - x \cos y}$$

21. Question

Find , when:

$$\text{If } xy = \tan (xy), \text{ show that } \frac{dy}{dx} = \frac{-y}{x}.$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\tan x)}{dx} = \sec^2 x$$

According to product rule of differentiation

$$\frac{d(xy)}{dx} = \frac{xdy}{dx} + \frac{yd(x)}{dx} = x \times \frac{dy}{dx} + y$$

Therefore ,

$$\frac{d(xy)}{dx} = \frac{d(\tan xy)}{dx}$$

$$x \frac{dy}{dx} + y = \sec^2(xy) \times \frac{d(xy)}{dx}$$

$$x \frac{dy}{dx} + y = \sec^2(xy) \times [x \frac{dy}{dx} + y]$$

$$\frac{dy}{dx} [x - x \sec^2(xy)] = y \sec^2(xy) - y$$

$$x \frac{dy}{dx} (1 - \sec^2 xy) = y(\sec^2(xy) - 1)$$

$$\frac{dy}{dx} = \frac{-y(1 - \sec^2(xy))}{x(1 - \sec^2 xy)}$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

22. Question

Find , when:

$$\text{If } y \log x = (x - y), \text{ prove that } \frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}.$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(x^n)}{dx} = n \times x^{(n-1)}, \frac{d(\log x)}{dx} = \frac{1}{x}$$

According to product rule of differentiation

$$\frac{d(y \log x)}{dx} = \frac{\log x d(y)}{dx} + \frac{y d(\log x)}{dx} = \log x \times \frac{dy}{dx} + \frac{y}{x}$$

Therefore ,

$$\frac{d(y \times \log x)}{dx} = \frac{d(x - y)}{dx}$$

$$\log x \times \frac{d(y)}{dx} + \frac{y}{x} = 1 - \frac{d(y)}{dx}$$

$$\frac{dy}{dx} [\log x + 1] = 1 - \frac{y}{x}$$

$$\frac{dy}{dx} [(1 + \log x)^2] = 1 - \frac{y}{x} (1 + \log x)$$

(Multiply by $1 + \log x$ on both sides)

$$\frac{dy}{dx} [(1 + \log x)^2] = 1 + \log x - \frac{y}{x} - \frac{y}{x} \log x$$

$$\frac{dy}{dx} [(1 + \log x)^2] = 1 + \log x - \frac{y}{x} - \frac{(x-y)}{x} \quad (y \log x = x - y)$$

$$\frac{dy}{dx} [(1 + \log x)^2] = 1 + \log x - \frac{y}{x} - 1 + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$$

23. Question

Find , when:

If $\cos y = x \cos (y + a)$, prove that $\frac{dy}{dx} = \frac{\cos^2 (y + a)}{\sin a}$.

Answer

Let us differentiate the whole equation w.r.t x

Formula : $\frac{d(\cos x)}{dx} = -\sin x$

According to chain rule of differentiation

$$\frac{d(\cos y)}{dx} = \frac{d(\cos y)}{dy} \times \frac{dy}{dx} = -\sin y \times \frac{dy}{dx}$$

According to product rule of differentiation

$$\frac{d(x \cos (y + a))}{dx} = x \frac{d(\cos y + a)}{dx} + \cos(y + a)$$

Therefore ,

$$\frac{d(\cos y)}{dx} = \frac{d(x \cos(y + a))}{dx}$$

$$-\sin y \frac{dy}{dx} = x \frac{d(\cos(y + a))}{dx} + \cos(y + a)$$

$$-\sin y \frac{dy}{dx} = x(-\sin(y+a) \frac{dy}{dx}) + \cos(y+a)$$

$$\frac{dy}{dx} [-\sin y + x \sin(y+a)] = \cos(y+a)$$

$$\frac{dy}{dx} = \frac{\cos(y+a)}{x \sin(y+a) - \sin y}$$

$$\frac{dy}{dx} = \frac{\cos^2(y+a)}{x \cos(y+a) \sin(y+a) - \cos(y+a) \sin y}$$

(Multiply and divide by $\cos(y+a)$)

$$\frac{dy}{dx} = \frac{\cos^2(y+a)}{\sin(y+a) \cos y - \cos(y+a) \sin y} \text{ (Since } \cos y = x \cos(y+a) \text{)}$$

$$\frac{dy}{dx} = \frac{\cos^2(y+a)}{\sin(y+a-y)} \text{ (Formula } \sin(a-b) = \sin a \cos b - \cos a \sin b \text{)}$$

$$\frac{dy}{dx} = \frac{\cos^2(y+a)}{\sin a}$$

24. Question

Find y , when:

$$\text{If } \cos^{-1} \left(\frac{x^2 - y^2}{x^2 + y^2} \right) = \tan^{-1} a, \text{ prove that } \frac{dy}{dx} = \frac{y}{x}.$$

Answer

Let us differentiate the whole equation w.r.t x

$$\text{Formula : } \frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

According to the chain rule of differentiation

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$$

Therefore ,

$$\frac{d(\cos^{-1} \frac{x^2 - y^2}{x^2 + y^2})}{dx} = \frac{d(\tan^{-1} a)}{dx}$$

$$-\frac{1}{\sqrt{1 - \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2}} \times \frac{d\left(\frac{x^2 - y^2}{x^2 + y^2}\right)}{dx} = 0$$

$$\frac{d\left(\frac{x^2 - y^2}{x^2 + y^2}\right)}{dx} = 0$$

$$\frac{x^2 + y^2 \left[\frac{d(x^2 - y^2)}{dx} \right] - (x^2 - y^2) \left[\frac{d(x^2 + y^2)}{dx} \right]}{(x^2 + y^2)^2} = 0$$

$$x^2 + y^2 \left[\frac{d(x^2 - y^2)}{dx} \right] - (x^2 - y^2) \left[\frac{d(x^2 + y^2)}{dx} \right] = 0$$

$$(x^2 + y^2) \left(2x - 2y \frac{dy}{dx} \right) - (x^2 - y^2) \left(2x + 2y \frac{dy}{dx} \right) = 0$$

$$(x^2 + y^2) \left(x - y \frac{dy}{dx} \right) = (x^2 - y^2) \left(x + y \frac{dy}{dx} \right)$$

$$\frac{dy}{dx} [-x^2y - y^3 - x^2y + y^3] = x^3 - xy^2 - x^3 - xy^2$$

$$\frac{dy}{dx} [-2x^2y] = -2xy^2$$

$$\frac{dy}{dx} = \frac{-2xy^2}{-2yx^2}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

Exercise 10F

1. Question

Find $\frac{dy}{dx}$, when:

$$\square y = x^{1/x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = \frac{\ln x}{x} \quad \{\ln(x^m) = m(\ln x)\}$$

Now differentiating both sides by x we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{x \left(\frac{1}{x}\right) - \ln x(1)}{x^2}$$

$$\frac{dy}{dx} = \frac{1 - \ln x}{x^2} \times y \quad \left\{ \text{divide rule } \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right\}$$

$$\frac{dy}{dx} = \frac{1 - \ln x}{x^2} \times x^{\frac{1}{x}} \quad \{y = x^{\frac{1}{x}}\}$$

2. Question

Find $\frac{dy}{dx}$, when:

$$y = x^{\sqrt{x}}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = \sqrt{x} \ln x$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \sqrt{x} \left(\frac{1}{x}\right) + \ln x \left(\frac{1}{2\sqrt{x}}\right) \quad \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} \left(1 + \frac{\ln x}{2}\right) \times y$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} \times \left(1 + \frac{\ln x}{2}\right) \times (x^{\sqrt{x}})$$

3. Question

Find $\frac{dy}{dx}$, when:

$$y = (\log x)^x$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = x \ln(\ln x)$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = x \left(\frac{1}{\ln x} \times \frac{1}{x} \right) + \ln(\ln x) \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\}$$

$$\frac{dy}{dx} = \left(\frac{1}{\ln x} + \ln(\ln x) \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{1}{\ln x} + \ln(\ln x) \right) \times (\ln x)^x$$

4. Question

Find $\frac{dy}{dx}$, when:

$$y = x^{\sin x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = \sin x \ln x$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \sin x \times \frac{1}{x} + \ln x \times \cos x \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\}$$

$$\frac{dy}{dx} = \left(\sin x \times \frac{1}{x} + \ln x \times \cos x \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{\sin x}{x} + \cos x (\ln x) \right) \times x^{\sin x}$$

5. Question

Find $\frac{dy}{dx}$, when:

$$y = x^{(\cos^{-1} x)}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = \cos^{-1} x \ln x$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= \cos^{-1} x \times \left(\frac{1}{x}\right) + \ln x \times \left(-\frac{1}{\sqrt{1-x^2}}\right) \left\{ \text{product rule, } \frac{d(uv)}{dx} \right. \\ &= u \frac{dv}{dx} + v \frac{du}{dx} \left. \right\} \end{aligned}$$

$$\frac{dy}{dx} = \cos^{-1} x \times \left(\frac{1}{x}\right) + \ln x \times \left(-\frac{1}{\sqrt{1-x^2}}\right) \times y$$

$$\frac{dy}{dx} = \left(\frac{\cos^{-1} x}{x} - \frac{\ln x}{\sqrt{1-x^2}} \right) \times x^{(\cos^{-1} x)}$$

6. Question

Find $\frac{dy}{dx}$, when:

$$y = (\tan x)^{1/x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = \left(\frac{1}{x}\right) \ln(\tan x)$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= \left(\frac{1}{x}\right) \times \left(\frac{1}{\tan x} \times \sec^2 x\right) \\ &+ \ln(\tan x) \times \left(-\frac{1}{x^2}\right) \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\} \end{aligned}$$

$$\frac{dy}{dx} = \left(\frac{\sec^2 x}{x \times \tan x} - \frac{\ln(\tan x)}{x^2} \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{\sec^2 x}{x \times \tan x} - \frac{\ln(\tan x)}{x^2} \right) \times \tan x^{\frac{1}{x}}$$

7. Question

Find $\frac{dy}{dx}$, when:

$$y = (\sin x)^{\cos x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = (\cos x) \ln(\sin x)$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= (\cos x) \times \left(\frac{1}{\sin x} \times \cos x \right) \\ &\quad + \ln(\sin x) \times (-\sin x) \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\} \end{aligned}$$

$$\frac{dy}{dx} = \left(\frac{\cos^2 x}{\sin x} - \sin x (\ln(\sin x)) \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{\cos^2 x}{\sin x} - \sin x (\ln(\sin x)) \right) \times \sin x^{\cos x}$$

8. Question

Find $\frac{dy}{dx}$, when:

$$y = (\log x)^{\sin x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = (\sin x) \ln(\ln x)$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= (\sin x) \times \left(\frac{1}{\ln x} \times \frac{1}{x} \right) \\ &\quad + \ln(\ln x) \times (\cos x) \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\} \end{aligned}$$

$$\frac{dy}{dx} = \left(\frac{\sin x}{x \times \ln x} - \cos x (\ln(\ln x)) \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{\sin x}{x \times \ln x} - \cos x (\ln(\ln x)) \right) \times \ln x^{\sin x}$$

9. Question

Find $\frac{dy}{dx}$, when:

$$y = (\cos x)^{\log x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = (\ln x) \ln(\cos x)$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= (\ln x) \times \left(\frac{1}{\cos x} \times (-\sin x) \right) \\ &\quad + \ln(\cos x) \times \left(\frac{1}{x} \right) \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\} \end{aligned}$$

$$\frac{dy}{dx} = \left(-\frac{\sin x \times \ln x}{\cos x} + \frac{(\ln \cos x)}{x} \right) \times y$$

$$\frac{dy}{dx} = \left(-\frac{\sin x \times \ln x}{\cos x} + \frac{(\ln \cos x)}{x} \right) \times \cos x^{\ln x}$$

10. Question

Find $\frac{dy}{dx}$, when:

$$y = (\tan x)^{\sin x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = (\sin x) \ln(\tan x)$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= (\sin x) \times \left(\frac{1}{\tan x} \times (\sec^2 x) \right) \\ &\quad + \ln(\tan x) \times (\cos x) \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\} \end{aligned}$$

$$\frac{dy}{dx} = \left(\frac{\sin x \times \sec^2 x}{\tan x} + \ln(\tan x) \cos x \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{\sin x \times \sec^2 x}{\tan x} + \ln(\tan x) \cos x \right) \times \tan x^{\sin x}$$

11. Question

Find $\frac{dy}{dx}$, when:

$$y = (\cos x)^{\cos x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = (\cos x) \ln(\cos x)$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= (\cos x) \times \left(\frac{1}{\cos x} \times (-\sin x) \right) \\ &+ \ln(\cos x) \times (-\sin x) \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\} \end{aligned}$$

$$\frac{dy}{dx} = (-\sin x - \ln(\cos x) \sin x) \times y$$

$$\frac{dy}{dx} = (-\sin x - \ln(\cos x) \sin x) \times \cos x^{\cos x}$$

12. Question

Find $\frac{dy}{dx}$, when:

$$y = (\tan x)^{\cot x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = (\cot x) \ln(\tan x)$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= (\cot x) \times \left(\frac{1}{\tan x} \times (-\sec^2 x) \right) \\ &+ \ln(\tan x) \times (-\operatorname{cosec}^2 x) \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\} \end{aligned}$$

$$\frac{dy}{dx} = (-\operatorname{cosec}^2 x - \ln(\tan x) \operatorname{cosec}^2 x) \times y$$

$$\frac{dy}{dx} = -\operatorname{cosec}^2 x \times (1 + \ln(\cos x)) \times \tan x^{\cot x}$$

13. Question

Find $\frac{dy}{dx}$, when:

$$y = x^{\sin 2x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = (\sin 2x) \ln(x)$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= (\sin 2x) \times \left(\frac{1}{x}\right) \\ &+ \ln(x) \times (\cos 2x \times 2) \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\} \end{aligned}$$

$$\frac{dy}{dx} = \left(\frac{\sin 2x}{x} + 2 \cos 2x \times \ln x \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{\sin 2x}{x} + 2 \cos 2x \times \ln x \right) \times x^{\sin 2x}$$

14. Question

Find $\frac{dy}{dx}$, when:

$$y = (\sin^{-1} x)^x$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = (x) \ln(\sin^{-1} x)$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = (x) \times \left(\frac{1}{\sin^{-1} x} \times \frac{1}{\sqrt{1-x^2}} \right) + \ln(\sin^{-1} x) \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\}$$

$$\frac{dy}{dx} = \left(\frac{x}{\sin^{-1} x \times \sqrt{1-x^2}} \times \ln \sin^{-1} x \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{x}{\sin^{-1} x \times \sqrt{1-x^2}} \times \ln \sin^{-1} x \right) \times \sin^{-1} x^x$$

15. Question

Find $\frac{dy}{dx}$, when:

$$y = \sin(x^x)$$

Answer

Here, the argument of the sinusoidal function has exponent as x itself.

For that, we will consider $x^x = u$ for simplicity.

$$y = \sin u$$

Differentiating both the sides,

$$\frac{dy}{dx} = \cos u \times \frac{du}{dx}. \dots\dots(1)$$

Now we have to find $\frac{du}{dx}$, where $u = x^x$

take log both the sides

$$\ln u = x \ln x$$

Now differentiating both sides by x, we get,

$$\frac{1}{u} \times \frac{du}{dx} = x \left(\frac{1}{x} \right) + \ln x$$

$$\frac{du}{dx} = (1 + \ln x) \times u$$

$$\frac{du}{dx} = (1 + \ln x) \times x^x$$

Substituting the value in equation 1,

$$\frac{dy}{dx} = \cos x (1 + \ln x) \times x^x$$

16. Question

Find $\frac{dy}{dx}$, when:

$$y = (3x + 5)^{(2x-3)}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = (2x - 3) \ln(3x - 5)$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= (2x - 3) \times \left(\frac{1}{3x - 5} \times 3 \right) \\ &\quad + \ln(3x - 5) \times 2 \left\{ \text{product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right\} \end{aligned}$$

$$\frac{dy}{dx} = \left(\frac{2x - 3}{3x - 5} \times 3 + \ln(3x - 5) \times 2 \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{2x - 3}{3x - 5} \times 3 + 2 \times \ln 3x - 5 \right) \times (3x - 5)^{2x-3}$$

17. Question

Find $\frac{dy}{dx}$, when:

$$y = (x + 1)^3 (x + 2)^4 (x + 3)^5$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = 3 \ln(x + 1) + 4 \ln(x + 2) + 5 \ln(x + 3) \quad \{\ln(mn) = \ln n + \ln m\}$$

Now differentiating both sides by x, we get,

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= \frac{3}{x + 1} + \frac{4}{x + 2} + \frac{5}{x + 3} \\ \frac{dy}{dx} &= \left(\frac{3}{x + 1} + \frac{4}{x + 2} + \frac{5}{x + 3} \right) \times y \end{aligned}$$

$$\frac{dy}{dx} = \left(\frac{3}{x+1} + \frac{4}{x+2} + \frac{5}{x+3} \right) \times (x+1)^3(x+2)^4(x+3)^5$$

18. Question

Find $\frac{dy}{dx}$, when:

$$y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = \frac{1}{2} (\ln(x-1) + \ln(x-2) - \ln(x-3) - \ln(x-4) - \ln(x-5))$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right) \times y$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right) \times \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

19. Question

Find $\frac{dy}{dx}$, when:

$$y = (2-x)^3(3+2x)^5$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = 3 \ln(2-x) + 5 \ln(3+2x)$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = 3 \left(\frac{-1}{2-x} \right) + 5 \left(\frac{1}{3+2x} \times 2 \right)$$

$$\frac{dy}{dx} = \left(\frac{3}{x-2} + \frac{10}{3+2x} \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{3}{x-2} + \frac{10}{3+2x} \right) \times (2-x)^3 (3+2x)^5$$

20. Question

Find $\frac{dy}{dx}$, when:

$$y = \cos x \cos 2x \cos 3x$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = \ln(\cos x) + \ln(\cos 2x) + \ln \cos 3x$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\}$$

Now differentiating both sides by x , we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{\cos x} \times (-\sin x) + \frac{1}{\cos 2x} \times (-2 \sin 2x) + \frac{1}{\cos 3x} (-3 \sin 3x)$$

$$\frac{dy}{dx} = \left(\frac{-\sin x}{\cos x} - \frac{2 \sin 2x}{\cos 2x} - \frac{3 \sin 3x}{\cos 3x} \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{-\sin x}{\cos x} - \frac{2 \sin 2x}{\cos 2x} - \frac{3 \sin 3x}{\cos 3x} \right) \times \cos x \cos 2x \cos 3x$$

$$\frac{dy}{dx} = (-\tan x - 2 \tan 2x - 3 \tan 3x) \times \cos x \cos 2x \cos 3x$$

21. Question

Find $\frac{dy}{dx}$, when:

$$y = \frac{x^5 \sqrt{x+4}}{(2x+3)^2}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = 5\ln(x) + \frac{1}{2}\ln(x+4) - 2\ln(2x+3)$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{5}{x} + \frac{1}{2(x+4)} - \frac{4}{2x+3}$$

$$\frac{dy}{dx} = \left(\frac{5}{x} + \frac{1}{2(x+4)} - \frac{4}{2x+3} \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{5}{x} + \frac{1}{2(x+4)} - \frac{4}{2x+3} \right) \times \frac{x^5\sqrt{x+4}}{(2x+3)^2}$$

22. Question

Find $\frac{dy}{dx}$, when:

$$y = \frac{(x+1)^2\sqrt{x-1}}{(x+4)^3 \cdot e^x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = 2\ln(x+1) + \frac{1}{2}\ln(x-1) - 3\ln(x+4) - x$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\} \{\ln e = 1\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{2}{x+1} + \frac{1}{2(x-1)} - \frac{3}{x+4} - 1$$

$$\frac{dy}{dx} = \left(\frac{2}{x+1} + \frac{1}{2(x-1)} - \frac{3}{x+4} - 1 \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{2}{x+1} + \frac{1}{2(x-1)} - \frac{3}{x+4} - 1 \right) \times \frac{(x+1)^2\sqrt{x-1}}{(x+4)^3 \cdot e^x}$$

23. Question

Find $\frac{dy}{dx}$, when:

$$y = \frac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = 2\ln(3x+5) + \frac{1}{2}\ln(x) - \frac{1}{2}\ln(x+1)$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{\ln\left(\frac{m}{n}\right) = \ln m - \ln n\right\} \{\ln e = 1\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{2}{3x+5} + \frac{1}{2x} - \frac{1}{2(x+1)}$$

$$\frac{dy}{dx} = \left(\frac{2}{3x+5} + \frac{1}{2x} - \frac{1}{2(x+1)} \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{2}{3x+5} + \frac{1}{2x} - \frac{1}{2(x+1)} \right) \times \frac{(3x+5)^2\sqrt{x}}{\sqrt{x+1}}$$

24. Question

Find $\frac{dy}{dx}$, when:

$$y = \frac{x^2\sqrt{1+x}}{(1+x^2)^{3/2}}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = 2\ln(x) + \frac{1}{2}\ln(x+1) - \frac{3}{2}\ln(x^2+1)$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{\ln\left(\frac{m}{n}\right) = \ln m - \ln n\right\} \{\ln e = 1\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{2}{x} + \frac{1}{2(x+1)} - \frac{3}{2(x^2+1)} \times 2x$$

$$\frac{dy}{dx} = \left(\frac{2}{x} + \frac{1}{2(x+1)} - \frac{6x}{2(x^2+1)} \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{2}{x} + \frac{1}{2(x+1)} - \frac{6x}{2(x^2+1)} \right) \times \frac{(x)^2 \sqrt{x+1}}{(1+x^2)^{\frac{3}{2}}}$$

25. Question

Find $\frac{dy}{dx}$, when:

$$y = \sqrt{(x-2)(2x-3)(3x-4)}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = \frac{1}{2} (\ln(x-2) + \ln(2x-3) + \ln(3x-4))$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\} \{\ln e = 1\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x-2} + \frac{2}{2x-3} + \frac{3}{3x-4} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x-2} + \frac{2}{2x-3} + \frac{3}{3x-4} \right) \times y$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x-2} + \frac{2}{2x-3} + \frac{3}{3x-4} \right) \times \sqrt{(x-2)(2x-3)(3x-4)}$$

26. Question

Find $\frac{dy}{dx}$, when:

$$y = \sin 2x \sin 3x \sin 4x$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = (\ln(\sin 2x) + \ln(\sin 3x) + \ln(\sin 4x))$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{\ln\left(\frac{m}{n}\right) = \ln m - \ln n\right\} \{\ln e = 1\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \left(\frac{2}{\sin 2x} + \frac{3}{\sin 3x} + \frac{4}{\sin 4x} \right)$$

$$\frac{dy}{dx} = \left(\frac{2}{\sin 2x} + \frac{3}{\sin 3x} + \frac{4}{\sin 4x} \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{2}{\sin 2x} + \frac{3}{\sin 3x} + \frac{4}{\sin 4x} \right) \times \sin 2x \sin 3x \sin 4x$$

27. Question

Find $\frac{dy}{dx}$, when:

$$y = \frac{x^3 \sin x}{e^x}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = 3 \ln x + \ln \sin x - x$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{\ln\left(\frac{m}{n}\right) = \ln m - \ln n\right\} \{\ln e = 1\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \left(\frac{1}{\sin x} \times \cos x + \frac{3}{x} - 1 \right)$$

$$\frac{dy}{dx} = \left(\frac{\cos x}{\sin x} + \frac{3}{x} - 1 \right) \times y$$

$$\frac{dy}{dx} = \left(\cot x + \frac{3}{x} - 1 \right) \times \frac{x^3 \sin x}{e^x}$$

28. Question

Find $\frac{dy}{dx}$, when:

$$y = \frac{e^x \log x}{x^2}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = x + \ln(\ln x) - 2 \ln x$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\} \{\ln e = 1\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \left(1 + \frac{1}{x \ln x} - \frac{2}{x}\right)$$

$$\frac{dy}{dx} = \left(1 + \frac{1}{x \ln x} - \frac{2}{x}\right) \times y$$

$$\frac{dy}{dx} = \left(1 + \frac{1}{x \ln x} - \frac{2}{x}\right) \times \frac{e^x \log x}{x^2}$$

29. Question

Find $\frac{dy}{dx}$, when:

$$y = \frac{x \cos^{-1} x}{\sqrt{1-x^2}}$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = \ln \cos^{-1} x + \ln(x) - \frac{1}{2} \ln(1-x^2)$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\} \{\ln e = 1\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \left(-\frac{1}{\sqrt{1-x^2}} + \frac{1}{x} + \frac{2x}{2(1-x^2)}\right)$$

$$\frac{dy}{dx} = \left(-\frac{1}{\sqrt{1-x^2}} + \frac{1}{x} + \frac{2x}{2(1-x^2)}\right) \times y$$

$$\frac{dy}{dx} = \left(-\frac{1}{\sqrt{1-x^2}} + \frac{1}{x} + \frac{x}{(1-x^2)}\right) \times \frac{x \cos^{-1} x}{\sqrt{1-x^2}}$$

30. Question

Find $\frac{dy}{dx}$, when:

$$y = (1+x)(1+x^2)(1+x^4)(1+x^6)$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = \ln(1+x) + \ln(1+x^2) + \ln(1+x^4) + \ln(1+x^6)$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\} \{\ln e = 1\}$$

Now differentiating both sides by x , we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{6x^5}{1+x^6} \right)$$

$$\frac{dy}{dx} = \left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{6x^5}{1+x^6} \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{6x^5}{1+x^6} \right) \times (1+x)(1+x^2)(1+x^4)(1+x^6)$$

31. Question

Find $\frac{dy}{dx}$, when:

$$y = x^x - 2^{\sin x}$$

Answer

simply taking log both sides would not help more.

For that let us assume $u = x^x$ and $v = 2^{\sin x}$

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$$u = x^x$$

Take log both sides

$$\ln u = x \ln x$$

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = x \left(\frac{1}{x} \right) + \ln x$$

$$\frac{du}{dx} = (1 + \ln x) \times u$$

$$\frac{du}{dx} = (1 + \ln x) \times x^x \dots\dots\dots(1)$$

$$v = 2^{\sin x}$$

Take log both sides,

$$\ln v = \sin x \cdot \ln 2$$

Differentiate ,

$$\frac{1}{v} \times \frac{dv}{dx} = \sin x(0) + \ln 2 \cdot \cos x$$

$$\frac{dv}{dx} = \ln 2 \cdot \cos x \times v$$

$$\frac{dv}{dx} = \ln 2 \cdot \cos x \times 2^{\sin x} \dots\dots\dots(2)$$

$$\frac{dy}{dx} = (1 + \ln x) \times x^x - \ln 2 \cdot \cos x \times 2^{\sin x}$$

32. Question

Find $\frac{dy}{dx}$, when:

$$y = (\log x)^x + x^{\log x}$$

Answer

simply taking log both sides would not help more.

For that let us assume $u = (\ln x)^x$ and $v = x^{\ln x}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (\ln x)^x$$

Take log both sides

$$\ln u = x \ln(\ln x)$$

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = x \left(\frac{1}{x} \times \frac{1}{\ln x} \right) + \ln(\ln x)$$

$$\frac{du}{dx} = \left(\frac{1}{\ln x} + \ln(\ln x) \right) \times u$$

$$\frac{du}{dx} = \left(\frac{1}{\ln x} + \ln(\ln x) \right) \times (\ln x)^x \dots\dots\dots(1)$$

$$v = x^{\ln x}$$

Take log both sides,

$$\ln v = \ln x \cdot \ln x$$

Differentiate ,

$$\frac{1}{v} \times \frac{dv}{dx} = 2 \cdot \ln x \times \frac{1}{x}$$

$$\frac{dv}{dx} = \frac{2 \cdot \ln x}{x} \times v$$

$$\frac{dv}{dx} = \frac{2 \cdot \ln x}{x} \times x^{\ln x} \dots\dots(2)$$

$$\frac{dy}{dx} = \left(\frac{1}{\ln x} + \ln(\ln x) \right) \times (\ln x)^x + \frac{2 \cdot \ln x}{x} \times x^{\ln x}$$

33. Question

Find $\frac{dy}{dx}$, when:

$$y = x^{\sin x} + (\sin x)^{\cos x}$$

Answer

simply taking log both sides would not help more.

For that let us assume $u = x^{\sin x}$ and $v = \sin x^{\cos x}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (x)^{\sin x}$$

Take log both sides

$$\ln u = \sin x \ln(x)$$

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = \sin x \left(\frac{1}{x} \right) + \ln(x) \times \cos x$$

$$\frac{du}{dx} = \left(\frac{\sin x}{x} + \ln(x) \times \cos x \right) \times u$$

$$\frac{du}{dx} = \left(\frac{\sin x}{x} + \ln(x) \times \cos x \right) \times (x^{\sin x} + (\sin x)^{\cos x}) \dots\dots\dots(1)$$

$$v = (\sin x)^{\cos x}$$

Take log both sides,

$$\ln v = \cos x \ln(\sin x)$$

Differentiate ,

$$\frac{1}{v} \times \frac{dv}{dx} = \cos x \left(\frac{1}{\sin x} \times \cos x \right)$$

$$\frac{dv}{dx} = \frac{\cos^2 x}{\sin x} \times v$$

$$\frac{dv}{dx} = \frac{\cos^2 x}{\sin x} \times (\sin x)^{\cos x} \dots\dots(2)$$

$$\frac{dy}{dx} = \frac{\cos^2 x}{\sin x} \times (\sin x)^{\cos x} + \left(\frac{\sin x}{x} + \ln(x) \times \cos x \right) \times (x^{\sin x} + (\sin x)^{\cos x})$$

34. Question

Find $\frac{dy}{dx}$, when:

$$y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$$

Answer

simply taking log both sides would not help more.

For that let us assume $u = (x \cdot \cos x)^x$ and $v = (x \cdot \sin x)^{\frac{1}{x}}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (x \cdot \cos x)^x$$

Take log both sides

$$\ln u = x (\ln(x) + \ln \cos x)$$

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = x \left(\frac{1}{x} - \frac{\sin x}{\cos x} \right) + (\ln(x) + \ln \cos x)$$

$$\frac{du}{dx} = \left(x \left(\frac{1}{x} - \frac{\sin x}{\cos x} \right) + (\ln(x) + \ln \cos x) \right) \times u$$

$$\frac{du}{dx} = \left(x \left(\frac{1}{x} - \frac{\sin x}{\cos x} \right) + (\ln(x) + \ln \cos x) \right) \times ((x \cdot \cos x)^x) \dots\dots\dots(1)$$

$$v = (x \cdot \sin x)^{\frac{1}{x}}$$

Take log both sides,

$$\ln v = \frac{1}{x} \times (\ln x + \ln \sin x)$$

Differentiate ,

$$\frac{1}{v} \times \frac{dv}{dx} = \frac{1}{x} \left(\frac{1}{x} + \frac{\cos x}{\sin x} \right) - \frac{1}{x^2} \times (\ln x + \ln \sin x)$$

$$\frac{dv}{dx} = \left(\frac{1}{x} \left(\frac{1}{x} + \frac{\cos x}{\sin x} \right) - \frac{1}{x^2} \times (\ln x + \ln \sin x) \right) \times v$$

$$\frac{dv}{dx} = \left(\frac{1}{x} \left(\frac{1}{x} + \frac{\cos x}{\sin x} \right) - \frac{1}{x^2} \times (\ln x + \ln \sin x) \right) \times (x \cdot \sin x)^{\frac{1}{x}} \dots\dots\dots(2)$$

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{1}{x} \left(\frac{1}{x} + \frac{\cos x}{\sin x} \right) - \frac{1}{x^2} \times (\ln x + \ln \sin x) \right) \times (x \cdot \sin x)^{\frac{1}{x}} \\ &\quad + \left(x \left(\frac{1}{x} - \frac{\sin x}{\cos x} \right) + (\ln(x) + \ln \cos x) \right) \times ((x \cdot \cos x)^x) \end{aligned}$$

35. Question

Find $\frac{dy}{dx}$, when:

$$y = (\sin x)^x + \sin^{-1} \sqrt{x}$$

Answer

simply taking log both sides would not help more.

For that let us assume $u = (\sin x)^x$ and $v = \sin^{-1} \sqrt{x}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (\sin x)^x$$

Take log both sides

$$\ln u = x \cdot \ln \sin x$$

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = x \left(\frac{\cos x}{\sin x} \right) + \ln \sin x$$

$$\frac{du}{dx} = \left(x \left(\frac{\cos x}{\sin x} \right) + \ln \sin x \right) \times u$$

$$\frac{du}{dx} = \left(x \left(\frac{\cos x}{\sin x} \right) + \ln \sin x \right) \times ((\sin x)^x) \dots\dots\dots(1)$$

for v we do not have to take log just simply differentiate it,

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}} \dots\dots\dots(2)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}} + \left(x \left(\frac{\cos x}{\sin x} \right) + \ln \sin x \right) \times ((\sin x)^x)$$

36. Question

Find $\frac{dy}{dx}$, when:

$$y = x^{x \cos x} + \left(\frac{x^2 + 1}{x^2 - 1} \right)$$

Answer

simply taking log both sides would not help more.

For that let us assume $u = (x)^{x \cos x}$ and $v = \frac{x^2 + 1}{x^2 - 1}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (x)^{x \cos x}$$

Take log both sides

$$\ln u = x \cdot \cos x \cdot \ln x$$

Here there are three terms to differentiate for this; we can take two term as one and then apply product rule, I am taking $x \cdot \ln x$ as a single term

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = \cos x \left(x \left(\frac{1}{x} \right) + \ln x \right) + x \cdot \ln x (-\sin x)$$

$$\frac{du}{dx} = (\cos x(1 + \ln x) - x \cdot \ln x \cdot \sin x) \times u$$

$$\frac{du}{dx} = (\cos x(1 + \ln x) - x \cdot \ln x \cdot \sin x) \times (x \cdot \cos x \cdot \ln x) \dots\dots\dots(1)$$

for v we do not have to take log just simply differentiate it,

$$\frac{dv}{dx} = \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{2x(-2)}{(x^2 - 1)^2} \dots\dots\dots(2)$$

$$\frac{dy}{dx} = (\cos x(1 + \ln x) - x \cdot \ln x \cdot \sin x) \times (x \cdot \cos x \cdot \ln x) + \frac{2x(-2)}{(x^2 - 1)^2}$$

37. Question

Find $\frac{dy}{dx}$, when:

$$y = e^x \sin^3 x \cos^4 x$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = x + 3 \cdot \ln \sin x + 4 \ln \cos x$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln \left(\frac{m}{n} \right) = \ln m - \ln n \right\} \{\ln e = 1\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \left(\frac{3 \cdot \cos x}{\sin x} - \frac{4 \sin x}{\cos x} + 1 \right)$$

$$\frac{dy}{dx} = \left(\frac{3 \cdot \cos x}{\sin x} - \frac{4 \sin x}{\cos x} + 1 \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{3 \cdot \cos x}{\sin x} - \frac{4 \sin x}{\cos x} + 1 \right) \times e^x \cdot \sin^3 x \cdot \cos^4 x$$

38. Question

Find $\frac{dy}{dx}$, when:

$$y = 2^x \cdot e^{3x} \sin 4x$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = x \cdot \ln 2 + 3x + \ln \sin 4x$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\} \{\ln e = 1\}$$

Now differentiating both sides by x , we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \ln 2 + 3 + \frac{\cos 4x}{\sin 4x} \times 4$$

$$\frac{dy}{dx} = \left(\ln 2 + 3 + \frac{\cos 4x}{\sin 4x} \times 4 \right) \times y$$

$$\frac{dy}{dx} = \left(\ln 2 + 3 + \frac{\cos 4x}{\sin 4x} \times 4 \right) \times e^{3x} \cdot \sin 4x \cdot 2^x$$

39. Question

Find $\frac{dy}{dx}$, when:

$$y = x^x \cdot e^{(2x+5)}$$

Answer

Here we need to take log both the sides to get that differentiation simple.

$$\ln y = x \cdot \ln x + 2x + 5$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\} \{\ln e = 1\}$$

Now differentiating both sides by x , we get,

$$\frac{1}{y} \times \frac{dy}{dx} = 1 + \ln x + 2$$

$$\frac{dy}{dx} = (\ln x + 3) \times y$$

$$\frac{dy}{dx} = (\ln x + 3) \times x^x \cdot e^{2x+5}$$

40. Question

Find $\frac{dy}{dx}$, when:

$$y = (2x + 3)^5 (3x - 5)^7 (5x - 1)^3$$

Answer

Here, we need to take log both the sides to get that differentiation simple.

$$\ln y = 5 \cdot \ln(2x + 3) + 7 \cdot \ln(3x - 5) + 3 \cdot \ln(5x - 1)$$

$$\{\ln(mn) = \ln n + \ln m\} \left\{ \ln\left(\frac{m}{n}\right) = \ln m - \ln n \right\} \{\ln e = 1\}$$

Now differentiating both sides by x, we get,

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{5 \times 2}{2x + 3} + \frac{7 \times 3}{3x - 5} + \frac{3 \times 5}{5x - 1}$$

$$\frac{dy}{dx} = \left(\frac{10}{2x + 3} + \frac{21}{3x - 5} + \frac{15}{5x - 1} \right) \times y$$

$$\frac{dy}{dx} = \left(\frac{10}{2x + 3} + \frac{21}{3x - 5} + \frac{15}{5x - 1} \right) \times (2x + 3)^5 (3x - 5)^7 (5x - 1)^3$$

41. Question

Find $\frac{dy}{dx}$, when:

$$(\cos x)^y = (\cos y)^x$$

Answer

. So the equation given is implicit, we will just take log both sides

$$y \cdot \ln(\cos x) = x \cdot \ln(\cos y)$$

Now differentiate it with respect to x and consider $\frac{dy}{dx} = y'$

$$y \left(\frac{-\sin x}{\cos x} \right) + \ln \cos x \cdot y' = x \left(\frac{-\sin y}{\cos y} \times y' \right) + \ln \cos y$$

Taking y' one side, we get

$$y'(\ln \cos x + x \cdot \tan x) = \ln \cos y + y \cdot \tan x$$

$$y' = \frac{\ln \cos y + y \cdot \tan x}{\ln \cos x + x \cdot \tan x}$$

42. Question

Find $\frac{dy}{dx}$, when:

$$(\tan x)^y = (\tan y)^x$$

Answer

. So the equation given is implicit, we will just take log both sides

$$y \cdot \ln(\tan x) = x \cdot \ln(\tan y)$$

Now differentiate it with respect to x and consider $\frac{dy}{dx} = y'$

$$y \left(\frac{\sec^2 x}{\tan x} \right) + \ln \tan x \cdot y' = x \left(\frac{\sec^2 y}{\tan y} \times y' \right) + \ln \tan y$$

Taking y' one side, we get

$$y' \left(\ln \tan x + \frac{x}{\sin y \cdot \cos y} \right) = \ln \tan y + \frac{y}{\sin x \cdot \cos x}$$

$$y' = \frac{\sin 2x \cdot \ln \tan y + 2y}{\sin 2y \cdot \ln \tan x + 2x}$$

43. Question

Find $\frac{dy}{dx}$, when:

$$y = ((\log x)^x + (x)^{\log x})$$

Answer

we can write this equation as,

$$y = e^{x \ln(\ln x)} + e^{\ln x \cdot \ln x}$$

Differentiate

$$y' = (\ln x)^x \left(x \left(\frac{1}{\ln x} \times \frac{1}{x} \right) + \ln(\ln x) \right) + x^{\ln x} \left(2 \cdot \frac{\ln x}{x} \right)$$

$$y' = (\ln x)^x \left(\frac{1}{\ln x} + \ln(\ln x) \right) + x^{\ln x} \left(\frac{2 \ln x}{x} \right)$$

44. Question

If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that $(1-x^2) \frac{dy}{dx} = (xy + 1)$.

Answer

differentiate the given y to get the result,

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2} \left(\frac{1}{\sqrt{1-x^2}} \right) - \sin^{-1} x \left(\frac{-x}{\sqrt{1-x^2}} \right)}{(\sqrt{1-x^2})^2}$$

$$\frac{dy}{dx} = \frac{1 + \frac{x \cdot \sin^{-1} x}{\sqrt{1-x^2}}}{1-x^2}$$

$$\frac{dy}{dx} (1-x^2) = 1 + xy$$

45. Question

If $y = \sqrt{x+y}$, prove that $\frac{dy}{dx} = \frac{1}{(2y-1)}$.

Answer

differentiate the given y to get the result,

$$\frac{dy}{dx} = \frac{1 + \frac{dy}{dx}}{2\sqrt{x+y}}$$

$$\text{let, } \frac{dy}{dx} = y'$$

$$y' = \frac{1+y'}{2\sqrt{x+y}} \text{ \{taking } y' \text{ one side\}}$$

$$y'(2\sqrt{x+y} - 1) = 1$$

$$\frac{dy}{dx} = \frac{1}{2y-1}$$

46. Question

If $x^a y^b = (x + y)^{(a+b)}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

Answer

taking log both sides,

$$a \ln x + b \ln y = (a + b) \cdot \ln(x + y)$$

differentiating both sides,

$$\frac{a}{x} + \frac{b}{y} \times y' = \frac{a + b}{x + y} \times (1 + y')$$

Take y' one side,

$$y' \left(\frac{b}{y} - \frac{a + b}{x + y} \right) = \frac{a + b}{x + y} - \frac{a}{x}$$

$$y' = \frac{ax + bx - (ax + ay)}{x \cdot (x + y)} \times \frac{y \cdot (x + y)}{bx + by - (ay + by)}$$

$$y' = \frac{bx - ay}{x} \times \frac{y}{bx - ay}$$

$$y' = \frac{y}{x}$$

47. Question

If $(x^x + y^x) = 1$, show that $\frac{dy}{dx} = - \left\{ \frac{x^x (1 + \log x) + y^x (\log y)}{xy^{x-1}} \right\}$

Answer

differentiate both sides,

$$x^x (1 + \ln x) + y^x \left(\frac{x}{y} \times y' + \ln y \right) = 0$$

Taking y' one side,

$$y' = \left(\frac{x^x (1 + \ln x)}{y^x} - \ln y \right) \times \frac{y}{x}$$

$$y' = \frac{x^x (1 + \ln x) - y^x \cdot \ln y}{x \cdot y^{x-1}}$$

48. Question

If $y = e^{\sin x} + (\tan x)^x$, prove that $\frac{dy}{dx} = e^{\sin x} \cos x + (\tan x)^x [2x \operatorname{cosec} 2x + \log \tan x]$.

Answer

differentiate both sides,

$$y' = e^{\sin x} (\cos x) + (\tan x)^x \left(x \left(\frac{\sec^2 x}{\tan x} \right) + \ln \tan x \right)$$

$$y' = e^{\sin x} (\cos x) + (\tan x)^x (2x \operatorname{cosec} 2x + \ln \tan x)$$

49. Question

If $y = \log(x + \sqrt{1+x^2})$, prove that $\frac{dy}{dx} = \frac{1}{\log(x + \sqrt{1+x^2})} \cdot \frac{1}{\sqrt{1+x^2}}$.

Answer

differentiate both sides,

$$y' = \frac{1}{x + \sqrt{1+x^2}} \times \left(1 + \frac{x}{\sqrt{1+x^2}} \right)$$

$$y' = \frac{1}{x + \sqrt{1+x^2}} \times \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}$$

$$y' = \frac{1}{\sqrt{1+x^2}}$$

50. Question

If $y = \log \sin \sqrt{x^2 + 1}$, prove that $\frac{dy}{dx} = \frac{x \cot \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}$.

Answer

differentiate both sides,

$$y' = \frac{1}{\sin(\sqrt{1+x^2})} \times \cos(\sqrt{1+x^2}) \times \frac{x}{\sqrt{1+x^2}}$$

$$y' = \frac{(\cot(\sqrt{1+x^2}) \cdot x)}{\sqrt{1+x^2}}$$

51. Question

If $y = \log \sqrt{\frac{1 - \cos x}{1 + \cos x}}$, show that $\frac{dy}{dx} = \operatorname{cosec} x$.

Answer

differentiate both sides,

$$y' = \sqrt{\frac{1 + \cos x}{1 - \cos x}} \times \frac{(1 + \cos x)(\sin x) - (1 - \cos x)(-\sin x)}{(1 + \cos x)^2}$$

$$y' = \sqrt{\frac{1 + \cos x}{1 - \cos x}} \times \frac{(\sin x + \sin x \cdot \cos x) + (\sin x - \sin x \cdot \cos x)}{(1 + \cos x)^2}$$

$$y' = \sqrt{\frac{1 + \cos x}{1 - \cos x}} \times \frac{2 \sin x}{(1 + \cos x)^2}$$

$$y' = \sqrt{\frac{2 \cos^2 \frac{x}{2}}{2 \sin^2 \frac{x}{2}}} \times \frac{4 \sin \frac{x}{2} \cdot \cos \frac{x}{2}}{(1 + \cos x)^2}$$

$$y' = 4 \cos \frac{x}{2} \times \frac{\cos \frac{x}{2}}{4 \cos^4 \frac{x}{2}}$$

$$y' = \frac{1}{\cos \frac{x}{2}}$$

$$y' = \sec^2 \frac{x}{2}$$

52. Question

If $y = \log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$, show that $\frac{dy}{dx} = \sec x$.

Answer

differentiate both sides,

$$y' = \frac{1}{\tan \left(\frac{\pi}{4} + \frac{x}{2} \right)} \times \sec^2 \left(\frac{\pi}{4} + \frac{x}{2} \right) \times \frac{1}{2}$$

$$y' = \frac{1}{2 \times \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$

$$y' = \frac{1}{\sin\left(\frac{\pi}{2} + x\right)}$$

$$y' = \sec x$$

53. Question

If $y = \sqrt{\frac{1 - \sin 2x}{1 + \sin 2x}}$, show that $\frac{dy}{dx} + \sec^2\left(\frac{\pi}{4} - x\right) = 0$.

Answer

differentiate both sides,

$$y' = \frac{1}{2} \times \sqrt{\frac{1 + \sin 2x}{1 - \sin 2x}} \times \frac{(1 + \sin 2x)(-2 \cos 2x) - (1 - \sin 2x)(2 \cos 2x)}{(1 + \sin 2x)^2}$$

$$y' = \frac{1}{2} \times \sqrt{\frac{1}{1 - \sin^2 2x}} \times \frac{2 \cos 2x(-1 - \sin 2x - 1 + \sin 2x)}{1 + \sin 2x}$$

$$y' = \frac{1}{2} \times \frac{-4}{1 + \sin 2x}$$

$$y' = \frac{-2}{(\cos x + \sin x)^2}$$

$$y' = \frac{-1}{\left(\frac{\cos x}{\sqrt{2}} + \frac{\sin x}{\sqrt{2}}\right)^2}$$

$$y' = \frac{-1}{\cos^2\left(\frac{\pi}{4} + x\right)}$$

$$\frac{dy}{dx} + \sec^2\left(\frac{\pi}{4} + x\right) = 0$$

54. Question

If $y = \log \sqrt{\frac{1 + \cos^2 x}{1 - e^{2x}}}$, show that $\frac{dy}{dx} = \frac{e^{2x}}{(1 - e^{2x})} - \frac{\sin x \cos x}{(1 + \cos^2 x)}$.

Answer

differentiate both sides,

$$y' = \sqrt{\frac{1 - e^{2x}}{1 + \cos^2 x}} \times \frac{(1 - e^{2x})(-2 \cos x \cdot \sin x) - (1 + \cos^2 x)(-2e^{2x})}{(1 - e^{2x})^2} \times \frac{1}{2} \\ \times \sqrt{\frac{1 - e^{2x}}{1 + \cos^2 x}}$$

$$y' = \sqrt{\frac{1 - e^{2x}}{\cos 2x}} \times \frac{(e^{2x} - 1)(\sin 2x) + 2e^{2x}(\cos 2x)}{(1 - e^{2x})^2} \times \frac{1}{2} \times \sqrt{\frac{1 - e^{2x}}{\cos 2x}}$$

$$y' = \frac{(e^{2x} - 1) \tan 2x + 2e^{2x}}{2 \cdot (1 - e^{2x})}$$

$$y' = \frac{e^{2x}}{(1 - e^{2x})} - \frac{\sin x \cdot \cos x}{(1 + \cos^2 x)}$$

55. Question

If $y = (x)^{\cos x} + (\sin x)^{\tan x}$, prove that $\frac{dy}{dx} = x^{\cos x} \left\{ \frac{\cos x}{x} - (\sin x) \log x \right\} + (\sin x)^{\tan x} \cdot \{1 + (\log \sin x) \sec^2 x\}$.

Answer

simply taking log both sides would not help more.

For that let us assume $u = (x)^{\cos x}$ and $v = (\sin x)^{\tan x}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (x)^{\cos x}$$

Take log both sides

$$\ln u = \cos x \cdot \ln x$$

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = \cos x \left(\frac{1}{x} \right) + \ln x (-\sin x)$$

$$\frac{du}{dx} = \left(\frac{\cos x}{x} - \ln x \cdot \sin x \right) \times u$$

$$\frac{du}{dx} = \left(\frac{\cos x}{x} - \ln x \cdot \sin x \right) \times (x^{\cos x}) \dots\dots\dots(1)$$

$$v = (\sin x)^{\tan x}$$

Take log both sides,

$$\ln v = \tan x \times (\ln \sin x)$$

Differentiate ,

$$\frac{1}{v} \times \frac{dv}{dx} = \tan x \left(\frac{\cos x}{\sin x} \right) + \ln \sin x (\sec^2 x)$$

$$\frac{dv}{dx} = \left(\tan x \left(\frac{\cos x}{\sin x} \right) + \ln \sin x (\sec^2 x) \right) \times v$$

$$\frac{dv}{dx} = (1 + \ln \sin x (\sec^2 x)) \times (\sin x)^{\tan x} \dots\dots(2)$$

$$\frac{dy}{dx} = (1 + \ln \sin x (\sec^2 x)) \times (\sin x)^{\tan x} + \left(\frac{\cos x}{x} - \ln x \cdot \sin x \right) \times (x^{\cos x})$$

56. Question

If $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$, prove that $\frac{dy}{dx} = (\sin x)^{\cos x} \cdot [\cot x \cos x - \sin x (\log \sin x)] + (\cos x)^{\sin x} \cdot [\cos x (\log \cos x) - \sin x \tan x]$.

Answer

simply taking log both sides would not help more.

For that let us assume $u = (\sin x)^{\cos x}$ and $v = (\cos x)^{\sin x}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (\sin x)^{\cos x}$$

Take log both sides

$$\ln u = \cos x \cdot \ln \sin x$$

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = \cos x \left(\frac{\cos x}{\sin x} \right) + \ln(\sin x) (-\sin x)$$

$$\frac{du}{dx} = \left(\frac{\cos^2 x}{\sin x} - \ln \sin x \cdot \sin x \right) \times u$$

$$\frac{du}{dx} = \left(\frac{\cos^2 x}{\sin x} - \ln \sin x \cdot \sin x \right) \times (\sin x^{\cos x}) \dots\dots\dots(1)$$

$$v = (\cos x)^{\sin x}$$

Take log both sides,

$$\ln v = \sin x \times (\ln \cos x)$$

Differentiate ,

$$\frac{1}{v} \times \frac{dv}{dx} = \sin x \left(\frac{-\sin x}{\cos x} \right) + \ln \cos x (\cos x)$$

$$\frac{dv}{dx} = \left(\sin x \left(\frac{-\sin x}{\cos x} \right) + \ln \cos x (\cos x) \right) \times v$$

$$\frac{dv}{dx} = \left(\sin x \left(\frac{-\sin x}{\cos x} \right) + \ln \cos x (\cos x) \right) \times (\cos x)^{\sin x} \dots\dots(2)$$

$$\begin{aligned} \frac{dy}{dx} &= \left(\sin x \left(\frac{-\sin x}{\cos x} \right) + \ln \cos x (\cos x) \right) \times (\cos x)^{\sin x} \\ &\quad + \left(\frac{\cos^2 x}{\sin x} - \ln \sin x \cdot \sin x \right) \times (\sin x^{\cos x}) \end{aligned}$$

57. Question

If $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$, $\frac{dy}{dx} = (\tan x)^{\cot x} \cdot \operatorname{cosec}^2 x (1 - \log \tan x)$
 $+ (\cot x)^{\tan x} \cdot \sec^2 x [\log(\cot x) - 1]$.

Answer

simply taking log both sides would not help more.

For that let us assume $u = (\tan x)^{\cot x}$ and $v = (\cot x)^{\tan x}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (\tan x)^{\cot x}$$

Take log both sides

$$\ln u = \cot x \cdot \ln \tan x$$

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = \cot x \left(\frac{\sec^2 x}{\tan x} \right) + \ln(\tan x) (-\operatorname{cosec}^2 x)$$

$$\frac{du}{dx} = \left(\cot x \left(\frac{\sec^2 x}{\tan x} \right) - \ln(\tan x) (\operatorname{cosec}^2 x) \right) \times u$$

$$\frac{du}{dx} = (\operatorname{cosec}^2 x (1 - \ln(\tan x))) \times (\tan x)^{\cot x} \dots\dots\dots(1)$$

$$v = (\cot x)^{\tan x}$$

Take log both sides,

$$\ln v = \tan x \times (\ln \cot x)$$

Differentiate ,

$$\frac{1}{v} \times \frac{dv}{dx} = \tan x \left(\frac{-\operatorname{cosec}^2 x}{\cot x} \right) + \ln \cot x (\sec^2 x)$$

$$\frac{dv}{dx} = (\sec^2 x (\ln \cot x - 1)) \times v$$

$$\frac{dv}{dx} = (\sec^2 x (\ln \cot x - 1)) \times (\cot x)^{\tan x} \dots\dots\dots(2)$$

$$\begin{aligned} \frac{dy}{dx} &= (\sec^2 x (\ln \cot x - 1)) \times (\cot x)^{\tan x} \\ &\quad + (\operatorname{cosec}^2 x (1 - \ln(\tan x))) \times (\tan x)^{\cot x} \end{aligned}$$

58. Question

If $y = x^{\cos x} + (\cos x)^x$, prove that $\frac{dy}{dx} = x^{\cos x} \cdot \left\{ \frac{\cos x}{x} - (\sin x) \log x \right\} + (\cos x)^x [(\log \cos x) - x \tan x]$.

Answer

simply taking log both sides would not help more.

For that let us assume $u = (x)^{\cos x}$ and $v = (\cos x)^x$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (x)^{\cos x}$$

Take log both sides

$$\ln u = \cos x \cdot \ln x$$

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = \cos x \left(\frac{1}{x} \right) + \ln(x)(-\sin x)$$

$$\frac{du}{dx} = \left(\cos x \left(\frac{1}{x} \right) + \ln(x)(-\sin x) \right) \times u$$

$$\frac{du}{dx} = \left(\cos x \left(\frac{1}{x} \right) + \ln(x)(-\sin x) \right) \times (x)^{\cos x} \dots\dots\dots(1)$$

$$v = (\cos x)^x$$

Take log both sides,

$$\ln v = x \times (\ln \cos x)$$

Differentiate ,

$$\frac{1}{v} \times \frac{dv}{dx} = x \left(\frac{-\sin x}{\cos x} \right) + \ln \cos x. 1$$

$$\frac{dv}{dx} = \left(x \left(\frac{-\sin x}{\cos x} \right) + \ln \cos x. 1 \right) \times v$$

$$\frac{dv}{dx} = \left(x \left(\frac{-\sin x}{\cos x} \right) + \ln \cos x. 1 \right) \times (\cos x)^x \dots\dots(2)$$

$$\begin{aligned} \frac{dy}{dx} &= \left(x \left(\frac{-\sin x}{\cos x} \right) + \ln \cos x. 1 \right) \times (\cos x)^x \\ &\quad + \left(\cos x \left(\frac{1}{x} \right) + \ln(x)(-\sin x) \right) \times (x)^{\cos x} \end{aligned}$$

59. Question

If $y = x^{\log x} + (\log x)^x$, prove that $\frac{dy}{dx} = x^{(\log x)} \left\{ \frac{2 \log x}{x} \right\} + (\log x)^x \cdot \left\{ \frac{1}{\log x} + \log(\log x) \right\}$.

Answer

simply taking log both sides would not help more.

For that let us assume $u = (x)^{\ln x}$ and $v = (\ln x)^x$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = (x)^{\ln x}$$

Take log both sides

$$\ln u = \ln x. \ln x$$

Differentiate

$$\frac{1}{u} \times \frac{du}{dx} = 2 \ln x \left(\frac{1}{x} \right)$$

$$\frac{du}{dx} = \left(2 \ln x \left(\frac{1}{x} \right) \right) \times u$$

$$\frac{du}{dx} = \left(2 \ln x \left(\frac{1}{x} \right) \right) \times (x)^{\ln x} \dots\dots\dots(1)$$

$$v = (\ln x)^x$$

Take log both sides,

$$\ln v = x \times (\ln x)$$

Differentiate ,

$$\frac{1}{v} \times \frac{dv}{dx} = x \left(\frac{1}{x} \right) + \ln x$$

$$\frac{dv}{dx} = (1 + \ln x) \times v$$

$$\frac{dv}{dx} = (1 + \ln x) \times (\ln x)^x \dots\dots\dots(2)$$

$$\frac{dy}{dx} = (1 + \ln x) \times (\ln x)^x + \left(2 \ln x \left(\frac{1}{x} \right) \right) \times (x)^{\ln x}$$

60. Question

$$\text{If } y - x^{(x^2-3)} + (x-3)^{x^2}, \text{ find } \frac{dy}{dx}.$$

Answer

equality is not given but we may assume that it is equal to 0.

We can also write this equation as

$$y - e^{(x^2-3) \ln x} + e^{x^2 \ln(x-3)} = 0$$

Now differentiating it,

$$y' - x^{x^2-3} \left(\frac{x^2-3}{x} + \ln x \cdot 2x \right) + (x-3)^{x^2} \cdot \left(\frac{x^2}{x-3} + \ln(x-3) \cdot 2x \right) = 0$$

$$y' = x^{x^2-3} \left(\frac{x^2-3}{x} + \ln x \cdot 2x \right) - (x-3)^{x^2} \cdot \left(\frac{x^2}{x-3} + \ln(x-3) \cdot 2x \right)$$

61. Question

If $f(x) = \left(\frac{3+x}{1+x}\right)^{2+3x}$, find $f'(0)$.

Answer

take log both the side,

$$\ln f(x) = (2+3x) \cdot \ln\left(\frac{3+x}{1+x}\right)$$

Now differentiate it,

$$\frac{1}{f(x)} \times f'(x) = (2+3x) \left(\frac{1+x}{3+x}\right) \left(\frac{1+x-(3+x)}{(1+x)^2}\right) + \ln\left(\frac{3+x}{1+x}\right) \cdot 3$$

$$f'(x) = \left(\frac{(2+3x)(-2)}{(3+x)(1+x)} + 3 \ln\left(\frac{3+x}{1+x}\right)\right) \times f(x)$$

To get $f'(0)$ we need to find $f(0)$,

Putting $x=0$ in f

$$f(0) = \left(\frac{3}{1}\right)^2$$

$$f(0) = 9$$

Now put $x=0$ in $f'(x)$,

$$f'(0) = \left(\left(\frac{2 \times (-2)}{3}\right) + 3 \ln 3\right) \times 9$$

$$f'(0) = 9\left(3 \ln 3 - \frac{4}{3}\right)$$

62. Question

If $y = (\sin x)^x + \sin^{-1} \sqrt{x}$, find $\frac{dy}{dx}$.

Answer

we can write this equation as,

$$y = e^{x \ln(\sin x)} + \sin^{-1} \sqrt{x}$$

Differentiate it,

$$y' = (\sin x)^x \left(\frac{x \times \cos x}{\sin x} + \ln(\sin x) \right) + \frac{1}{\sqrt{1 - \sqrt{x}^2}} \times \frac{1}{2\sqrt{x}}$$

$$y' = (\sin x)^x (x \cdot \cot x + \ln \sin x) + \frac{1}{2\sqrt{x} \cdot \sqrt{1 - x}}$$

63. Question

If $(x^2 + y^2)^2 = xy$, find $\frac{dy}{dx}$.

Answer

simply differentiate both sides,

$$2(x^2 + y^2)(2x + 2y \cdot y') = x \cdot y' + y$$

Take y' one side

$$4x^3 + 4x^2 \cdot y \cdot y' + 4y^2 \cdot x + 4y^3 \cdot y' = x \cdot y' + y$$

$$y'(4x^2 \cdot y + 4y^3 - x) = y - 4x^3 - 4y^2 x$$

$$y' = \frac{y - 4x^3 - 4y^2 x}{4x^2 \cdot y + 4y^3 - x}$$

64. Question

$$y = x^{\cot x} + \frac{2x^2 - 3}{x^2 + x + 2}, \text{ find } \frac{dy}{dx}.$$

Answer

we can write this as,

$$y = e^{\cot x \cdot \ln x} + \frac{2x^2 - 3}{x^2 + x + 2}$$

Differentiate ,

$$y' = x^{\cot x} \left(\frac{\cot x}{x} + \ln x (-\operatorname{cosec}^2 x) \right) + \frac{(x^2 + x + 2)(4x) - (2x^2 - 3)(2x + 1)}{(x^2 + x + 2)^2}$$

$$y' = x^{\cot x} \left(\frac{\cot x}{x} + \ln x (-\operatorname{cosec}^2 x) \right) + \frac{2x^2 + 14x + 3}{(x^2 + x + 2)^2}$$

65. Question

Find , when:

If $y = \tan^{-1} \frac{a}{x} + \log \sqrt{\frac{x-a}{x+a}}$, prove that $\frac{dy}{dx} = \frac{2a^3}{(x^4 - a^4)}$.

Answer

Differentiate it,

$$y' = \frac{1}{1 + \frac{a^2}{x^2}} \times \left(-\frac{a}{x^2}\right) + \sqrt{\frac{x+a}{x-a}} \times \frac{1}{2} \times \sqrt{\frac{x+a}{x-a}} \times \frac{(x+a) - (x-a)}{(x+a)^2}$$

$$y' = \frac{-a}{x^2 + a^2} + \frac{x-a}{2(x+a)} \times \frac{2a}{(x-a)^2}$$

$$y' = -\frac{a}{(x^2 + a^2)} + \frac{a}{x^2 - a^2}$$

$$y' = \frac{ax^2 + a^3 - ax^2 + a^3}{x^4 - a^4}$$

$$y' = \frac{2a^3}{x^4 - a^4}$$

66. Question

If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

Answer

taking log both sides,

$$m \ln x + n \ln y = (m+n) \cdot \ln(x+y)$$

differentiating both sides,

$$\frac{m}{x} + \frac{n}{y} \times y' = \frac{m+n}{x+y} \times (1+y')$$

Take y' one side,

$$y' \left(\frac{n}{y} - \frac{m+n}{x+y} \right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$y' = \frac{mx + nx - (mx + my)}{x \cdot (x+y)} \times \frac{y \cdot (x+y)}{nx + ny - (my + ny)}$$

$$y' = \frac{nx - my}{x} \times \frac{y}{nx - my}$$

$$y' = \frac{y}{x}$$

Exercise 10G

1. Question

If $y = (\sin x)^{(\sin x)^{(\sin x)^{(\sin x)^{\dots\infty}}}}$, prove that $\frac{dy}{dx} = \frac{y^2 \cot x}{(1 - y \log \sin x)}$.

Answer

Given : $y = (\sin x)^{(\sin x)^{(\sin x)^{(\sin x)^{\dots\infty}}}}$

To prove : $\frac{dy}{dx} = \frac{y^2 \cot x}{1 - y \log \sin x}$

Formula used : $\log a = \log b^m$

$$\log a = m \log b$$

$$\frac{d(\log y)}{dx} = \frac{1}{y} \frac{dy}{dx}$$

$$\frac{d(\sin x)}{dx} = \cos x$$

If u and v are functions of x , then $\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

$$y = (\sin x)^y$$

taking log on both sides

$$\log y = \log (\sin x)^y$$

$$\log y = y \log (\sin x)$$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dx} = \frac{d[y \log(\sin x)]}{dx}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{dy}{dx} \log(\sin x) + y \frac{d \log(\sin x)}{dx}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{dy}{dx} \log(\sin x) + y \frac{1}{\sin x} \times \frac{d(\sin x)}{dx}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{dy}{dx} \log(\sin x) + y \frac{\cos x}{\sin x}$$

$$\left(\frac{1}{y} - \log \sin x\right) \frac{dy}{dx} = y \cot x$$

$$\frac{1 - y \log \sin x}{y} \frac{dy}{dx} = y \cot x$$

$$\frac{dy}{dx} = \frac{y^2 \cot x}{1 - y \log \sin x}$$

$$\frac{dy}{dx} = \frac{y^2 \cot x}{1 - y \log \sin x}$$

2. Question

If $y = (\cos x)^{(\cos x)^{(\cos x) \dots \infty}}$, prove that $\frac{dy}{dx} = \frac{-y^2 \tan x}{1 - y \log \cos x}$.

Answer

Given : $y = (\cos x)^{(\cos x)^{(\cos x)^{\dots \infty}}}$

To prove : $\frac{dy}{dx} = \frac{-y^2 \tan x}{1 - y \log \cos x}$

Formula used : $\log a = \log b^m$

$$\log a = m \log b$$

$$\frac{d(\log y)}{dx} = \frac{1}{y} \frac{dy}{dx}$$

$$\frac{d(\cos x)}{dx} = -\sin x$$

If u and v are functions of x , then $\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

Given that $y = (\cos x)^y$

taking log on both sides

$$\log y = \log (\cos x)^y$$

$$\log y = y \log (\cos x)$$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dx} = \frac{d[y \log(\cos x)]}{dx}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{dy}{dx} \log(\cos x) + y \frac{d \log(\cos x)}{dx}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{dy}{dx} \log(\cos x) + y \frac{1}{\cos x} \times \frac{d(\cos x)}{dx}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{dy}{dx} \log(\cos x) + y \frac{-\sin x}{\cos x}$$

$$\left(\frac{1}{y} - \log \cos x\right) \frac{dy}{dx} = -y \tan x$$

$$\frac{dy}{dx} = \frac{-y^2 \tan x}{1 - y \log \cos x}$$

$$\frac{dy}{dx} = \frac{-y^2 \tan x}{1 - y \log \cos x}$$

3. Question

If $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$, prove that $\frac{dy}{dx} = \frac{1}{(2y-1)}$.

Answer

$$\text{Given : } y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$$

$$\text{To prove : } \frac{dy}{dx} = \frac{1}{2y-1}$$

$$\text{Formula used : } \log a = \log b^m$$

$$\log a = m \log b$$

$$\frac{d(\log y)}{dx} = \frac{1}{y} \frac{dy}{dx}$$

$$\frac{dx}{dx} = 1$$

$$\text{If } u \text{ and } v \text{ are functions of } x, \text{ then } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$$

$$y = \sqrt{x + y}$$

squaring on both sides

$$y^2 = x + y$$

Differentiating with respect to x

$$2y \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$(2y - 1) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y-1}$$

$$\frac{dy}{dx} = \frac{1}{2y-1}$$

4. Question

If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}}$, prove that $\frac{dy}{dx} = \frac{\sin x}{(1-2y)}$.

Answer

$$\text{Given : } y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}}$$

$$\text{To prove : } \frac{dy}{dx} = \frac{\sin x}{2y-1}$$

Formula used : $\log a = \log b^m$

$$\log a = m \log b$$

$$\frac{d(\log y)}{dx} = \frac{1}{y} \frac{dy}{dx}$$

$$\frac{d(\cos x)}{dx} = -\sin x$$

$$\text{If } u \text{ and } v \text{ are functions of } x, \text{ then } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

$$y = \sqrt{\cos x + y}$$

squaring on both sides

$$y^2 = \cos x + y$$

Differentiating with respect to x

$$2y \frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$(2y - 1) \frac{dy}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{-\sin x}{2y-1} = \frac{\sin x}{1-2y}$$

$$\frac{dy}{dx} = \frac{\sin x}{1-2y}$$

$$\frac{dy}{dx} = \frac{\sin x}{1-2y}$$

5. Question

If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}}$, prove that $\frac{dy}{dx} = \frac{\sec^2 x}{(2y-1)}$.

Answer

$$\text{Given : } y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}}$$

$$\text{To prove : } \frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$$

Formula used : $\log a = \log b^m$

$$\log a = m \log b$$

$$\frac{d(\log y)}{dx} = \frac{1}{y} \frac{dy}{dx}$$

$$\frac{d(\tan x)}{dx} = \sec^2 x$$

If u and v are functions of x, then $\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)).g'(x)$

$$y = \sqrt{\tan x + y}$$

squaring on both sides

$$y^2 = \tan x + y$$

Differentiating with respect to x

$$2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$(2y - 1) \frac{dy}{dx} = \sec^2 x$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{2y-1} = \frac{\sec^2 x}{2y-1}$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$$

6. Question

If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots \infty}}}$, show that $(2y - 1) \cdot \frac{dy}{dx} = \frac{1}{x}$.

Answer

$$\text{Given : } y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots \dots \dots \infty}}}$$

$$\text{To show : } (2y - 1) \cdot \frac{dy}{dx} = \frac{1}{x}$$

$$\text{Formula used : } \log a = \log b^m$$

$$\log a = m \log b$$

$$\frac{d(\log y)}{dx} = \frac{1}{y} \frac{dy}{dx}$$

$$\text{If } u \text{ and } v \text{ are functions of } x, \text{ then } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

$$y = \sqrt{\log x + y}$$

squaring on both sides

$$y^2 = \log x + y$$

Differentiating with respect to x

$$2y \frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$$

$$(2y - 1) \frac{dy}{dx} = \frac{1}{x}$$

$$(2y - 1) \frac{dy}{dx} = \frac{1}{x}$$

7. Question

If $y = a^{x^{a^{x^{\dots\infty}}}}$, prove that $\frac{dy}{dx} = \frac{y^2 (\log y)}{x[1 - y(\log x)(\log y)]}$.

Answer

Given : $y = a^{x^{a^{x^{\dots\infty}}}}$

To show : $\frac{dy}{dx} = \frac{y^2 (\log y)}{x[1 - y(\log x)(\log y)]}$

Formula used : $\log a = \log b^m$

$$\log a = m \log b$$

$$\frac{d(\log y)}{dx} = \frac{1}{y} \frac{dy}{dx}$$

If u and v are functions of x, then $\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

$$y = a^{x^y}$$

taking log on both sides

$$\log y = \log a^{x^y}$$

$$\log y = x^y \cdot \log a$$

taking log on both sides

$$\log(\log y) = \log(x^y \cdot \log a)$$

$$\log(\log y) = y \cdot \log x + \log(\log a)$$

Differentiating both sides with respect to x

$$\frac{d(\log [\log y])}{dx} = \frac{d(y \cdot \log x)}{dx} + 0 \text{ (as differentiation of } \log(\log a) \text{ [constant] is zero)}$$

$$\frac{1}{\log y} \frac{d \log y}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d \log x}{dx}$$

$$\frac{1}{\log y} \cdot \frac{1}{y} \frac{dy}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{1}{x}$$

$$\left(\frac{1}{\log y} \cdot \frac{1}{y} - \log x\right) \frac{dy}{dx} = \frac{y}{x}$$

$$\left(\frac{1-y(\log x)(\log y)}{y(\log y)}\right) \frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y^2(\log y)}{x[1-y(\log x)(\log y)]}$$

$$\frac{dy}{dx} = \frac{y^2(\log y)}{x[1-y(\log x)(\log y)]}$$

8. Question

If $y = x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \dots \infty}}}$ prove that $\frac{dy}{dx} = \frac{y}{(2y-x)}$.

Answer

Given : $y = x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \dots \infty}}}$

To show : $\frac{dy}{dx} = \frac{y}{(2y-x)}$

Formula used : $\log a = \log b^m$

$$\log a = m \log b$$

$$\frac{d(\log y)}{dx} = \frac{1}{y} \frac{dy}{dx}$$

If u and v are functions of x, then $\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

$$y = x + \frac{1}{y}$$

$$y^2 = xy + 1$$

Differentiating with respect to x

$$\frac{d(y^2)}{dx} = \frac{d(xy)}{dx} + 0 \text{ (as differentiation of constant is zero)}$$

$$2y \cdot \frac{dy}{dx} = x \cdot \frac{dy}{dx} + y$$

$$(2y - x) \frac{dy}{dx} = y$$

$$\frac{dy}{dx} = \frac{y}{(2y-x)}$$

$$\frac{dy}{dx} = \frac{y}{(2y-x)}$$

Exercise 10H

1. Question

Differentiate x^6 with respect to $(1/\sqrt{x})$.

Answer

Given : Let $u = x^6$ and $v = \frac{1}{\sqrt{x}}$

To differentiate : x^6 with respect to $(1/\sqrt{x})$.

Formula used : $\frac{d(x^n)}{dx} = n \cdot x^{n-1}$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

Let $u = x^6$ and $v = \frac{1}{\sqrt{x}}$

Differentiating u with respect to x

$$\frac{du}{dx} = 6x^5$$

Differentiating v with respect to x

$$\frac{dv}{dx} = \frac{-1}{2} x^{-\frac{3}{2}}$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\frac{du}{dv} = \frac{6x^5}{\frac{-1}{2} x^{-\frac{3}{2}}}$$

$$\frac{du}{dv} = -12x^{5+\frac{3}{2}}$$

$$\frac{du}{dv} = -12x^{\frac{13}{2}}$$

Ans. $-12x^{13/2}$

2. Question

Differentiate $\log x$ with respect to $\cot x$.

Answer

Given : Let $u = \log x$ and $v = \cot x$

To differentiate : $\log x$ with respect to $\cot x$

Formula used : $\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

$$\frac{d(\log x)}{dx} = \frac{1}{x}$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

Let $u = \log x$ and $v = \cot x$

Differentiating u with respect to x

$$\frac{du}{dx} = \frac{1}{x}$$

Differentiating v with respect to x

$$\frac{dv}{dx} = -\operatorname{cosec}^2 x$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\frac{du}{dv} = \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x}$$

$$\frac{du}{dv} = \frac{-1}{x \operatorname{cosec}^2 x}$$

3. Question

Differentiate $e^{\sin x}$ with respect to $\cos x$.

Answer

Given : Let $u = e^{\sin x}$ and $v = \cos x$

To differentiate : $e^{\sin x}$ with respect to $\cos x$

Formula used : $\frac{d(e^x)}{dx} = e^x$

$$\frac{d(\cos x)}{dx} = -\sin x$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)).g'(x)$

Let $u = e^{\sin x}$ and $v = \cos x$

Differentiating u with respect to x

$$\frac{du}{dx} = \frac{d(e^{\sin x})}{dx} = \cos x \cdot e^{\sin x}$$

Differentiating v with respect to x

$$\frac{dv}{dx} = -\sin x$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\frac{du}{dv} = \frac{\cos x \cdot e^{\sin x}}{-\sin x}$$

$$\frac{du}{dv} = -e^{\sin x} \cdot \cot x$$

Ans. $-e^{\sin x} \cot x$

4. Question

Differentiate $\tan^{-1} \sqrt{\frac{1-x^2}{1+x^2}}$ with respect to $\cos^{-1} x^2$.

Answer

Given : Let $u = \tan^{-1} \sqrt{\frac{1-x^2}{1+x^2}}$ and $v = \cos^{-1} x^2$.

To differentiate : $\tan^{-1} \sqrt{\frac{1-x^2}{1+x^2}}$ with respect to $\cos^{-1} x^2$.

Formula used : $\frac{d(x^n)}{dx} = n \cdot x^{n-1}$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)).g'(x)$

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{vdu - u dv}{v^2}$$

$$\text{Let } u = \tan^{-1} \sqrt{\frac{1-x^2}{1+x^2}} \text{ and } v = \cos^{-1} x^2.$$

Differentiating u with respect to x

$$\frac{du}{dx} = \frac{d\left(\tan^{-1} \sqrt{\frac{1-x^2}{1+x^2}}\right)}{dx} = \frac{1}{1+\frac{1-x^2}{1+x^2}} \cdot \frac{d\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)}{dx}$$

$$\frac{du}{dx} = \frac{1+x^2}{1+x^2+1-x^2} \cdot \frac{1}{2} \left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}} \cdot \frac{-2x(1+x^2) - 2x(1-x^2)}{(1+x^2)^2}$$

$$\frac{du}{dx} = \frac{1+x^2}{2} \cdot \frac{1}{2} \left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}} \cdot \frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2} = \frac{1+x^2}{2} \cdot \frac{1}{2} \left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}} \cdot \frac{-4x}{(1+x^2)^2}$$

$$\frac{du}{dx} = \left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}} \cdot \frac{-x}{(1+x^2)} = \sqrt{\frac{1+x^2}{1-x^2}} \cdot \frac{-x}{(1+x^2)} = \frac{-x}{\sqrt{(1-x^2)(1+x^2)}} = \frac{-x}{\sqrt{1-x^4}}$$

$$\frac{du}{dx} = \frac{-x}{\sqrt{1-x^4}}$$

Differentiating v with respect to x

$$\frac{dv}{dx} = -\frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d(x^2)}{dx} = \frac{-2x}{\sqrt{1-x^4}}$$

$$\frac{dv}{dx} = \frac{-2x}{\sqrt{1-x^4}}$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\frac{du}{dv} = \frac{\frac{-x}{\sqrt{1-x^4}}}{\frac{-2x}{\sqrt{1-x^4}}} = \frac{1}{2}$$

$$\frac{du}{dv} = \frac{1}{2}$$

$$\text{Ans. } \frac{1}{2}$$

5. Question

Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Answer

Given : Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

To differentiate : $\tan^{-1}\frac{2x}{1-x^2}$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Formula used : $\frac{d(x^n)}{dx} = n \cdot x^{n-1}$

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{vdu - u dv}{v^2}$$

Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Differentiating u with respect to x

$$\frac{du}{dx} = \frac{d(\tan^{-1}\frac{2x}{1-x^2})}{dx} = \frac{1}{1+(\frac{2x}{1-x^2})^2} \cdot \frac{d(\frac{2x}{1-x^2})}{dx} = \frac{1}{1+\frac{4x^2}{1-x^2}} \cdot \frac{2(1-x^2)+2x(2x)}{(1-x^2)^2}$$

$$\frac{du}{dx} = \frac{(1-x^2)^2}{1+x^4-2x^2+4x^2} \cdot \frac{2-2x^2+4x^2}{(1-x^2)^2} = \frac{(1-x^2)^2}{1+x^4+2x^2} \cdot \frac{2+2x^2}{(1-x^2)^2} = \frac{2(1+x^2)}{(1+x^2)^2} = \frac{2}{(1+x^2)}$$

$$\frac{du}{dx} = \frac{2}{(1+x^2)}$$

Differentiating v with respect to x

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\frac{2x}{1+x^2})^2}} \cdot \frac{d(\frac{2x}{1+x^2})}{dx} = \frac{1+x^2}{\sqrt{1+x^4+2x^2-4x^2}} \cdot \frac{2(1+x^2)-2x(2x)}{(1+x^2)^2}$$

$$\frac{dv}{dx} = \frac{1+x^2}{\sqrt{1+x^4-2x^2}} \cdot \frac{2+2x^2-4x^2}{(1+x^2)^2} = \frac{1+x^2}{\sqrt{(1-x^2)^2}} \cdot \frac{2-2x^2}{(1+x^2)^2} = \frac{1+x^2}{1-x^2} \cdot \frac{2(1-x^2)}{(1+x^2)^2} = \frac{2}{1+x^2}$$

$$\frac{dv}{dx} = \frac{2}{1+x^2}$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\frac{du}{dv} = \frac{\frac{2}{(1+x^2)}}{\frac{2}{(1+x^2)}} = 1$$

$$\frac{du}{dv} = 1$$

Ans. 1

6. Question

Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1}(2x^2 - 1)$.

Answer

Given : Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ and $v = \cos^{-1}(2x^2 - 1)$.

To differentiate : $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ with respect to $\cos^{-1}(2x^2 - 1)$

Formula used : $\frac{d(x^n)}{dx} = n \cdot x^{n-1}$

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{vdu - u dv}{v^2}$$

Let $u = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$ and $v = \cos^{-1}(2x^2 - 1)$

Differentiating u with respect to x

$$\frac{du}{dx} = \frac{d\left(\tan^{-1}\frac{x}{\sqrt{1-x^2}}\right)}{dx} = \frac{1}{1+\left(\frac{x}{\sqrt{1-x^2}}\right)^2} \cdot \frac{d\left(\frac{x}{\sqrt{1-x^2}}\right)}{dx} = \frac{1}{1+\frac{x^2}{1-x^2}} \cdot \frac{1(\sqrt{1-x^2}) + x\left(\frac{-2x}{2\sqrt{1-x^2}}\right)}{1-x^2}$$

$$\frac{du}{dx} = \frac{1-x^2}{1-x^2+x^2} \cdot \frac{\sqrt{1-x^2} - \frac{-x^2}{\sqrt{1-x^2}}}{1-x^2} = (1-x^2) \cdot \frac{\frac{1-x^2+x^2}{\sqrt{1-x^2}}}{(1-x^2)^2} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Differentiating v with respect to x

$$\frac{dv}{dx} = \frac{d[\cos^{-1}(2x^2-1)]}{dx} = \frac{-1}{\sqrt{1-(2x^2-1)^2}} \cdot \frac{d(2x^2-1)}{dx} = \frac{-1}{\sqrt{1-4x^4-1+4x^2}} \cdot 4x$$

$$\frac{dv}{dx} = \frac{-4x}{\sqrt{4x^2-4x^4}} = \frac{-4x}{2x\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

$$\frac{dv}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\frac{du}{dv} = \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{-2}{\sqrt{1-x^2}}} = \frac{-1}{2}$$

$$\frac{du}{dv} = \frac{-1}{2}$$

$$\text{Ans. } \frac{-1}{2}$$

7. Question

Differentiate $\sin^3 x$ with respect to $\cos^3 x$.

Answer

Given : Let $u = \sin^3 x$ and $v = \cos^3 x$

To differentiate : $\sin^3 x$ with respect to $\cos^3 x$

Formula used : $\frac{d(x^n)}{dx} = n \cdot x^{n-1}$

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\frac{d(\cos x)}{dx} = -\sin x$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

Let $u = \sin^3 x$ and $v = \cos^3 x$

Differentiating u with respect to x

$$\frac{du}{dx} = 3\sin^2 x \cdot \frac{d(\sin x)}{dx} = 3\sin^2 x \cos x$$

$$\frac{du}{dx} = 3\sin^2 x \cos x$$

Differentiating v with respect to x

$$\frac{dv}{dx} = 3\cos^2 x \cdot \frac{d(\cos x)}{dx} = -3\cos^2 x \sin x$$

$$\frac{dv}{dx} = -3\cos^2 x \sin x$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\frac{du}{dv} = \frac{3\sin^2 x \cos x}{-3\cos^2 x \sin x} = \frac{\sin x}{-\cos x} = -\tan x$$

$$\frac{du}{dv} = -\tan x$$

Ans. $-\tan x$

8. Question

Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

Answer

Given : Let $u = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and $v = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

To differentiate : $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

Formula used : $\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \tan 3\theta$

$$\frac{d(x^n)}{dx} = n \cdot x^{n-1}$$

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{vdu - u dv}{v^2}$$

$$\text{Let } u = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \text{ and } v = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right).$$

Differentiating u with respect to x

$$\frac{du}{dx} = \frac{d \cos^{-1} \frac{1-x^2}{1+x^2}}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{d\left(\frac{1-x^2}{1+x^2}\right)}{dx} = \frac{-(1+x^2)}{\sqrt{(1+x^2)^2-(1-x^2)^2}} \cdot \frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}$$

$$\frac{du}{dx} = \frac{-(1+x^2)}{\sqrt{1+x^4+2x^2-1-x^4+2x^2}} \cdot \frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2} = \frac{-(1+x^2)}{\sqrt{4x^2}} \cdot \frac{-4x}{(1+x^2)^2} = \frac{+2}{1+x^2}$$

$$\frac{du}{dx} = \frac{+2}{1+x^2}$$

$$\text{For } v = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right).$$

Let $x = \tan \theta$

$$\tan^{-1} \frac{3x-x^3}{1-3x^2} = \tan^{-1} \frac{3 \tan \theta - \tan^3 \theta}{1-3 \tan^2 \theta} = \tan^{-1} (\tan 3\theta) = 3\theta = 3 \tan^{-1} x$$

$$\tan^{-1} \frac{3x-x^3}{1-3x^2} = 3 \tan^{-1} x$$

Differentiating v with respect to x ,

$$\frac{dv}{dx} = \frac{d(3 \tan^{-1} x)}{dx} = \frac{3}{1+x^2}$$

$$\frac{dv}{dx} = \frac{3}{1+x^2}$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\frac{du}{dv} = \frac{\frac{+2}{1+x^2}}{\frac{3}{1+x^2}} = \frac{2}{3}$$

$$\frac{du}{dv} = \frac{2}{3}$$

$$\text{Ans. } \frac{2}{3}$$

9. Question

Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Answer

Given : Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

To differentiate : $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Formula used : $\frac{d(x^n)}{dx} = n \cdot x^{n-1}$

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Put $x = \cot \theta$ or $\theta = \cot^{-1} x$ in u

$$\tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = \tan^{-1} \frac{\sqrt{1+\cot^2 \theta}-1}{\cot \theta} = \tan^{-1} \frac{\operatorname{cosec} \theta - 1}{\cot \theta}$$

$$\tan^{-1} \frac{\operatorname{cosec} \theta - 1}{\cot \theta} = \tan^{-1} \frac{\frac{1}{\sin \theta} - 1}{\frac{1}{\cos \theta}} = \tan^{-1} \frac{\frac{1-\sin \theta}{\sin \theta}}{\frac{1}{\cos \theta}} = \tan^{-1} \frac{1-\sin \theta}{\frac{\sin \theta}{\cos \theta}}$$

$$\tan^{-1} \frac{1-\sin \theta}{\frac{\sin \theta}{\cos \theta}} = \tan^{-1} \frac{1-\sin \theta}{\cos \theta}$$

We know that $1 - \sin \theta = \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} - 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ and $\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}$

$$1 - \sin \theta = \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)^2$$

Substituting the above values in $\tan^{-1} \frac{1-\sin \theta}{\cos \theta}$, we get

$$\tan^{-1} \frac{1-\sin \theta}{\cos \theta} = \tan^{-1} \frac{(\cos \frac{\theta}{2} - \sin \frac{\theta}{2})^2}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} = \tan^{-1} \frac{(\cos \frac{\theta}{2} - \sin \frac{\theta}{2})^2}{(\cos \frac{\theta}{2} - \sin \frac{\theta}{2})(\cos \frac{\theta}{2} + \sin \frac{\theta}{2})}$$

$$\tan^{-1} \frac{1-\sin \theta}{\cos \theta} = \tan^{-1} \frac{(\cos \frac{\theta}{2} - \sin \frac{\theta}{2})}{(\cos \frac{\theta}{2} + \sin \frac{\theta}{2})}$$

Dividing by $\cos \frac{\theta}{2}$ on numerator and denominator, we get

$$\tan^{-1} \frac{(\cos \frac{\theta}{2} - \sin \frac{\theta}{2})}{(\cos \frac{\theta}{2} + \sin \frac{\theta}{2})} = \tan^{-1} \frac{1 - \tan \frac{\theta}{2}}{1 + \tan \frac{\theta}{2}} = \tan^{-1} \tan(\frac{\pi}{4} - \frac{\theta}{2}) = \frac{\pi}{4} - \frac{\theta}{2}$$

$$\tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = \frac{\pi}{4} - \frac{\theta}{2} = \frac{\pi}{4} - \frac{\cot^{-1} x}{2}$$

Differentiating u with respect to x

$$\frac{d(\tan^{-1} \frac{\sqrt{1+x^2}-1}{x})}{dx} = \frac{d(\frac{\pi}{4} - \frac{\cot^{-1} x}{2})}{dx} = \frac{1}{2(1+x^2)}$$

$$\frac{du}{dx} = \frac{1}{2(1+x^2)}$$

$$v = \sin^{-1} \frac{2x}{1+x^2}$$

Put $x = \tan \theta$

$$V = \sin^{-1} \frac{2x}{1+x^2} = \sin^{-1} \frac{2 \tan \theta}{1 + \tan^2 \theta} = \sin^{-1} \frac{2 \frac{\sin \theta}{\cos \theta}}{\sec^2 \theta} = \sin^{-1} \frac{2 \frac{\sin \theta}{\cos \theta}}{\frac{1}{\cos^2 \theta}} = \sin^{-1} (2 \sin \theta \cos \theta)$$

$$V = \sin^{-1} \frac{2x}{1+x^2} = \sin^{-1} (2 \sin \theta \cos \theta) = \sin^{-1} (\sin 2\theta) = 2\theta = 2 \tan^{-1} x$$

$$V = \sin^{-1} \frac{2x}{1+x^2} = 2 \tan^{-1} x$$

Differentiating v with respect to x

$$\frac{dv}{dx} = \frac{2}{1+x^2}$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\frac{du}{dv} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}} = \frac{1}{4}$$

$$\frac{du}{dv} = \frac{1}{4}$$

$$\text{Ans. } \frac{1}{4}$$

10. Question

Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$ when $x \neq 0$.

Answer

$$\text{Given : Let } u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \text{ and } v = \cos^{-1}(2x\sqrt{1-x^2})$$

$$\text{To differentiate : } \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \text{ with respect to } \cos^{-1}(2x\sqrt{1-x^2})$$

$$\text{Formula used : } \frac{d(x^n)}{dx} = n \cdot x^{n-1}$$

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

The CHAIN RULE states that the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$

$$\text{Let } u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \text{ and } v = \cos^{-1}(2x\sqrt{1-x^2})$$

Substitute $x = \cos\theta$ in u

$$u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right) = \tan^{-1}\left(\frac{\sqrt{\sin^2\theta}}{\cos\theta}\right)$$

$$u = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta$$

$$u = \theta = \cos^{-1} x$$

Differentiating u with respect to x

$$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

Substitute $x = \sin\theta$ in v ,

$$v = \cos^{-1}(2x\sqrt{1-x^2}) = \cos^{-1}(2\sin\theta\sqrt{1-\sin^2\theta}) = \cos^{-1}(2\sin\theta\sqrt{\cos^2\theta})$$

$$v = \cos^{-1}(2\sin\theta\sqrt{\cos^2\theta}) = \cos^{-1}(2\sin\theta \cdot \cos\theta) = \cos^{-1}(\sin 2\theta)$$

$$v = \cos^{-1}(\sin 2\theta) = \cos^{-1}(\cos[\frac{\pi}{2} - 2\theta]) = \frac{\pi}{2} - 2\theta$$

$$v = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\sin^{-1}x$$

$$v = \frac{\pi}{2} - 2\sin^{-1}x$$

Differentiating v with respect to x

$$\frac{dv}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$\frac{du}{dv} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{-2}{\sqrt{1-x^2}}} = \frac{1}{2}$$

$$\text{Ans. } -\frac{1}{2}$$

Exercise 10I

1. Question

Find $\frac{dy}{dx}$, when

$$x = at^2, y = 2at$$

Answer

Theorem: y and x are given in a different variable that is t . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$$\frac{dy}{dx} = \frac{d(2at)}{dt}$$

$$= 2a \dots (1)$$

$$\frac{dx}{dt} = \frac{d(at^2)}{dt}$$

$$= 2at \dots\dots(2)$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{2a}{2at}$$

$$= \frac{1}{t}$$

2. Question

Find $\frac{dy}{dx}$, when

$$x = a \cos \theta, y = b \sin \theta$$

Answer

y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\frac{dy}{d\theta} = \frac{db \sin \theta}{d\theta} \left(\frac{d \sin \theta}{d\theta} = \cos \theta \right)$$

$$= b \cos \theta. \dots\dots(1)$$

$$\frac{dx}{d\theta} = \frac{d(a \cos \theta)}{d\theta} \left(\frac{d \cos \theta}{d\theta} = -\sin \theta \right)$$

$$= -a \sin \theta \dots\dots(2)$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{b \cos \theta}{-a \sin \theta} \left(\frac{\cos \theta}{\sin \theta} = \cot \theta \right)$$

$$= \frac{-b \cot \theta}{a}.$$

3. Question

Find $\frac{dy}{dx}$, when

$$x = a \cos^2 \theta, y = b \sin^2 \theta$$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\frac{dy}{d\theta} = \frac{db \sin^2 \theta}{d\theta}$$

$$= b \times 2 \sin \theta \times \cos \theta \text{ (using the chain rule } \frac{d \sin^2 \theta}{d\theta} = 2 \sin \theta \times \frac{d \sin \theta}{d\theta} = 2 \sin \theta \times \cos \theta \text{)}$$

$$= 2b \sin \theta \cos \theta \text{(1)}$$

$$\frac{dx}{d\theta} = \frac{da \cos^2 \theta}{d\theta}$$

$$= a \times (2 \cos \theta) \times (-\sin \theta) \text{ (using chain rule } \frac{d \cos^2 \theta}{d\theta} = 2 \cos \theta \times \frac{d \cos \theta}{d\theta} = 2 \cos \theta \times (-\sin \theta) \text{)}$$

$$= -2a \sin \theta \cos \theta.$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{2b \sin \theta \cos \theta}{-2a \sin \theta \cos \theta}$$

$$= \frac{-b}{a}.$$

4. Question

Find $\frac{dy}{dx}$, when

$$x = a \cos^3 \theta, y = a \sin^3 \theta$$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\frac{dy}{d\theta} = \frac{da \sin^3 \theta}{d\theta}$$

$$= a \times 3 \sin^2 \theta \times \cos \theta \text{ (using the chain rule } \frac{d \sin^3 \theta}{d\theta} = 3 \sin^2 \theta \times \frac{d \sin \theta}{d\theta} = 3 \sin^2 \theta \times \cos \theta \text{)}$$

$$= 3a \sin^2 \theta \cos \theta \text{(1)}$$

$$\frac{dx}{d\theta} = \frac{da \cos^3 \theta}{d\theta}$$

$$= a \times (3 \cos^2 \theta) \times (-\sin \theta) \text{ (using chain rule } \frac{d \cos^3 \theta}{d\theta} = 3 \cos^2 \theta \times \frac{d \cos \theta}{d\theta} = 3 \cos^2 \theta \times (-\sin \theta) \text{)}$$

$$= -3a \sin \theta \cos^2 \theta.$$

Dividing (1) and (2), we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{3a \sin^2 \theta \cos \theta}{-3a \sin \theta \cos^2 \theta} \\ &= \frac{-\sin \theta}{\cos \theta} \\ &= -\tan \theta\end{aligned}$$

5. Question

Find $\frac{dy}{dx}$, when

$$x = a(1 - \cos \theta), y = a(\theta + \sin \theta)$$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{da(\theta + \sin \theta)}{d\theta} \\ &= a \times (1 + \cos \theta) \dots\dots\dots(1)\end{aligned}$$

$$\begin{aligned}\frac{dx}{d\theta} &= \frac{da(1 - \cos \theta)}{d\theta} \\ &= a \sin \theta. \dots\dots\dots(2)\end{aligned}$$

Dividing (1) and (2), we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{a(1 + \cos \theta)}{a \sin \theta} \\ &= \frac{1 + \cos \theta}{\sin \theta} \\ &= \frac{2 \cos^2(\theta/2)}{2 \sin(\theta/2) \cos(\theta/2)} \quad (1 + \cos \theta = 2 \cos^2 \theta/2 \text{ and } \sin \theta = 2 \sin(\theta/2) \cos(\theta/2)) \\ &= \cot(\theta/2)\end{aligned}$$

6. Question

Find $\frac{dy}{dx}$, when

$$x = a \log t, y = b \sin t$$

Answer

Theorem: y and x are given in a different variable that is t . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$$\frac{dy}{dt} = \frac{dbsint}{dt}$$

$$= bcost \dots\dots\dots(1)$$

$$\frac{dx}{dt} = \frac{d(a logt)}{dt}$$

$$= \frac{a}{t} \dots\dots\dots(2)$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{b cost}{a/t}$$

$$= \frac{bt cost}{a}.$$

7. Question

Find $\frac{dy}{dx}$, when

$$x = (\log t + \cos t), y = (e^t + \sin t)$$

Answer

Theorem: y and x are given in a different variable that is t . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$$\frac{dy}{dt} = \frac{d(e^t + \sin t)}{dt}$$

$$= e^t + \cos t \dots\dots\dots(1) \left(\frac{de^t}{dt} = e^t \right)$$

$$\frac{dx}{dt} = \frac{d(\log t + \cos t)}{dt}$$

$$= \frac{1}{t} - \sin t. \dots\dots\dots(2) \left(\frac{d \log t}{dt} = \frac{1}{t} \right)$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{e^t + \cos t}{\frac{1}{t} - \sin t}$$

$$= \frac{t(e^t + \cos t)}{1 - t \sin t}.$$

8. Question

Find $\frac{dy}{dx}$, when

$$x = \cos \theta + \cos 2\theta, y = \sin \theta + \sin 2\theta$$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\frac{dy}{d\theta} = \frac{d(\sin\theta + \sin 2\theta)}{d\theta}$$

$$= \cos\theta + \cos 2\theta \times 2 \dots\dots\dots(1) \text{ (using chain rule } \frac{d \sin 2\theta}{d\theta} = \cos 2\theta \times \frac{d 2\theta}{d\theta} \text{)}$$

$$\frac{dx}{d\theta} = \frac{d(\cos\theta + \cos 2\theta)}{d\theta}$$

$$= -\sin\theta - 2\sin 2\theta \dots\dots\dots(2) \text{ (using chain rule } \frac{d \cos 2\theta}{d\theta} = \sin 2\theta \times \frac{d 2\theta}{d\theta} \text{)}$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{\cos\theta + 2\cos 2\theta}{-(\sin\theta + 2\sin 2\theta)}$$

9. Question

Find $\frac{dy}{dx}$, when

$$x = \sqrt{\sin 2\theta}, y = \sqrt{\cos 2\theta}$$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\frac{dx}{d\theta} = \frac{d\sqrt{\sin 2\theta}}{d\theta}$$

$$= \frac{2\cos 2\theta}{2\sqrt{\sin 2\theta}} \text{ (using chain rule } \frac{d \sqrt{\sin 2\theta}}{d\theta} = \frac{1}{2\sqrt{\sin 2\theta}} \times \frac{d \sin 2\theta}{d\theta} \text{)}$$

$$\frac{dx}{d\theta} = \frac{\cos 2\theta}{\sqrt{\sin 2\theta}} \dots\dots\dots(1)$$

$$\frac{dy}{d\theta} = \frac{d(\sqrt{\cos 2\theta})}{d\theta}$$

$$= \frac{-2\sin 2\theta}{2\sqrt{\cos 2\theta}} \text{ (using chain rule } \frac{d \sqrt{\cos 2\theta}}{d\theta} = \frac{1}{2\sqrt{\cos 2\theta}} \times \frac{d \cos 2\theta}{d\theta} \text{)}$$

$$= \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}} \dots\dots(2)$$

Dividing (2) and (2), we get

$$\frac{dy}{dx} = - \frac{\sin 2\theta / \sqrt{\cos 2\theta}}{\cos 2\theta / \sqrt{\sin 2\theta}}$$

$$= - \frac{\sqrt{\sin^3 2\theta}}{\sqrt{\cos^3 2\theta}}$$

$$= -(\tan 2\theta)^{3/2}$$

10. Question

Find $\frac{dy}{dx}$, when

$$x = e^\theta (\sin \theta + \cos \theta), y = e^\theta (\sin \theta - \cos \theta)$$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\frac{dy}{d\theta} = \frac{d e^\theta (\sin \theta - \cos \theta)}{d\theta}$$

$$= e^\theta (\cos \theta + \sin \theta) + (\sin \theta - \cos \theta) e^\theta \dots\dots\dots(1) \text{ \{by using product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \}$$

$$\frac{dx}{d\theta} = \frac{d e^\theta (\sin \theta + \cos \theta)}{d\theta}$$

$$= e^\theta (\cos \theta - \sin \theta) + e^\theta (\sin \theta + \cos \theta) \dots\dots\dots(2) \text{ \{by using product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \}$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{e^\theta (2 \sin \theta)}{e^\theta (2 \cos \theta)}$$

$$= \tan \theta .$$

11 . Question

Find $\frac{dy}{dx}$, when

$$x = a (\cos \theta + \theta \sin \theta), y = a (\sin \theta - \theta \cos \theta)$$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\frac{dy}{d\theta} = \frac{d a(\sin\theta - \theta \cos\theta)}{d\theta}$$

$$= a(\cos\theta - (-\theta \sin\theta + \cos\theta)) \text{ \{by using product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \text{ while differentiating } \theta \cos\theta \}$$

$$= a(\theta \sin\theta) \dots\dots\dots(1)$$

$$\frac{dx}{d\theta} = \frac{d a(\cos\theta + \theta \sin\theta)}{d\theta}$$

$$= a(-\sin\theta + \theta \cos\theta + \sin\theta) \text{ \{by using product rule, } \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \text{ while differentiating } \theta \cos\theta \}$$

$$= a \times \theta \cos\theta \dots\dots\dots(2)$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{a \times \theta \sin\theta}{a \times \theta \cos\theta}$$

$$= \tan\theta \text{ ANS}$$

12. Question

Find $\frac{dy}{dx}$, when

$$x = \frac{3at}{(1+t^2)}, y = \frac{3at^2}{(1+t^2)}$$

Answer

Theorem: y and x are given in a different variable that is t . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$$\frac{dy}{dt} = \frac{d \frac{3at^2}{(1+t^2)}}{dt}$$

$$= \frac{(1+t^2)6at - 3at^2(2t)}{(1+t^2)^2} \text{ \{by using divide rule, } \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \}$$

$$= \frac{6at + 6at^3 - 6at^3}{(1+t^2)^2}$$

$$= \frac{6at}{(1+t^2)^2} \dots\dots\dots(1)$$

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d\left(\frac{3at}{1+t^2}\right)}{d\theta} \\
&= \frac{(1+t^2)3a - 3at(2t)}{(1+t^2)^2} \left\{ \text{by using divide rule, } \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right\} \\
&= \frac{3a + 3at^2 - 6at^2}{(1+t^2)^2} \\
&= \frac{3a - 3at^2}{(1+t^2)^2} \dots\dots\dots(2)
\end{aligned}$$

Dividing (1) and (2), we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{6at/(1+t^2)^2}{3a(1-t^2)/(1+t^2)^2} \\
&= \frac{2t}{(1-t^2)}
\end{aligned}$$

13. Question

Find $\frac{dy}{dx}$, when

$$x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$$

Answer

Theorem: y and x are given in a different variable that is t . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d\left(\frac{2t}{1+t^2}\right)}{dt} \\
&= \frac{(1+t^2)2 - 2t(2t)}{(1+t^2)^2} \left\{ \text{by using divide rule, } \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right\} \\
&= \frac{2 + 2t^2 - 4t^2}{(1+t^2)^2} \\
&= \frac{2 - 2t^2}{(1+t^2)^2} \dots\dots\dots(1)
\end{aligned}$$

$$\frac{dx}{dt} = \frac{d\left(\frac{1-t^2}{1+t^2}\right)}{d\theta}$$

$$\begin{aligned}
&= \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} \left\{ \text{by using divide rule, } \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right\} \\
&= \frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2} \\
&= \frac{-4t}{(1+t^2)^2} \dots\dots\dots(2)
\end{aligned}$$

Dividing (1) and (2), we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2-2t^2/(1+t^2)^2}{-4t/(1+t^2)^2} \\
&= \frac{t^2-1}{(2t)}
\end{aligned}$$

14. Question

Find $\frac{dy}{dx}$, when

$$x = \cos^{-1} \frac{1}{\sqrt{1+t^2}}, y = \sin^{-1} \frac{t}{\sqrt{1+t^2}}$$

Answer

Theorem: y and x are given in a different variable that is t . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

Let us assume $u = \frac{t}{\sqrt{1+t^2}}$

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d \sin^{-1}(u)}{dt} \\
&= \frac{1}{\sqrt{1-u^2}} \times \frac{du}{dt} \\
&= \frac{1}{\sqrt{1-u^2}} \times \frac{\sqrt{1+t^2} \times 1 - t(2t/2\sqrt{1+t^2})}{(\sqrt{1+t^2})^2} \left\{ \text{by using divide rule, } \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right\}
\end{aligned}$$

Putting value of u

$$\begin{aligned}
&= \frac{\sqrt{1+t^2}}{1} \times \frac{1}{(1+t^2)^{\frac{3}{2}}} \\
&= \frac{1}{1+t^2} \dots\dots\dots(1)
\end{aligned}$$

Let assume $v = \frac{1}{\sqrt{1+t^2}}$

$$\frac{dx}{dt} = \frac{d(\cos^{-1} v)}{dv} \times \frac{dv}{dt}$$

$$= \frac{-1}{\sqrt{1-v^2}} \times \left(\frac{-1}{(\sqrt{1+t^2})^2} \right) \times \frac{2t}{2\sqrt{1+t^2}} \left\{ \text{by using divide rule, } \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right\}$$

Putting value of v

$$= \frac{t\sqrt{1+t^2}}{t \times (1+t^2)^{\frac{3}{2}}}$$

$$= \frac{\sqrt{1+t^2}}{(1+t^2)^{\frac{3}{2}}}$$

$$= \frac{1}{(1+t^2)} \dots\dots\dots(2)$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{1}{1+t^2} \times \frac{(1+t^2)}{1}$$

$$= 1$$

15. Question

If $x = 2 \cos t - 2 \cos^3 t$, $y = \sin t - 2 \sin^3 t$, show that $\frac{dy}{dx} = \cot t$.

Answer

Theorem: y and x are given in a different variable that is t . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$$\frac{dy}{dt} = \frac{d(\sin t - 2 \sin^3 t)}{dt}$$

$$= \cos t - 6 \sin^2 t \times \cos t \dots\dots\dots(1) \text{ (using chain rule)}$$

$$\frac{dx}{dt} = \frac{d(2 \cos t - 2 \cos^3 t)}{dt}$$

$$= -2 \sin t + 6 \cos^2 t \times \sin t \dots\dots\dots(2) \text{ (using chain rule)}$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{\cos t (1 - 6 \sin^2 t)}{2 \sin t (3 \cos^2 t - 1)}$$

$$= \frac{t(e^t + \cos t)}{1 - t \sin t}$$

16. Question

$$\text{If } x = \frac{1 + \log t}{t^2} \text{ and } y = \frac{3 + 2 \log t}{t} \frac{dy}{dx} = t.$$

Answer

Theorem: y and x are given in a different variable that is t . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$$\begin{aligned} \frac{dy}{dt} &= \frac{d(3+2\log t)/t}{dt} \\ &= \frac{t\left(\frac{2}{t}\right) - (3+2\log t) \times 1}{t^2} \quad \left\{ \text{by using divide rule, } \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right\} \\ &= -\frac{1+2\log t}{t^2} \dots\dots\dots(1) \end{aligned}$$

$$\begin{aligned} \frac{dx}{dt} &= \frac{d(1 + \log t)/t^2}{dt} \\ &= \frac{t^2\left(\frac{1}{t}\right) - (2t+2t \log t)}{t^4} \quad \left\{ \text{by using divide rule, } \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right\} \\ &= -\frac{2\log t + 1}{t^3} \dots\dots\dots(2) \end{aligned}$$

Dividing (1) and (2), we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{-(1+2 \log t)/t^2}{-(1+2 \log t)/t^3} \\ &= t. \end{aligned}$$

17. Question

$$\text{If } x = a(\theta - \sin \theta), y = a(1 - \cos \theta), \text{ find } \frac{dy}{dx} \text{ at } \theta = \frac{\pi}{2}.$$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\begin{aligned} \frac{dy}{d\theta} &= \frac{d a(1 - \cos \theta)}{d\theta} \\ &= a \sin \theta \dots\dots\dots(1) \end{aligned}$$

$$\frac{dx}{d\theta} = \frac{d a(\theta - \sin \theta)}{d\theta}$$

$$= a(1 - \cos\theta) \dots\dots\dots(2)$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{a \sin\theta}{a \times (1 - \cos\theta)}$$

Putting $\theta = \pi/2$

$$= \frac{\sin(\pi/2)}{1 - \cos(\pi/2)}$$

$$= 1.$$

18. Question

If $x = 2 \cos\theta - \cos 2\theta$ and $y = 2 \sin\theta - \sin 2\theta$, show that $\frac{dy}{dx} = \tan \frac{3\theta}{2}$.

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\frac{dy}{d\theta} = \frac{d(2\sin\theta - \sin 2\theta)}{d\theta}$$

$$= 2\cos\theta - 2\cos 2\theta \dots\dots\dots(1)$$

$$\frac{dx}{d\theta} = \frac{d(2\cos\theta - \cos 2\theta)}{d\theta}$$

$$= -2\sin\theta + 2\sin 2\theta \dots\dots\dots(2)$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{2\cos\theta - 2\cos 2\theta}{2\sin 2\theta - 2\sin\theta}$$

$$= \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}$$

$$= \frac{\cos\theta - (2\cos^2\theta - 1)}{2\sin\theta\cos\theta - \sin\theta} \{ \sin 2t = 2\sin t \cos t \} \{ \cos 2t = 2\cos^2 t - 1 \}$$

By factorising numerator, we get

$$= \frac{(1 - \cos\theta)(\cos\theta + \frac{1}{2})}{2\sin\theta(\cos\theta - \frac{1}{2})}$$

$$= \frac{1 - \cos\theta}{2\sin\theta} \times \frac{\cos\theta + \frac{1}{2}}{\cos\theta - \frac{1}{2}} \left\{ \frac{1 - \cos\theta}{\sin\theta} = \tan\left(\frac{\theta}{2}\right) \right\}$$

$$= \frac{\tan\left(\frac{\theta}{2}\right)}{1} \times \frac{(2(1-\tan^2\left(\frac{\theta}{2}\right)) + (1+\tan^2\left(\frac{\theta}{2}\right)))}{2(1-\tan^2\left(\frac{\theta}{2}\right)) - (1+\tan^2\left(\frac{\theta}{2}\right))}$$

For simplicity let's take $\theta/2$ as x .

$$\begin{aligned} &= \frac{\tan x}{2} \times \frac{2-2\tan^2 x + 1 + \tan^2 x}{2-2\tan^2 x - 1 - \tan^2 x} \\ &= \frac{\tan x}{2} \times \frac{3-\tan^2 x}{1-3\tan^2 x} \\ &= \frac{3\tan x - \tan^3 x}{1-3\tan^2 x} \cdot \frac{3\tan x - \tan^3 x}{1-3\tan^2 x} = \tan 3x \\ &= \frac{\tan 3x}{2} \cdot x = \frac{\theta}{2} \\ &= \frac{\tan\left(\frac{3\theta}{2}\right)}{2}. \end{aligned}$$

19. Question

If $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$, find $\frac{dy}{dx}$.

Answer

Theorem: y and x are given in a different variable that is t . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$$\begin{aligned} \frac{dx}{dt} &= \frac{d\left(\frac{\sin^3 t}{\sqrt{\cos 2t}}\right)}{dt} \\ &= \frac{\sqrt{\cos 2t}(3\sin^2 t \times \cos t) - \sin^3 t \left(\frac{-\sin 2t}{\sqrt{\cos 2t}}\right)}{\cos 2t} \quad \left\{ \text{by using divide rule, } \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right\} \\ &= \frac{\cos 2t \times (3\sin^2 t \times \cos t) + \sin^3 t \times (2\sin t \cos t)}{(\cos 2t)^{\frac{3}{2}}} \quad \{ \sin 2t = 2\sin t \cos t \} \\ &= \frac{\sin^2 t \cos t (3\cos 2t + 2\sin^2 t)}{(\cos 2t)^{\frac{3}{2}}} \quad \{ \cos 2t = 1 - 2\sin^2 t \} \\ &= \frac{\sin^2 t \cos t (3 - 4\sin^2 t)}{(\cos 2t)^{\frac{3}{2}}} \\ &= \frac{\sin t \cos t (3\sin t - 4\sin^3 t)}{2(\cos 2t)^{\frac{3}{2}}} \quad \{ \sin 3t = 3\sin t - 4\sin^3 t \} \\ &= \frac{\sin 2t \times \sin 3t}{(\cos 2t)^{\frac{3}{2}}} \dots\dots\dots(1) \end{aligned}$$

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d \frac{\cos^3 t}{\sqrt{\cos 2t}}}{dv} \\
&= \frac{\sqrt{\cos 2t}(3 \cos^2 t \times (-\sin t) - \cos^3 t \left(\frac{-\sin 2t}{\sqrt{\cos 2t}}\right))}{\cos 2t} \left\{ \text{by using divide rule, } \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right\} \\
&= \frac{\cos 2t \times (-3 \cos^2 t \times \sin t) + \cos^3 t \times (2 \sin t \cos t)}{(\cos 2t)^{\frac{3}{2}}} \{ \sin 2t = 2 \sin t \cos t \} \\
&= \frac{\cos^2 t \sin t (-3 \cos 2t + 2 \cos^2 t)}{(\cos 2t)^{\frac{3}{2}}} \{ \cos 2t = 2 \cos^2 t - 1 \} \\
&= \frac{\cos^2 t \sin t (3 - 4 \cos^2 t)}{(\cos 2t)^{\frac{3}{2}}} \\
&= \frac{\sin t \cos t (3 \cos t - 4 \cos^3 t)}{(\cos 2t)^{\frac{3}{2}}} \{ \cos 3t = 4 \cos^3 t - 3 \cos t \} \\
&= -\frac{\sin 2t \times \cos 3t}{2(\cos 2t)^{\frac{3}{2}}} \dots\dots\dots(1)
\end{aligned}$$

Dividing (1) and (2), we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{\sin 2t \times \cos 3t}{2(\cos 2t)^{\frac{3}{2}}}}{\frac{\sin 2t \times \sin 3t}{(\cos 2t)^{\frac{3}{2}}}} \\
&= -\cot 3t
\end{aligned}$$

20. Question

If $x = (2 \cos \theta - \cos 2\theta)$ and $y = (2 \sin \theta - \sin 2\theta)$, find $\left(\frac{d^2 y}{dx^2} \right)_{\theta = \frac{\pi}{2}}$.

Answer

here we have to find the double derivative, so to find double derivative we will just differentiate the first derivative once again with a similar method.

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\begin{aligned}
\frac{dy}{d\theta} &= \frac{d(2 \sin \theta - \sin 2\theta)}{d\theta} \\
&= 2 \cos \theta - 2 \cos 2\theta \dots\dots\dots(1)
\end{aligned}$$

$$\frac{dx}{d\theta} = \frac{d(2\cos\theta - \cos 2\theta)}{d\theta}$$

$$= -2\sin\theta + 2\sin 2\theta \dots\dots\dots(2)$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}$$

$$= \tan\left(\frac{3\theta}{2}\right) \text{ \{as shown in question no. 18\}}$$

$$\text{Let } \frac{dy}{dx} = f'$$

$$\frac{d^2 y}{dx^2} = f''$$

⇒ To find f'' we will differentiate f' with θ and then divide with equation (2).

$$\frac{d\frac{dy}{dx}}{d\theta} = \frac{d\tan\left(\frac{3\theta}{2}\right)}{d\theta}$$

$$= \frac{\sec^2\left(\frac{3\theta}{2}\right)}{1} \times \frac{3}{2}$$

Now divide by equation (2).

$$\frac{d^2 y}{dx^2} = \frac{3\sec^2\left(\frac{3\theta}{2}\right)}{4} \times \frac{1}{(\sin 2\theta - \sin\theta)}$$

Putting $\theta = \pi/2$

$$\frac{d^2 y}{dx^2} = \frac{3}{4} \times (-2)$$

$$= -\frac{3}{2}.$$

21. Question

If $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$, find $\frac{d^2 y}{dx^2}$.

Answer

here we have to find the double derivative, so to find double derivative we will just differentiate the first derivative once again with a similar method.

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$$\frac{dy}{d\theta} = \frac{d a(1+\cos\theta)}{d\theta}$$

$$= -a\sin\theta \dots\dots\dots(1)$$

$$\frac{dx}{d\theta} = \frac{d a(\theta - \sin\theta)}{d\theta}$$

$$= a(1-\cos\theta) \dots\dots\dots(2)$$

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{-a\sin\theta}{a \times (1-\cos\theta)}$$

$$= \frac{-2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{2\sin^2\frac{\theta}{2}} \{ \sin 2t = 2\sin t \cos t \} \{ \cos 2t = 1 - 2\sin^2 t \}$$

$$= -\cot(\theta/2)$$

⇒ To find f'' we will differentiate f' with θ and then divide with equation (2).

$$\frac{d\frac{dy}{dx}}{d\theta} = \frac{\operatorname{cosec}^2(\frac{\theta}{2})}{2} \times \frac{1}{a(1-\cos\theta)}$$

$$= \frac{-1}{2a\sin^2(\frac{\theta}{2}) \times (2\sin^2(\frac{\theta}{2}))} \left\{ 1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right) \right\} \left\{ \operatorname{cosec}^2\theta = \frac{1}{\sin^2\theta} \right\}$$

$$= \frac{1}{4a} \operatorname{cosec}^4\left(\frac{\theta}{2}\right).$$

Exercise 10J

1. Question

Find the second derivate of :

(i) x^{11} (ii) 5^x

(iii) $\tan x$ (iv) $\cos^{-1}x$

Answer

(i) x^{11}

Differentiating with respect to x

$$f'(x) = 11x^{11-1}$$

$$f'(x) = 11x^{10}$$

Differentiating with respect to x

$$f''(x) = 110x^{10-1}$$

$$f''(x) = 110x^9$$

$$(ii) 5^x$$

Differentiating with respect to x

$$f'(x) = 5^x \log_e 5 \quad [\text{Formula: } a^x = a^x \log_e a]$$

Differentiating with respect to x

$$f''(x) = \log_e 5 \cdot 5^x \log_e 5$$

$$= 5^x (\log_e 5)^2$$

$$(iii) \tan x$$

Differentiating with respect to x

$$f'(x) = \sec^2 x$$

Differentiating with respect to x

$$f''(x) = 2 \sec x \cdot \sec x \tan x$$

$$= 2 \sec^2 x \tan x$$

$$(iv) \cos^{-1} x$$

Differentiating with respect to x

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

Differentiating with respect to x

$$f''(x) = \frac{-1}{2} \times \frac{-1}{(1-x^2)^{\frac{3}{2}}} \times -2x$$

$$= \frac{-x}{(1-x^2)^{\frac{3}{2}}}$$

2. Question

Find the second derivative of:

$$(i) x \sin x$$

$$(ii) e^{2x} \cos 3x$$

$$(iii) x^3 \log x$$

Answer

Differentiating with respect to x

$$f'(x) = \sin x + x \cos x$$

Differentiating with respect to x

$$f''(x) = \cos x + \cos x - x \sin x$$

$$= -\sin x + 2\cos x$$

(ii) $e^{2x} \cos 3x$

Differentiating with respect to x

$$f'(x) = 2e^{2x}\cos 3x + e^{2x}(-\sin 3x).3$$

$$= 2e^{2x}\cos 3x - 3e^{2x}\sin 3x$$

Differentiating with respect to x

$$f''(x) = 2.2e^{2x}\cos 3x + 2e^{2x}(-\sin 3x).3 - 3.2e^{2x}\sin 3x - 3e^{2x}\cos 3x.3$$

$$= 4e^{2x}\cos 3x - 6e^{2x}\sin 3x - 6e^{2x}\sin 3x - 9e^{2x}\cos 3x$$

$$= -12e^{2x}\sin 3x - 5e^{2x}\cos 3x$$

(iii) $x^3 \log x$

Differentiating with respect to x

$$f'(x) = 3x^2 \log x + \frac{x^3}{x}$$

$$f'(x) = 3x^2 \log x + x^2$$

Differentiating with respect to x

$$f''(x) = 6x \log x + \frac{3x^2}{x} + 2x$$

$$= 6x \log x + 3x + 2x$$

$$= 6x \log x + 5x$$

3. Question

If $y = x + \tan x$, show that $\cos^2 x \cdot \frac{d^2 y}{dx^2} - 2y + 2x = 0$.

Answer

$$y = x + \tan x, \Rightarrow \tan x = y - x \dots (i)$$

Differentiating with respect to x

$$\frac{dy}{dx} = 1 + \sec^2 x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = 2 \sec x \cdot \sec x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \sec^2 x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2 \tan x}{\cos^2 x}$$

$$\Rightarrow \cos^2 x \frac{d^2y}{dx^2} = 2 \tan x \text{ [putting value of } \tan x \text{ from (i)]}$$

$$\Rightarrow \cos^2 x \frac{d^2y}{dx^2} = 2y - 2x$$

$$\Rightarrow \cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$$

4. Question

If $y = 2 \sin x + 3 \cos x$, show that $y + \frac{d^2y}{dx^2} = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = 2 \cos x - 3 \sin x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = -2 \sin x - 3 \cos x$$

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

Hence Proved

5. Question

If $y = 3 \cos (\log x) + 4 \sin (\log x)$, prove that $x^2 y_2 + x y_1 + y = 0$.

Answer

Differentiating with respect to x

$$y_1 = -3 \sin(\log x) \frac{1}{x} + 4 \cos(\log x) \frac{1}{x}$$

$$\Rightarrow y_1 = \frac{-3\sin(\log x) + 4\cos(\log x)}{x} \quad [\text{we can also write this as } xy_1 = -3\sin(\log x) + 4\cos(\log x)]$$

Differentiating with respect to x

$$y_2 = \frac{x\left(-3\cos(\log x)\frac{1}{x} - 4\sin(\log x)\frac{1}{x}\right) - (-3\sin(\log x) + 4\cos(\log x))}{x^2}$$

$$\Rightarrow x^2 y_2 = \frac{-x}{x} (3\cos(\log x) - 4\sin(\log x)) - (y_1 x)$$

$$\Rightarrow x^2 y_2 = -y - xy_1$$

$$\Rightarrow x^2 y_2 + xy_1 + y = 0$$

Hence Proved

6. Question

If $y = e^{-x} \cos x$, show that $\frac{d^2 y}{dx^2} = 2e^{-x} \sin x$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = -e^{-x} \cos x + e^{-x}(-\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -e^{-x} \cos x - e^{-x} \sin x$$

$$\Rightarrow \frac{dy}{dx} = -e^{-x}(\cos x + \sin x)$$

Differentiating with respect to x

$$\frac{d^2 y}{dx^2} = e^{-x}(\cos x + \sin x) - e^{-x}(-\sin x + \cos x)$$

$$\Rightarrow \frac{d^2 y}{dx^2} = e^{-x}(\cos x + \sin x - (-\sin x) - \cos x)$$

$$\Rightarrow \frac{d^2 y}{dx^2} = e^{-x}(\sin x + \sin x)$$

$$\Rightarrow \frac{d^2 y}{dx^2} = 2e^{-x} \sin x$$

Hence proved

7. Question

If $y = \sec x - \tan x$. show that $(\cos x) \frac{d^2y}{dx^2} = y^2$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = \sec x \tan x - \sec^2 x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \sec x \tan x \times \tan x + \sec x \times \sec^2 x - 2 \sec x \times \sec x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec x \tan^2 x + \sec^3 x - 2 \sec^2 x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec x (\tan^2 x + \sec^2 x - 2 \sec x \tan x)$$

$$\Rightarrow \frac{1}{\sec x} \frac{d^2y}{dx^2} = (\sec x - \tan x)^2$$

$$\Rightarrow \cos x \frac{d^2y}{dx^2} = y^2$$

Hence Proved

8. Question

If $y = (\operatorname{cosec} x + \cot x)$, prove that $(\sin x) \frac{d^2y}{dx^2} - y^2 = 0$.

Answer

$$\frac{dy}{dx} = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \operatorname{cosec} x \cot^2 x + \operatorname{cosec}^3 x + 2 \operatorname{cosec} x \times \operatorname{cosec} x \cot x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \operatorname{cosec} x (\cot^2 x + \operatorname{cosec}^2 x + 2 \operatorname{cosec} x \cot x)$$

$$\Rightarrow \frac{1}{\operatorname{cosec} x} \frac{d^2y}{dx^2} = (\cot x + \operatorname{cosec} x)^2$$

$$\Rightarrow \sin x \frac{d^2y}{dx^2} = y^2$$

$$\Rightarrow \sin x \frac{d^2 y}{dx^2} - y^2 = 0$$

Hence proved

9. Question

If $y = \tan^{-1} x$, show that $(1 + x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$\Rightarrow (1 + x^2) \frac{dy}{dx} = 1$$

Differentiating with respect to x

$$(1 + x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = 0$$

Hence Proved

10. Question

If $y = \sin(\sin x)$, prove that $\frac{d^2 y}{dx^2} + (\tan x) \frac{dy}{dx} + y \cos^2 x = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = \cos(\sin x) \cos x$$

Differentiating with respect to x

$$\frac{d^2 y}{dx^2} = -\sin(\sin x) \cos x \cos x - \sin x \cos(\sin x)$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -y \cos^2 x - \sin x \frac{dy}{\cos x}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -y \cos^2 x - \tan x \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2 y}{dx^2} + y \cos^2 x + \tan x \frac{dy}{dx} = 0$$

Hence Proved

11. Question

If $y = a \cos (\log x) + b \sin (\log x)$, prove that $x^2 y_2 + x y_1 + y = 0$.

Answer

Differentiating with respect to x

$$y_1 = -a \sin(\log x) \frac{1}{x} \quad [\text{can also be written as } -x y_1 = a \sin(\log x)]$$

Differentiating with respect to x

$$y_2 = \frac{-x a \cos(\log x) \frac{1}{x} + a \sin(\log x)}{x^2}$$

$$\Rightarrow x^2 y_2 = -y - x y_1$$

$$\Rightarrow x^2 y_2 + x y_1 + y = 0$$

Hence Proved

12. Question

Find the second derivative of $e^{3x} \sin 4x$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = 3e^{3x} \sin 4x + 4e^{3x} \cos 4x$$

Differentiating with respect to x

$$\Rightarrow \frac{d^2 y}{dx^2} = 9e^{3x} \sin 4x + 12e^{3x} \cos 4x + 12e^{3x} \cos 4x - 16e^{3x} \sin 4x$$

$$\Rightarrow \frac{d^2 y}{dx^2} = 24e^{3x} \cos 4x - 7e^{3x} \sin x$$

$$\Rightarrow \frac{d^2 y}{dx^2} = e^{3x} (24 \cos x - 7 \sin x)$$

13. Question

Find the second derivative of $\sin 3x \cos 5x$.

Answer

$$y = \frac{1}{2} [\sin(5x + 3x) + \sin(5x - 3x)]$$

$$y = \frac{1}{2} \sin 8x + \frac{1}{2} \sin 2x$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{8}{2} \cos 8x + \frac{2}{2} \cos 2x$$

$$\Rightarrow \frac{dy}{dx} = 4 \cos 8x + \cos 2x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = -32 \sin 8x - 2 \sin 2x$$

Hence Proved

14. Question

If $y = e^{\tan x}$, prove that $(\cos^2 x) \frac{d^2y}{dx^2} - (1 + \sin 2x) \frac{dy}{dx} = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = \sec^2 x e^{\tan x}$$

$$\Rightarrow \frac{1}{\sec^2 x} \frac{dy}{dx} = e^{\tan x}$$

$$\Rightarrow \cos^2 x \frac{dy}{dx} = e^{\tan x}$$

Differentiating with respect to x

$$(\cos^2 x) \frac{d^2y}{dx^2} - (2 \cos x \sin x) \frac{dy}{dx} = \sec^2 x e^{\tan x}$$

$$\Rightarrow (\cos^2 x) \frac{d^2y}{dx^2} - \sin 2x \frac{dy}{dx} = \frac{dy}{dx}$$

$$\Rightarrow (\cos^2 x) \frac{d^2y}{dx^2} - \sin 2x \frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\Rightarrow (\cos^2 x) \frac{d^2y}{dx^2} - (\sin 2x + 1) \frac{dy}{dx} = 0$$

Hence Proved

15. Question

If $y = \frac{\log x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{(2 \log x - 3)}{x^3}$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{\frac{1}{x} \times x - \log x}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - \log x}{x^2}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{\frac{-1}{x} \times x^2 - 2x(1 - \log x)}{x^4}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x - 2x(1 - \log x)}{x^4}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1 - 2 + 2 \log x}{x^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(2 \log x - 3)}{x^3}$$

Hence proved

16. Question

If $y = e^{ax} \cos bx$, show that $\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = ae^{ax} \cos bx - be^{ax} \sin bx$$

$$be^{ax} \sin bx = ae^{ax} \cos bx - \frac{dy}{dx}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = a^2 e^{ax} \cos bx - abe^{ax} \sin bx - abe^{ax} \sin bx - b^2 e^{ax} \cos bx$$

$$\Rightarrow \frac{d^2y}{dx^2} = a^2 e^{ax} \cos bx - 2abe^{ax} \sin bx - b^2 e^{ax} \cos bx$$

$$\Rightarrow \frac{d^2y}{dx^2} = a^2 e^{ax} \cos bx - 2a \left(a e^{ax} \cos bx - \frac{dy}{dx} \right) - b^2 e^{ax} \cos bx$$

$$\Rightarrow \frac{d^2y}{dx^2} = a^2 e^{ax} \cos bx - 2a^2 e^{ax} \cos bx + 2a \frac{dy}{dx} - b^2 e^{ax} \cos bx \Rightarrow$$

$$\frac{d^2y}{dx^2} = -a^2 e^{ax} \cos bx - b^2 e^{ax} \cos bx + 2a \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(a^2 + b^2)(e^{ax} \cos bx) + 2a \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(a^2 + b^2)y + 2a \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$$

Hence Proved

17. Question

If $y = e^{a \cos^{-1}x}$, $-1 \leq x \leq 1$, show that $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$.

Answer

Taking log on both sides

$$\log y = a \cos^{-1}x \log e$$

$$\log y = a \cos^{-1}x$$

Differentiating with respect to x

$$\frac{1}{y} \frac{dy}{dx} = \frac{-a}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ae^{a \cos^{-1}x}}{\sqrt{1-x^2}}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{\frac{a^2 e^{a \cos^{-1}x}}{\sqrt{1-x^2}} \times \sqrt{1-x^2} - a e^{\cos^{-1}x} \times \frac{2x}{2\sqrt{1-x^2}}}{(1-x^2)}$$

$$\Rightarrow (1-x^2) \frac{d^2y}{dx^2} = a^2 e^{\cos^{-1}x} - \frac{ax e^{\cos^{-1}x}}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2) \frac{d^2y}{dx^2} = a^2 y + x \frac{dy}{dx}$$

$$\Rightarrow (1-x^2) \frac{d^2y}{dx^2} - a^2 y - x \frac{dy}{dx} = 0$$

Hence Proved

18. Question

If $x = at^2$ and $y = 2at$, find $\frac{d^2y}{dx^2}$ at $t = 2$.

Answer

Differentiating with t

$$\frac{dx}{dt} = 2at \quad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{-1}{t^2} \frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4} \times \frac{1}{2at}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4} \times \frac{1}{4a}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{16a}$$

19. Question

If $x = a(\theta - \sin \theta)$ and $y = a(1 - \cos \theta)$, find $\frac{d^2y}{dx^2}$ at $\theta = \pi$.

Answer

Differentiating with respect to θ

$$\frac{dx}{d\theta} = a(1 - \cos \theta) \quad \frac{dy}{d\theta} = a \sin \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{\cos \theta (1 - \cos \theta) - \sin^2 \theta}{(1 - \cos \theta)^2} \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cos \theta - \cos^2 \theta - \sin^2 \theta}{(1 - \cos \theta)^2} \times \frac{1}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cos \theta - 1}{(1 - \cos \theta)^2} \times \frac{1}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-(1 - \cos \theta)}{(1 - \cos \theta)^2} \times \frac{1}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{a(1 - \cos \theta)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{a(1 - (-1))^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4a}$$

20. Question

If $y = \sin(\log x)$, prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$.

Answer

Differentiating with respect to

$$\frac{dy}{dx} = \cos(\log x) \frac{1}{x}$$

$$\Rightarrow x \frac{dy}{dx} = \cos(\log x)$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{-\sin(\log x) \frac{1}{x} x - \cos(\log x)}{x^2}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = -\sin(\log x) - \cos(\log x)$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = -y - x \frac{dy}{dx}$$

$$\Rightarrow x^2 \frac{d^2 y}{dx^2} + y + x \frac{dy}{dx} = 0$$

Hence Proved

21. Question

If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that $(1-x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} - y = 0$.

Answer

$$\sqrt{1-x^2} y = \sin^{-1} x$$

Differentiating with respect to x

$$\sqrt{1-x^2} \frac{dy}{dx} - \frac{2xy}{2\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2) \frac{dy}{dx} - xy = 1$$

Differentiating with respect to x

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - x \frac{dy}{dx} - y = 0$$

$$\Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} - y = 0$$

Hence Proved

22. Question

If $y = e^x \sin x$, prove that $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$.

Answer

$$y = e^x \sin x$$

Differentiating with respect to x

$$\frac{dy}{dx} = e^x \sin x + e^x \cos x$$

$$\left[e^x \cos x = \frac{dy}{dx} - e^x \sin x \right]$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2e^x \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \frac{dy}{dx} - 2e^x \sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \frac{dy}{dx} - 2y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$$

23. Question

If $x = a \left(\cos \theta + \log \tan \frac{\theta}{2} \right)$ and $y = a \sin \theta$, show that the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$ is $\frac{2\sqrt{2}}{a}$.

Answer

$$\frac{dx}{d\theta} = a \left(-\sin \theta + \frac{\sec^2 \frac{\theta}{2}}{2 \tan \frac{\theta}{2}} \right) \frac{dy}{d\theta} = a \cos \theta$$

$$= a \left(-\sin \theta + \frac{1}{\sin \theta} \right)$$

$$= a \left(\frac{-\sin^2 \theta + 1}{\sin \theta} \right)$$

$$= \frac{a \cos^2 \theta}{\sin \theta}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\frac{dy}{dx} = a \cos \theta \times \frac{\sin \theta}{a \cos^2 \theta}$$

$$\frac{dy}{dx} = \tan \theta$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \sec^2 \theta \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = (\sqrt{2})^2 \times \frac{\sin \theta}{a \cos^2 \theta}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \times \frac{\frac{1}{\sqrt{2}}}{a\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2\sqrt{2}}{a}$$

24. Question

If $x = \cos t + \log \tan \frac{t}{2}$, $y = \sin t$ then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.

Answer

$$\frac{dx}{dt} = -\sin t + \frac{\sec^2 \frac{t}{2}}{2 \tan \frac{t}{2}} \frac{dy}{dt} = \cos t$$

$$= -\sin t + \frac{1}{\sin t}$$

$$= \frac{-\sin^2 t + 1}{\sin t}$$

$$= \frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = \cos t$$

Differentiating with respect to t

$$\Rightarrow \frac{d^2y}{dt^2} = -\sin t \text{ [Putting } t = \pi/4 \text{]}$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{1}{\sqrt{2}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \cos t \times \frac{\sin t}{\cos^2 t}$$

$$\Rightarrow \frac{dy}{dx} = \tan t$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \sec^2 t \frac{dt}{dx} \text{ [Putting } t = \pi/4 \text{]}$$

$$\Rightarrow \frac{d^2y}{dx^2} = (\sqrt{2}) \times \frac{\sin t}{\cos^2 t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \times \frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\sqrt{2}$$

25. Question

If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - \frac{y}{x} = 0$.

Answer

$$y = x^x$$

Taking log on both sides

$$\log y = x \log x$$

Differentiating with respect to x

$$\frac{1}{y} \frac{dy}{dx} = 1 + \log x \dots (i)$$

$$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{y}{x} + (1 + \log x) \frac{dy}{dx} \text{ [putting value of } (1 + \log x) \text{ from (i)]}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y} \left(\frac{dy}{dx} \right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{y}{x} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 = 0$$

Hence Proved

26. Question

If $y = (\cot^{-1} x)^2$, then show that $(x^2 + 1)^2 \frac{d^2y}{dx^2} + 2x(x^2 + 1) \frac{dy}{dx} = 2$.

Answer

$$y = (\cot^{-1} x)^2$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{-2 \cot^{-1} x}{1+x^2}$$

$$\Rightarrow -2 \cot^{-1} x = (1+x^2) \frac{dy}{dx}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{2+4x \cot^{-1} x}{(1+x^2)^2}$$

$$\Rightarrow (1+x^2)^2 \frac{d^2y}{dx^2} - 4x \cot^{-1} x = 2$$

$$\Rightarrow (1+x^2)^2 \frac{d^2y}{dx^2} - 2x \left(-(1+x^2) \frac{dy}{dx} \right) = 2$$

$$\Rightarrow (1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2) \frac{dy}{dx} = 2$$

Hence Proved

27. Question

If $y = \left\{ x + \sqrt{x^2 + 1} \right\}^m$, then show that $(x^2 + 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - m^2 y = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = m \left\{ x + \sqrt{x^2 + 1} \right\}^{m-1} \left(1 + \frac{2x}{2\sqrt{x^2 + 1}} \right)$$

$$\Rightarrow \frac{dy}{dx} = m \left\{ x + \sqrt{x^2 + 1} \right\}^{m-1} \left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right)$$

$$\Rightarrow \frac{dy}{dx} = m \frac{\left\{ x + \sqrt{x^2 + 1} \right\}^m}{\sqrt{x^2 + 1}}$$

$$\Rightarrow \frac{dy}{dx} = m \frac{y}{\sqrt{x^2 + 1}}$$

$$\left[\frac{dy}{dx} \sqrt{x^2 + 1} = my \right]$$

Differentiating with respect to x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{m \frac{dy}{dx} \sqrt{1+x^2} - \frac{2xmy}{2\sqrt{x^2+1}}}{(1+x^2)}$$

$$\Rightarrow (1 + x^2) \frac{d^2y}{dx^2} = m^2y - x \frac{dy}{dx}$$

$$\Rightarrow (1 + x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - m^2y = 0$$

Hence Proved

28. Question

If $y = \log \left[x + \sqrt{x^2 + a^2} \right]$, then prove that $(x^2 + a^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0$

Answer

$$\frac{dy}{dx} = \frac{1 + \frac{2x}{2\sqrt{x^2 + a^2}}}{x + \sqrt{x^2 + a^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sqrt{x^2 + a^2} + 2x}{2\sqrt{x^2 + a^2}} \times \frac{1}{x + \sqrt{x^2 + a^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a^2}}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{-2x}{2(x^2 + a^2)\sqrt{x^2 + a^2}}$$

$$\Rightarrow (x^2 + a^2) \frac{d^2y}{dx^2} = \frac{-x}{\sqrt{x^2 + a^2}}$$

$$\Rightarrow (x^2 + a^2) \frac{d^2y}{dx^2} = -x \frac{dy}{dx}$$

$$\Rightarrow (x^2 + a^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0$$

Hence Proved

29. Question

If $x = a(\cos \theta + \theta \sin \theta)$ and $y = a(\sin \theta - \theta \cos \theta)$, show that $\frac{d^2y}{dx^2} = \frac{1}{a} \left(\frac{\sec^3 \theta}{\theta} \right)$

Answer

Differentiating with respect to θ

$$\frac{dx}{d\theta} = a(-\sin \theta + \sin \theta + \theta \cos \theta) \quad \frac{dy}{d\theta} = a(\cos \theta - \cos \theta + \theta \sin \theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a\theta \cos \theta \Rightarrow \frac{dy}{d\theta} = a\theta \sin \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a\theta \sin \theta}{a\theta \cos \theta} = \frac{\sin \theta}{\cos \theta}$$

$$\Rightarrow \frac{dy}{dx} = \tan \theta$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \sec^2 \theta \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 \theta \times \frac{1}{a\theta \cos \theta}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 \theta \times \frac{\sec \theta}{a\theta}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^3 \theta}{a\theta}$$

Hence Proved

30. Question

If $x = a \cos \theta + b \sin \theta$ and $y = a \sin \theta - b \cos \theta$, show that $y^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0$.

Answer

$$\frac{dx}{d\theta} = -a \sin \theta + b \cos \theta \quad \frac{dy}{d\theta} = a \cos \theta + b \sin \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a \cos \theta + b \sin \theta}{-a \sin \theta + b \cos \theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

Differentiating with respect to x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y - x \frac{dy}{dx}}{y^2}$$

$$\Rightarrow y^2 \frac{d^2y}{dx^2} = y - x \frac{dy}{dx}$$

$$\Rightarrow y^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0$$

Hence Proved