PART I: Section M (Common for Both Streams)

(Answer any **TWO** questions)

1. (a) Find the number of ways in which an arrangement of four letters can be made from the letters of the word "PROMOTION".

   (b) Show that the sum of cubes of any three consecutive numbers is divisible by 9.

   (c) From 50 students taking examinations in mathematics, statistics and engineering, 37 passed in mathematics, 24 in statistics and 43 in engineering. At most 19 passed in mathematics and statistics, at most 29 in mathematics and engineering, and, at most 20 in statistics and engineering. Find the largest possible number of students that could have passed all the three examinations.

   [10 + 5 + 5 = 20]

2. (a) Find the sum of the following series up to (n+1) terms:

   $$2 + 3 \times \binom{n}{1} + 5 \times \binom{n}{2} + 9 \times \binom{n}{3} + \cdots\cdots .$$

   (b) If a, b, c are positive integers such that

   $$a^2 + b^2 - ab = c^2 ,$$

   then show that $(a - c)(b - c) \leq 0$.

   (c) Test for convergence of the series:

   $$\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}, \quad x \in \mathbb{R}^+.$$

   [7 + 8 + 5 = 20]

3. (a) Solve the differential equation

   $$(2x - y + 4)dy = (x - 2y + 5)dx,$$

   assuming that the curve passes through the point $(1, 2)$.

(b) A circle of radius *r* touches a straight line at a point M. Two points A and B are chosen on the line on opposite sides of M such that $\mathrm{MA} = \mathrm{MB} = a$. Find the radius of the circle passing through A and B and touching the given circle.

[10 + 10 = 20]

4. (a) A company wants to purchase 25 vehicles with a combined capacity of 28,000 cubic feet. Three different types of vehicles are available in the market: a 10-foot vehicle with a capacity of 350 cubic feet, a 12-foot vehicle with a capacity of 700 cubic feet, and an 18-foot vehicle with a capacity of 1,400 cubic feet. How many of each type of vehicle should the company purchase?

(b) Consider a triangle *ABC* with *a, b, c* as opposite sides of the angles *A, B, C* respectively. Suppose *p, q, r* are the lengths of the internal bisectors of the angles *A, B, C* respectively. Find the value of

$$\frac{1}{p}\mathrm{Cos}\frac{A}{2}+\frac{1}{q}\mathrm{Cos}\frac{B}{2}+\frac{1}{r}\mathrm{Cos}\frac{C}{2}$$

in terms of the sides *a, b, c*.

[10 + 10 = 20]

PART II: Section S (Statistics Stream)

(Answer any **FOUR** questions)

5. (a) Suppose that two integers are chosen randomly from 11 to 99. Find the probability that their difference is divisible by 3.

   (b) It is estimated that 50% of the emails are spam. A software can detect 99% spam and the probability of false positive (i.e., a non-spam email is detected as spam) is 5%. If an email is detected as spam, what is the probability that it is in fact a non-spam?

[10 + 10 = 20]

6. (a) Suppose that accidents on a highway can be modelled by Poisson process with a rate of 35 per week, 20% of which being fatal and 80% being non-fatal. If the total number of accidents is 50 during a 5-day period, what is the probability that exactly 10 of them are fatal?

   (b) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the limiting distribution of $\sqrt{n}(\bar{X}_n - 1)$ as $n \to \infty$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

[8 + 12 = 20]

7. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the normal population with mean 0 and variance σ^2. Examine whether $\frac{\sum_{i=1}^{n} X_i^2}{n}$ is MVUE of σ^2.

(b) An engineer wishes to compare three test methods T_1, T_2 and T_3 to examine the strength of three different materials in accordance with a randomized block design of experiment such that each material acts like a block, with single replicate.

i) Write down the layout of the experiment.

ii) Write down the associated linear model stating the assumptions, if any.

iii) State the procedure to test the hypothesis for comparing T_1, T_2 and T_3.

[10 + (2+2+6) = 20]

8. Suppose X and Y are independent and identically distributed $N(0,1)$. Let $Z = X + \sqrt{\theta}Y,\ \ \theta > 0$.

(a) Find the joint distribution of (X, Z).

(b) Let (X_1, Z_1), (X_2, Z_2), …, (X_n, Z_n) be a random sample of size n from the joint distribution of (X, Z). Find the maximum likelihood estimator (MLE) of θ. Find mean and variance of the MLE.

[8 + (5+2+5) = 20]

9. (a) Let $X_{11}, X_{12}, \ldots, X_{1n_1}$ and $X_{21}, X_{22}, \ldots, X_{2n_2}$ be independent random samples from two continuous distributions F_1 and F_2 respectively. The observations from both distributions are combined and ranked (average rank in case of tie). Let R_1, R_2, …, R_{n_1} denote the ranks corresponding to the first sample. Find $\mathrm{E}\left(\sum_{i=1}^{n_1} R_i\right)$ and $\mathrm{Var}\left(\sum_{i=1}^{n_1} R_i\right)$.

(b) Consider a finite population of N units. Suppose a random sample of n units is selected from the population by SRSWOR.

Show that the probability of drawing a specified unit at the rth $(1 < r \leq n)$ draw is same as the probability of drawing it at the first draw.

[(4+6) + 10 = 20]

10. (a) In a university, entrance test is conducted for the four postgraduate programmes A, B, C and D. In this year, 10000 males and 9000 females have applied for admission, combining all these four programmes. There is a strong feeling everywhere that the entrance process is biased towards males. The admission data for the four postgraduate programmes is given below:

Programme	Males		Females	
	Applied	Admitted	Applied	Admitted
A	4000	2500	1000	700
B	4000	2000	2000	1000
C	1000	250	3000	1000
D	1000	250	3000	900
Total	10000	5000	9000	3600

As a statistician, your job is to test whether there exists any bias towards males candidates.

i) Write down the null hypothesis and alternative hypothesis you want to test.

ii) Define the test procedure, along with the test statistic (numerical calculations are not needed).

(b) Consider a population with probability density function $f(x, \theta) = \theta e^{-\theta x},\ x > 0,\ \theta > 0$. Let X be a single observation from the population. If $X > 1$ is the critical region for testing H_0: $\theta = 2$ against H_1: $\theta = 1$, find the size and power of the test.

[(4+8) + (4+4) = 20]

PART II: Section E (Engineering Stream)

(Answer any **FOUR** questions)

5. (a) A train of weight W starts from rest and is accelerated with a force equal to F_a. It keeps on accelerating till such a point of its journey from where a retarding force equal to F_r will bring it to a stop at a point d units away from its starting point. Show that the velocity of the train when it commences its retardation is

$$\sqrt{\frac{2dg}{W\left(\frac{F_a + F_r}{F_a \times F_r}\right)}}.$$

(b) Two smooth spheres each of weight W and radius r are in equilibrium in a horizontal channel of width $b(< 4r)$, as shown below in Figure 1.

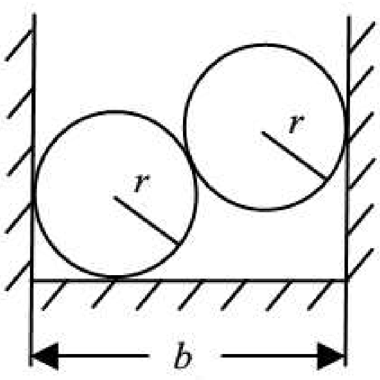

Figure 1

Find the three reactions from the sides of the channel, which are all smooth and the force exerted by the spheres on each other.

$[10 + 10 = 20]$

6. (a) Draw the PV cycle diagram for an Otto cycle.

Using the PV cycle diagram and considering the working substance as an ideal gas, show that the thermal efficiency of an engine operating in an idealized Otto cycle is

$$\eta = 1 - r^{1-\gamma},$$

where r is the compression ratio and γ is the molar specific heat ratio.

(b) Air is contained in a vertical piston-cylinder assembly fitted with an electrical resistor. The atmosphere exerts a pressure of 1 bar on the top of the piston, which has a mass of 45 kg and a face area of 0.09 m^2. Electric current passes through the resistor, and the volume of the air slowly increases by 0.045 m^3 while its pressure remains constant. The mass of the air in the cylinder is 0.27 kg, and its specific internal energy increases by 42 kJ/kg. Consider the friction between the piston and the cylinder wall to be negligible, and the value of acceleration due to gravity as $g = 10$ m/s^2.

Determine the heat transfer from the resistor in kJ for a system consisting of (i) the air alone (ii) the air and the piston.

[(2+8) + (6+4) = 20]

7. (a) A gas contained in a cylinder, fitted with a frictionless piston, expands against a constant external pressure of 1 atm from a volume of 5 litres to a volume of 10 litres. In doing so, it absorbs 400 J of thermal energy from its surroundings. Determine the change in internal energy of the system in Joule.

(b) Sketch the device energy interaction and determine if a tray of ice cubes could remain frozen when placed in a food freezer having a COP of 9, operating in a room where the temperature is 32^0C.

(c) An apparatus that liquefies helium is in a laboratory at 296 K. The helium in the apparatus is at 4 K. If 150 mJ of heat is transferred from the helium, find the minimum amount of heat delivered to the laboratory.

(d) From a circular disc of radius R and mass $9M$, a small disc of radius $\frac{R}{3}$, in contact with the periphery of the circular disc of radius R, is removed. Find the moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through the center of the disc.

[4 + (2+3) + 3 + 8 = 20]

8. (a) Use nodal analysis method to find currents in the various resistors of the circuit shown below in Figure 2.

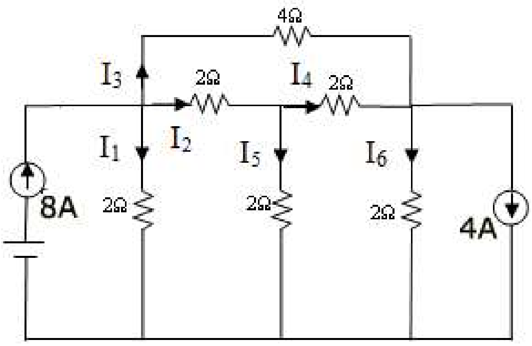

Figure 2

(b) A 6-pole DC motor is lap-wound with 600 conductors. The armature diameter is 20 cm and the pole shoe is 15 cm. The motor is drawing 50 A and running at 1000 rpm. The torque developed is 60 Nm. Find the average flux density per pole.

[12 + 8 = 20]

9. (a) Derive the expression for closed-loop gain in terms of reverse transmission factor β and transfer gain A for a negative feedback amplifier.

(b) An amplifier has a voltage gain of -500. This gain is changed to -100 when negative feedback is applied. Determine the reverse transmission factor β.

(c) Consider the circuit shown in Figure 3. When $R_2 = 10\ \Omega$ and $R_1 = R_3 = R_4 = R_5 = 1\ \text{k}\Omega$, find the differential mode gain $V_0/(V_1 - V_2)$.

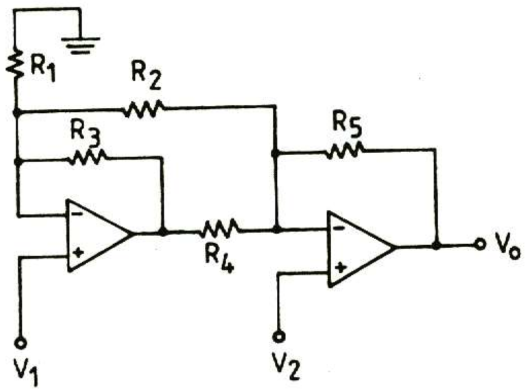

Figure 3

[6 + 4 + 10 = 20]

10. (a) A right hexagonal pyramid of 25 mm base and 50 mm height is placed with its axis making 30^0 with the horizontal plane and parallel to the vertical plane. Draw the front and top views of the pyramid.

(b) Draw the isometric view of a cylinder with 50 mm diameter and 100 mm height.

[14 + 6 = 20]