CSS

GS-2017

(Computer & Systems Sciences)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **COMPUTER & SYSTEMS SCIENCES - December 11, 2016**
Duration : Three hours (3 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine readable. Use only Black/Blue ball point pen to fill-in the answer sheet.

2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. Do not mark more than one circle for any question : this will be treated as a wrong answer.

3. This question paper consists of three (3) parts. **Part-A** contains fifteen (15) questions and **must be attempted** by all candidates. **Part-B & Part-C** contain fifteen (15) questions each, directed towards candidates for (B) Computer Science and (C) Systems Science (including Communcations and Applied Probability), respectively. **STUDENTS MAY ATTEMPT EITHER PART-B OR PART-C. In case, a student attempts both Parts B & C (no extra time will be given) and qualifies for interview in both B & C, he/she will have opportunity to be interviewed in both areas.** All questions carry equal marks. A correct answer for a question will give you +4 marks, a wrong answer will give you -1 mark, and a question not answered will not get you any marks.

4. We advise you to first mark the correct answers in the QUESTION SHEET and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators, mobile phones, laptops, tablets (or other electronic devices, including those connecting to the internet) is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

Part A: Common Part

1. A suitcase weighs one kilogram plus half of its weight. How much does the suitcase weigh?

(a) 1.333... kilograms

(b) 1.5 kilograms

(c) 1.666... kilograms

(d) 2 kilograms

(e) cannot be determined from the given data

2. For vectors x, y in $\mathbb{R}^n$, define the inner product $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, and the length of x to be $\|x\| = \sqrt{\langle x, x\rangle}$. Let a, b be two vectors in $\mathbb{R}^n$ so that $\|b\| = 1$. Consider the following statements:

(i) $\langle a, b\rangle \leq \|b\|$

(ii) $\langle a, b\rangle \leq \|a\|$

(iii) $\langle a, b\rangle = \|a\|\|b\|$

(iv) $\langle a, b\rangle \geq \|b\|$

(v) $\langle a, b\rangle \geq \|a\|$

Which of the above statements must be TRUE of a, b? Choose from the following options.

(a) (ii) only

(b) (i) and (ii)

(c) (iii) only

(d) (iv) only

(e) (iv) and (v)

3. On planet TIFR, the acceleration of an object due to gravity is half that on planet earth. An object on planet earth dropped from a height h takes time t to reach the ground. On planet TIFR, how much time would an object dropped from height h take to reach the ground?

(a) $t/\sqrt{2}$

(b) $\sqrt{2}t$

(c) $2t$

(d) h/t

(e) $h/2t$

4. Which of the following functions asymptotically grows the fastest as n goes to infinity?

(a) $(\log \log n)!$

✓ (b) $(\log \log n)^{\log n}$

(c) $(\log \log n)^{\log \log \log n}$

(d) $(\log n)^{\log \log n}$

(e) $2^{\sqrt{\log \log n}}$

5. How many distinct ways are there to split 50 identical coins among three people so that each person gets at least 5 coins?

(a) 3^{35}

(b) $3^{50} - 2^{50}$

(c) $\binom{35}{2}$

(d) $\binom{50}{15} \cdot 3^{35}$

✓ (e) $\binom{37}{2}$

6. How many distinct words can be formed by permuting the letters of the word ABRACADABRA?

✓ (a) $\dfrac{11!}{5!\,2!\,2!}$

(b) $\dfrac{11!}{5!\,4!}$

(c) $11!\,5!\,2!\,2!$

(d) $11!\,5!\,4!$

(e) $11!$

7. Consider the sequence $S_0, S_1, S_2, \ldots$ defined as follows: $S_0 = 0, S_1 = 1$, and $S_n = 2S_{n-1} + S_{n-2}$ for $n \geq 2$. Which of the following statements is FALSE?

(a) for every $n \geq 1$, S_{2n} is even

(b) for every $n \geq 1$, S_{2n+1} is odd

✓ (c) for every $n \geq 1$, S_{3n} is a multiple of 3

(d) for every $n \geq 1$, S_{4n} is a multiple of 6

(e) none of the above

8. In a tutorial on geometrical constructions, the teacher asks a student to construct a right-angled triangle ABC where the hypotenuse BC is 8 inches and the length of the perpendicular dropped from A onto the hypotenuse is h inches, and offers various choices for the value of h. For which value of h can such a triangle NOT exist?

(a) 3.90 inches

(b) $2\sqrt{2}$ inches

(c) $2\sqrt{3}$ inches

✔ (d) 4.1 inches

(e) none of the above

9. Consider the *majority* function on three bits, $\mathsf{maj} : \{0,1\}^3 \to \{0,1\}$, where $\mathsf{maj}(x_1, x_2, x_3) = 1$ if and only if $x_1+x_2+x_3 \geq 2$. Let $p(\alpha)$ be the probability that the output is 1 when each input is set to 1 independently with probability α. What is $p'(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha}p(\alpha)$?

(a) 3α

(b) α^2

✔ (c) $6\alpha(1-\alpha)$

(d) $3\alpha^2(1-\alpha)$

(e) $6\alpha(1-\alpha)+\alpha^2$

10. For a set A, define $\mathcal{P}(A)$ to be the set of all subsets of A. For example, if $A = \{1,2\}$, then $\mathcal{P}(A) = \{\emptyset, \{1,2\}, \{1\}, \{2\}\}$. Let $f : A \to \mathcal{P}(A)$ be a function and A is not empty. Which of the following must be TRUE?

(a) f cannot be one-to-one (injective)

✔ (b) f cannot be onto (surjective)

(c) f is both one-to-one and onto (bijective)

(d) there is no such f possible

(e) if such a function f exists, then A is infinite

11. Let $f \circ g$ denote function composition such that $(f \circ g)(x) = f(g(x))$. Let $f : A \to B$ such that for all $g : B \to A$ and $h : B \to A$ we have $f \circ g = f \circ h \Rightarrow g = h$. Which of the following must be TRUE?

(a) f is onto (surjective)

✔ (b) f is one-to-one (injective)

(c) f is both one-to-one and onto (bijective)

(d) the range of f is finite

(e) the domain of f is finite

12. Consider the following program modifying an $n \times n$ square matrix A:

```
for i = 1 to n:
    for j = 1 to n:
        temp = A[i][j] + 10
        A[i][j] = A[j][i]
        A[j][i] = temp - 10
    end for
end for
```

Which of the following statements about the contents of the matrix `A` at the end of this program must be TRUE ?

(a) the new `A` is the transpose of the old `A`

(b) all elements above the diagonal have their values increased by 10 and all values below have their values decreased by 10

(c) all elements above the diagonal have their values decreased by 10 and all values below have their values increased by 10

(d) the new matrix A is symmetric, that is, `A[i][j] = A[j][i]` for all $1 \leq i, j \leq n$

✔ (e) `A` remains unchanged

13. A set of points $S \subseteq \mathbb{R}^2$ is convex if for any points x, $y \in S$, every point on the straight line joining x and y is also in S. For two sets of points $S, T \subset \mathbb{R}^2$, define the sum $S+T$ as the set of points obtained by adding a point in S to a point in T. That is, $S+T := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = y_1 + z_1,\ x_2 = y_2 + z_2,\ \ (y_1, y_2) \in S,\ (z_1, z_2) \in T\}$. Similarly, $S-T := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = y_1 - z_1,\ x_2 = y_2 - z_2,\ \ (y_1, y_2) \in S,\ (z_1, z_2) \in T\}$ is the set of points obtained by subtracting a point in T from a point in S. Which of the following statements is TRUE for all convex sets S, T?

(a) $S+T$ is convex, but not $S-T$

(b) $S-T$ is convex, but not $S+T$

(c) exactly one of $S+T$ and $S-T$ is convex, but it depends on S and T which one

(d) neither $S+T$ nor $S-T$ is convex

✔ (e) both $S+T$ and $S-T$ are convex

14. Consider the following game with two players, Aditi and Bharat. There are n tokens in a bag. The players know n, and take turns removing tokens from the bag. In each turn, a player can either remove one token or two tokens. The player that removes the last token from the bag loses. Assume that Aditi always goes first. Further, we say that a player has a winning strategy if she or he can win the game, no matter what the other player does. Which of the following statements is TRUE?

(a) For $n = 3$, Bharat has a winning strategy. For $n = 4$, Aditi has a winning strategy

✓ (b) For $n = 7$, Bharat has a winning strategy. For $n = 8$, Aditi has a winning strategy

(c) For both $n = 3$ and $n = 4$, Aditi has a winning strategy

(d) For both $n = 7$ and $n = 8$, Bharat has a winning strategy

(e) Bharat never has a winning strategy

15. Let $T(a, b)$ be the function with two arguments (both nonnegative integral powers of 2) defined by the following recurrence:

$$
\begin{aligned}
T(a,b) &= T\left(\frac{a}{2}, b\right) + T\left(a, \frac{b}{2}\right) && \text{if } a, b \geq 2;\\
T(a,1) &= T\left(\frac{a}{2}, 1\right) && \text{if } a \geq 2;\\
T(1,b) &= T\left(1, \frac{b}{2}\right) && \text{if } b \geq 2;\\
T(1,1) &= 1.
\end{aligned}
$$

What is $T(2^r, 2^s)$?

(a) rs

(b) $r + s$

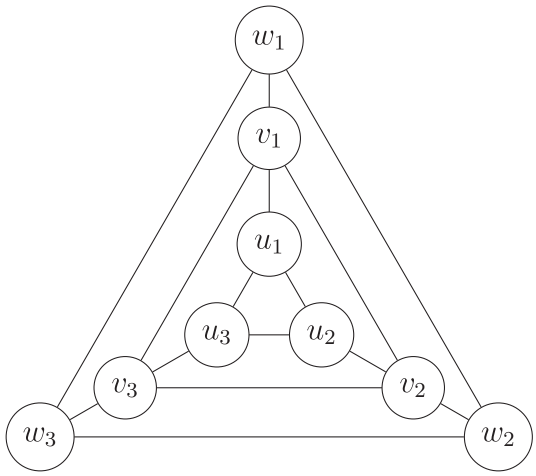
(c) $\binom{2^r + 2^s}{2^r}$

✓ (d) $\binom{r+s}{r}$

(e) 2^{r-s} if $r \geq s$, otherwise 2^{s-r}

Part B: Computer Science

1. A vertex colouring with three colours of a graph $G = (V, E)$ is a mapping $c : V \to \{R, G, B\}$ so that adjacent vertices receive distinct colours. Consider the following undirected graph.



How many vertex colourings with three colours does this graph have?

(a) 3^9

(b) 6^3

(c) 3×2^8

(d) 27

✔ (e) 24

2. Consider the following statements:

(i) Checking if a given *undirected* graph has a cycle is in P.

(ii) Checking if a given *undirected* graph has a cycle is in NP.

(iii) Checking if a given *directed* graph has a cycle is in P.

(iv) Checking if a given *directed* graph has a cycle is in NP.

Which of the above statements is/are TRUE? Choose from the following options.

(a) Only (i) and (ii)

(b) Only (ii) and (iv)

(c) Only (ii), (iii), and (iv)

(d) Only (i), (ii), and (iv)

✔ (e) All of them

3. We have an implementation that supports the following operations on a stack (in the instructions below, `s` is the name of the stack).

`isempty(s)`: returns `True` if s is empty, and `False` otherwise .

`top(s)`: returns the top element of the stack, but does not pop the stack; returns `null` if the stack is empty.

`push(s,x)`: places `x` on top of the stack.

`pop(s)`: pops the stack; does nothing if `s` is empty.

Consider the following code:

```
pop_ray_pop(x):
    s = empty
    for i = 1 to length(x):
        if (x[i] == ’(’):
            push(s,x[i])
        else:
            while (top(s) == ’(’):
                pop(s)
            end while
            push(s, ’)’)
        end if
    end for

    while not isempty(s):
        print top(s)
        pop(s)
    end while
```

What is the output of this program when

```
pop_ray_pop("((()((())((((")
```

is executed ?

(a) `((((`

(b) `)))((((`

(c) `)))`

(d) `((()))`

(e) `()()`

4. Let L be the language over the alphabet $\{1, 2, 3, (,)\}$ generated by the following grammar (with start symbol S, and non-terminals $\{A, B, C\}$):

$$\begin{aligned} S &\rightarrow A\,B\,C \\ A &\rightarrow (\\ B &\rightarrow 1\,B \mid 2\,B \mid 3\,B \\ B &\rightarrow 1 \mid 2 \mid 3 \\ C &\rightarrow) \end{aligned}$$

Then, which of the following is TRUE?

(a) L is finite

(b) L is not recursively enumerable

(c) L is regular

(d) L contains only strings of even length

(e) L is context-free but not regular

5. Consider the following pseudocode fragment, where y is an integer that has been initialized.

```
int i = 1
int j = 1
while (i < 10):
    j = j * i
    i = i + 1
    if (i==y):
        break
    end if
end while
```

Consider the following statements:

(i) $(i == 10)$ or $(i == y)$

(ii) If $y > 10$, then $i == 10$

(iii) If $j = 6$, then $y == 4$

Which of the above statements is/are TRUE at the end of the while loop? Choose from the following options.

(a) (i) only

(b) (iii) only

(c) (ii) and (iii) only

(d) (i), (ii), and (iii)

(e) None of the above

6. Consider First Order Logic (FOL) with equality and suitable function and relation symbols. Which one of the following is FALSE?

✓ (a) Partial orders cannot be axiomatized in FOL

(b) FOL has a complete proof system

(c) Natural numbers cannot be axiomatized in FOL

(d) Real numbers cannot be axiomatized in FOL

(e) Rational numbers cannot be axiomatized in FOL

7. An array of n distinct elements is said to be un-sorted if for every index i such that $2 \leq i \leq n-1$, either $A[i] > \max\{A[i-1], A[i+1]\}$, or $A[i] < \min\{A[i-1], A[i+1]\}$. What is the time-complexity of the fastest algorithm that takes as input a sorted array A with n distinct elements, and un-sorts A?

(a) $O(n \log n)$ but not $O(n)$

✓ (b) $O(n)$ but not $O(\sqrt{n})$

(c) $O(\sqrt{n})$ but not $O(\log n)$

(d) $O(\log n)$ but not $O(1)$

(e) $O(1)$

8. For any natural number n, an ordering of all binary strings of length n is a Gray code if it starts with 0^n, and any successive strings in the ordering differ in exactly one bit (the first and last string must also differ by one bit). Thus, for $n = 3$, the ordering (000, 100, 101, 111, 110, 010, 011, 001) is a Gray code. Which of the following must be TRUE for all Gray codes over strings of length n?

✓ (a) the number of possible Gray codes is even

(b) the number of possible Gray codes is odd

(c) in any Gray code, if two strings are separated by k other strings in the ordering, then they must differ in exactly $k+1$ bits

(d) in any Gray code, if two strings are separated by k other strings in the ordering, then they must differ in exactly k bits

(e) none of the above

9. Which of the following regular expressions correctly accepts the set of all 0/1-strings with an even (possibly zero) number of 1s?

(a) $(10^*10^*)^*$

(b) $(0^*10^*1)^*$

(c) $0^*1(10^*1)^*10^*$

(d) $0^*1(0^*10^*10^*)^*10^*$

✓ (e) $(0^*10^*1)^*0^*$

10. A vertex colouring of a graph $G = (V, E)$ with k colours is a mapping $c : V \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for every $(u, v) \in E$. Consider the following statements:

(i) If every vertex in G has degree at most d, then G admits a vertex colouring using $d + 1$ colours.

(ii) Every cycle admits a vertex colouring using 2 colours.

(iii) Every tree admits a vertex colouring using 2 colours.

Which of the above statements is/are TRUE? Choose from the following options.

(a) only (i)

(b) only (i) and (ii)

(c) only (i) and (iii)

(d) only (ii) and (iii)

(e) (i), (ii), and (iii)

11. Given that
$B(x)$ means "x is a bat",
$F(x)$ means "x is a fly", and
$E(x, y)$ means "x eats y",
what is the best English translation of

$$\forall x(F(x) \to \forall y(E(y, x) \to B(y)))?$$

(a) all flies eat bats

(b) every fly is eaten by some bat

(c) bats eat only flies

(d) every bat eats flies

(e) only bats eat flies

12. An undirected graph is complete if there is an edge between every pair of vertices. Given a complete undirected graph on n vertices, in how many ways can you choose a direction for the edges so that there are no directed cycles?

(a) n

(b) $\frac{n(n-1)}{2}$

(c) $n!$

(d) 2^n

(e) 2^m, where $m = \frac{n(n-1)}{2}$

13. For an undirected graph $G = (V, E)$, the line graph $G' = (V', E')$ is obtained by replacing each edge in E by a vertex, and adding an edge between two vertices in V' if the corresponding edges in G are incident on the same vertex. Which of the following is TRUE of line graphs?

(a) the line graph for a complete graph is complete

(b) the line graph for a connected graph is connected

(c) the line graph for a bipartite graph is bipartite

(d) the maximum degree of any vertex in the line graph is at most the maximum degree in the original graph

(e) each vertex in the line graph has degree one or two

14. Consider the following grammar G with terminals $\{[,]\}$, start symbol S, and non-terminals $\{A, B, C\}$:

$$\begin{aligned} S &\rightarrow AC \mid SS \mid AB \\ C &\rightarrow SB \\ A &\rightarrow [\\ B &\rightarrow] \end{aligned}$$

A language L is called prefix-closed if for every $x \in L$, every prefix of x is also in L. Which of the following is FALSE?

(a) $L(G)$ is context free

(b) $L(G)$ is infinite

(c) $L(G)$ can be recognized by a deterministic push down automaton

(d) $L(G)$ is prefix-closed

(e) $L(G)$ is recursive.

15. A multivariate polynomial in n variables with integer coefficients has a binary root if it is possible to assign each variable either 0 or 1, so that the polynomial evaluates to 0. For example, the multivariate polynomial $-2x_1^3 - x_1x_2 + 2$ has the binary root $(x_1 = 1,\, x_2 = 0)$. Then determining whether a multivariate polynomial, given as the sum of monomials, has a binary root:

(a) is trivial: every polynomial has a binary root

(b) can be done in polynomial time

(c) is NP-hard, but not in NP

(d) is in NP, but not in P and not NP-hard

(e) is both in NP and NP-hard

Part C: Systems Science

1. Consider a system which in response to input $x(t)$ outputs

$$y(t) = 2x(t-2) + x(2t-1) + 1.$$

Which of the following describes the system?

(a) linear, time-invariant, causal

(b) linear, time-invariant, non-causal

(c) non-linear, time-invariant, causal

(d) non-linear, time-invariant, non-causal

✔ (e) non-linear, time-variant

2. Suppose a 1µH inductor and a 1Ω resistor are connected in series to a 1V battery. What happens to the current in the circuit?

✔ (a) The current starts at 0A, and gradually rises to 1A

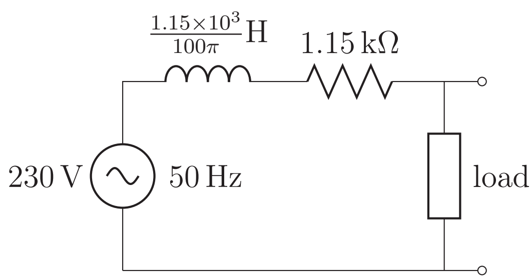
(b) The current rises instantaneously to 1A and stays there after that

(c) The current rises instantaneously to 1A and then falls to 0A

(d) The current oscillates over time between 0A and 1A

(e) The current oscillates over time between 1A and -1A

3. What is the maximum average power that can be dissipated by a load connected to the output terminals of the following circuit with an alternating current source?



(a) 23 W

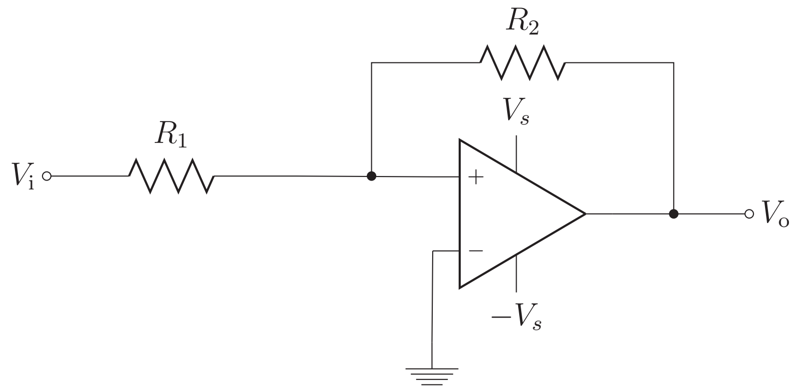
✔ (b) 11.5 W

(c) 8.1317 W

(d) 2.875 W

(e) None of the above

4. A Schmitt trigger circuit is a comparator circuit with a hysteresis. Consider the Schmitt trigger circuit in the figure implemented using an opamp. What are the trigger levels for this circuit?



(a) $\pm\dfrac{R_1}{R_2}V_s$

(b) $\pm\dfrac{R_2}{R_1}V_s$

(c) $\pm\dfrac{R_1}{R_1+R_2}V_s$

(d) $\pm\dfrac{R_2}{R_1+R_2}V_s$

(e) None of the above

5. Consider the inequality

$$n - \frac{1}{n} \geq \sqrt{n^2 - 1},$$

where n is an integer ≥ 1. Which of the following statements is TRUE?

(a) This inequality holds for all integers $n \geq 1$

(b) This inequality holds for all but finitely many integers $n \geq 1$

(c) This inequality holds for only finitely many integers $n \geq 1$

(d) This inequality does *not* hold for any integer $n \geq 1$

(e) $n - \frac{1}{n} = \sqrt{n^2 - 1}$ for infinitely many integers $n \geq 1$

6. Let $a, b \in \{0, 1\}$. Consider the following statements where $*$ is the AND operator, $\oplus$ is EXCLUSIVE-OR, and $^{\mathsf{c}}$ denotes the complement function.

(i) $\max\{a * b, b \oplus a^{\mathsf{c}}\} = 1$

(ii) $\max\{a \oplus b, b \oplus a^{\mathsf{c}}\} = 1$

(iii) $\min\{a * b, b * a^{\mathsf{c}}\} = 0$

(iv) $\min\{a \oplus b, b \oplus a^{\mathsf{c}}\} = 1$

Which of the above statements is/are always TRUE? Choose from the following options.

(a) (i) and (ii) only

✔ (b) (ii) and (iii) only

(c) (iii) and (iv) only

(d) (iv) and (i) only

(e) None of the above

7. A circulant matrix is a square matrix whose each row is the preceding row rotated to the right by one element, e.g., the following is a 3×3 circulant matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

For any $n \times n$ circulant matrix ($n > 5$), which of the following n-length vectors is always an eigenvector?

(a) A vector whose k-th element is k

(b) A vector whose k-th element is n^k

✔ (c) A vector whose k-th element is $\exp\left(j\dfrac{2\pi(n-5)k}{n}\right)$ where $j = \sqrt{-1}$

(d) A vector whose k-th element is $\sinh\left(\dfrac{2\pi k}{n}\right)$

(e) None of the above

8. Consider the two positive integer sequences, defined for a fixed positive integer $c \geq 2$

$$f(n) = \frac{1}{n}\left\lfloor \frac{n}{c} \right\rfloor, \quad g(n) = n\left\lfloor \frac{c}{n} \right\rfloor,$$

where $\lfloor t \rfloor$ denotes the largest integer with value at most t. Which of the following statements is TRUE as $n \to \infty$?

(a) Both sequences converge to zero

(b) The first sequence does not converge, while the second sequence converges to 0

(c) The first sequence converges to zero, while the second sequence does not converge

✓ (d) The first sequence converges to $1/c$, while the second sequence converges to 0

(e) The first sequence converges to $1/c$, while the second sequence converges to c

9. Recall that for a random variable X which takes values in $\mathbb{N}$, the set of natural numbers, its entropy in bits is defined as

$$H(X) = \sum_{n=1}^{\infty} p_n \log_2 \frac{1}{p_n},$$

where, for $n \in \mathbb{N}$, p_n denotes the probability that $X = n$. Now, consider a fair coin which is tossed repeatedly until a heads is observed. Let X be the random variable which indicates the number of tosses made. What is the entropy of X in bits?

(a) 1

(b) 1.5

(c) $\frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden ratio)

✓ (d) 2

(e) None of the above

10. Consider a single coin where the probability of heads is $p \in (0, 1)$ and probability of tails is $1 - p$. Suppose that this coin is flipped an infinite number of times. Let N_1 denote the number of flips till we see heads for the first time. Let N_2 denote the number of flips after the first N_1 flips, until a tails is observed for the first time (on flips observed after the first N_1 flips). What is the expected value of $N_1 + N_2$?

✓ (a) $\frac{1}{1-p} + \frac{1}{p}$

(b) $\frac{1-p}{p} + \frac{p}{1-p}$

(c) $\frac{2}{p}$

(d) $\frac{1}{p^2 + (1-p)^2}$

(e) $\frac{2}{p(1-p)}$

11. Consider a unit length interval $[0, 1]$ and choose a point X on it with uniform distribution. Let $L_1 = X$ and $L_2 = 1 - X$ be the length of the two sub-intervals created by this point on the unit interval. Let $L = \max\{L_1, L_2\}$. Consider the following statements where $\mathbf{E}$ denotes expectation.

(i) $\mathbf{E}[L] = 3/4$

(ii) $\mathbf{E}[L] = 2/3$

(iii) L is uniformly distributed over $[1/2, 1]$

(iv) L is uniformly distributed over $[1/3, 1]$

Which of the above statements is/are TRUE? Choose from the following options.

(a) Only (i)

(b) Only (ii)

✓ (c) Only (i) and (iii)

(d) Only (ii) and (iv)

(e) None of the above

12. Consider a signal X that can take two values, -1 with probability p and $+1$ with probability $1 - p$. Let $Y = X + N$, where N is mean zero random noise that has probability density function symmetric about 0. Given p and on observing Y, the detection problem is to decide on a value for X from -1 and $+1$. Let $\hat{X}$ denote the decision, then *error* is said to happen if $\hat{X}$ is not the true X. Consider the following statements about the optimal detector that minimizes the probability of error.

(i) If $p = 1/2$, then choosing $\hat{X} = +1$ if $Y > 0$ and $\hat{X} = -1$ if $Y < 0$ minimizes the probability of error.

(ii) The probability of error of the optimal detector for $p = 1/3$ is larger in comparison to the probability of error of the optimal detector for $p = 1/2$.

(iii) If $p = 0$, then choosing $\hat{X} = +1$ for any Y minimizes the probability of error.

Which of the above statements is/are TRUE? Choose from the following options.

(a) Only (i)

(b) Only (ii)

(c) Only (iii)

(d) Only (i) and (ii)

✓ (e) Only (i) and (iii)

13. Let A be an $n \times n$ matrix. Consider the following statements.

(i) A can have full-rank even if there exists two vectors $v_1 \neq v_2$ such that $Av_1 = Av_2$.

(ii) A can be similar to the identity matrix, when A is not the identity matrix. Recall that two matrices B and C are said to be similar if $B = S^{-1}CS$ for some matrix S.

(iii) If λ is an eigenvalue of A, then $\exists$ a vector $x \neq 0$ such that $(A - \lambda I)x = 0$.

Which of the above statements is/are TRUE? Choose from the following options.

(a) Only (i)

(b) Only (ii)

✓ (c) Only (iii)

(d) (i), (ii), and (iii)

(e) None of the above

14. Consider the positive integer sequence

$$x_n = n^{50} e^{-(\log(n))^{3/2}}, \quad n = 1, 2, 3, \ldots$$

Which of the following statements is TRUE?

(a) For every $M > 0$, there exists an n such that $x_n > M$

(b) Sequence $\{x_n\}$ first increases and then decreases to 1 as $n \to \infty$

(c) Sequence $\{x_n\}$ first decreases and then increases with $n \geq 1$

✓ (d) Sequence $\{x_n\}$ eventually converges to zero as $n \to \infty$

(e) None of the above

15. Suppose that $f(x)$ is a real valued continuous function such that $f(x) \to \infty$ as $x \to \infty$. Further, let

$$a_n = \sum_{j=1}^{n} 1/j$$

and

$$b_n = \sum_{j=1}^{n} 1/j^2.$$

Which of the following statements is true? (Hint: Consider the convergence properties of the sequences $\{a_n\}$ and $\{b_n\}$.)

(a) There exists a number M and a positive integer n_0 so that $f(a_n) \leq M$ and $f(b_n) \leq M$ for all $n \geq n_0$

✓ (b) There exists a number M and a positive integer n_0 so that $f(a_n) \geq M$ and $f(b_n) \leq M$ for all $n \geq n_0$

(c) There exists a number M and a positive integer n_0 so that $f(a_n) - f(b_n) \leq M$ for all $n \geq n_0$

(d) There does not exist any number M so that $f(b_n)$ and $f(a_n)$ are greater than M for all n

(e) None of the above