311

Intermediate (TOSS) Course
Senior Secondary Course

MATHEMATICS
2

(CORE MODULES)

Designed and developed

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A Word With You

Dear Learner,

Congratulations! for successfully completing Book-1.

In this book also you will be introduced to three core modules namely. **Functions, Calculus and Statistics.**

Every branch of science and engineering offers abundant examples of variable quantitites that are interrelated so that the value of one quantity determines the value of another variable. A relationship of this kind between two variables is called a functional relationship. The fourth module of **Functions** has been designed to highlight this concept in detail. Various trigonometric functions have also been discussed in this module.

The fifth module will introduce you to a very important branch of mathematics viz. **Calulus.** It has wide applications in Natural and Social Science, Engineering, Commerce, Economics to name a few areas. Calculus can be broadly divided into two parts, namely Differential Calculus and Integral Calculus.

In Differential Calculus, we will deal with limit, continuity and derivability of functions. We have tried to explain these concepts using graphical illustrations. You will be introduced to the derivatives of algebraic, trigonometric, exponential and logarithmic functions. Then we shall take up the applications of derivatives in the problems of tangents and normals, and maximizing a given function.

Lastly, you will learn about integral calculus. The definition of the integral calculus is motivated by the familiar notion of area. Although the method of plane geometry enables us to calculate the area of polygons, they do not provide the ways of finding the area of plane region whose boundaries are curves other than circles. By means of integral calculus, we can find areas of such regions.

This six module on **Statistics** enables you to make decisions in situations of uncertainties and risks. A decision about the number of units of a product to be produced is based upon estimates of the number of units expected to be sold. If the number to be sold were known in advance, the decision would help the manufacturer to produce precisely that quantity incurring neither shortage nor excess. However in actual decision-making situations, exact information is not always available. We shall take up these kinds of uncertainties in the module on Statistics.

We would suggest that you go through all the solved examples given in the learning material and then try to solve independently all questions included in **"Check Your Progress"** and **"Terminal Exercise"** given at the end of each lesson:

This book contains three **"Questions For Practice"** pages for self-check. Try to solve all of them and get it checked by the subject teacher at your study centre.

If you face any difficulty please do write to us. Your suggestions are also welcome.

Yours suggestions are also welcome.

Wishing you all success.

(Suvendu Sekhar Das)

Academic Officer (Mathematics)

FORE WARD

Education plays an important role in the modern society. Many innovations can be achieved through Education. Hence the Department of Education is giving equal importance to non-formal education through Open Distance Learning (ODL) mode on the lines of formal education. This is the first State Open School established in the country in the year 1991 offering courses up to Upper primary Level till 2008. From the academic year 2008-2009 SSC Course was introduced and Intermediate Course from the year 2010-2011. The qualified learners from the Open School are eligible for both higher studies and employment. So far 7,67,190 learners were enrolled in the Open Schools and 4,50,024 learners have successfully completed their courses.

With the aim of improving the administration at the grass-root level the Telangana Government re-organized the existing Districts and formed 31 new Districts. The formation of new Districts provide wide range of employment and Business opportunities besides self employment. Given the freedom and flexibilities available, the Open School system provides a second chance of learning for those who could not fulfill their dreams of formal education.

Government of Telangana is keen in providing quality education by supplying study materials along with the text books to enable the learners to take the exam with ease. Highly experienced professionals and subject experts are involved in preparing curriculum and study material based on subject wise blue prints. The study material for the academic year 2018-19 is being printed and supplied to all the learners throughout the state.

I wish the learners of Open School make best use of the study material to brighten their future opportunities and rise up to the occasion in building Bangaru Telangana.

With best wishes ....

S. Venkateshwara Sharma
DIRECTOR,
Telangana Open School Society,
Hyderabad

An Overview of the Learning Material

CORE MODULES

ALGEBRA

Complex Numbers
Quadratic Equations
Matrices
Determinants and their Applications
Inverse of a Matrix and its Application
Linear Inequations and their Applications
Permutations and Combinations
Binomial Theorem

COORDINATE GEOMETRY

Cartesian System of Coordinates
Straight Lines
Circles
Conic Sections

SEQUENCES AND SERIES

Arithmetic and Geometric Progressions
Special Types of Series

FUNCTIONS

Sets, Relations and Functions
Trigonometric Functions-I
Trigonometric Functions-II
Inverse Trigonometric Functions
Relations between Sides and Angles of a Triangle

CALCULUS

Limit and Continuity
Differentiation
Differentiation of Trigonometric Functions
Differentiation of Exponential and Logarithmic Functions
Tangents and Normals
Maxima and Minima
Integration
Definite Integrals
Differential Equations

STATISTICS

Measures of Dispersion
Random Experiments and Events
Probability

OPTIONAL MODULES

(You have to choose any one of the following modules)

VECTORS AND THREE DIMENSIONAL GEOMETRY

Vectors
Introduction to Three Dimensional Geometry
The Plane
The Straight Line
The Sphere

MATHEMATICS FOR COMMERCE, ECONOMICS AND BUSINESS

Shares and Debentures
Index Numbers
Insurance
Indirect Taxes
Application of Calculus in Commerce and Economics

Additional Curriculum

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CONTENTS

MODULE-IV

FUNCTIONS

15

SETS, RELATIONS AND FUNCTIONS

Let us consider the following situation :

One day Mrs. and Mr. Mehta went to the market. Mr. Mehta purchased the following objects/ items.

"a toy, one kg sweets and a magazine". Where as Mrs. Mehta purchased the following objects/ items.

"Lady fingers, Potatoes and Tomatoes".

In both the examples, objects in each collection are well defined. What can you say about the collection of students who speak the truth ? Is it well defined? Perhaps not.

A set is a collection of well defined objects. For a collection to be a set it is necessary that it should be well defined.

The word well defined was used by the German Mathematician George Cantor (1845- 1918 A.D) to define a set. He is known as father of set theory. Now-a-days set theory has become basic to most of the concepts in Mathematics.

In our everyday life we come across different types of relations between the objects. The concept of relation has been developed in mathematical form.

The word function was introduced by Leibnitz in 1694. Function is a special type of relation. Each function is a relation but each ***relation is not a function***. In this lesson we shall discuss some basic definitions and operations involving sets, Cartesian product of two sets, relation between two sets, the conditions under when a relation becomes a function, different types of function and their properties.

OBJECTIVES

After studying this lesson, you will be able to :

- define a set and represent the same in different forms;
- define different types of sets such as, finite and infinite sets, empty set, singleton set, equivalent sets, equal sets, sub sets and cite examples thereof;
- define and cite examples of universal set, complement of a set and difference between two sets;
- define union and intersection of two sets;
- represent union and intersection of two sets, universal sets, complement of a set, difference between two sets by Venn Diagram;
- solve real life problems using Venn Diagram;
- define Cartesian product of two sets;
- define relation, function and cite examples thereof;
- find domain and range of a function;
- define and cite examples of diferent types of functions (one-one, many-one, onto, into and bijection);
- determine wheather a function is one-one, many-one, onto or into;
- draw the graph of functions;
- define and cite examples of odd and even functions;
- determine wheather a function is odd or even or neither;
- define and cite examples of functions like $|x|$, $[x]$ the greatest integer function, polynomial functions, logarithmic and exponential functions;
- define composition of two functions;
- define the inverse of a function; and
- state the conditions for the inverse to exist.

EXPECTED BACKGROUND KNOWLEDGE

- Number systems, concept of ordered pairs.

15.1 SOME STANDARD NOTATIONS

Before defining different terms of this lesson let us consider the following examples:

(i) collection of tall students in your school.	(i) collection of those students of your school whose height is more than 180 cm.
(ii) collection of honest persons in your colony.	(ii) collection of those people in your colony who have never been found involved in any theft case.
(iii) collection of interesting books in your school library.	(iii) collection of Mathematics books in your school library.
(iv) collection of intelligent students in your school.	(iv) collection of those students in your school who have secured more than 80% marks in annual examination.

In all collections written on left hand side of the vertical line the term tallness, interesting, honesty, intelligence are not well defined. In fact these notions vary from individual to individual. Hence those collections can not be considered as sets.

While in all collections written on right hand side of the vertical line, 'height' 'more than 180 cm.' 'mathematics books' 'never been found involved in theft case,' ' marks more than 80%' are well defined properties. Therefore, these collections can be considered as sets.

If a collection is a set then each object of this collection is said to be an element of this set. A set is usually denoted by capital letters of English alphabet and its elements are denoted by small letters.

For example, A = Toy elephant, packet of sweets, magazines.

Some standard notations to represent sets :

N : the set of natural numbers

W : the set of whole numbers

Z or I : the set of integers

Z^+ : the set of positve integers

Z^- : the set of negative integers

Q : the set of rational numbers

R : the set of real numbers

C : the set of complex numbers

Other frequently used symbols are :

$\in$: 'belongs to'

$\notin$: 'does not belong to'

$\exists$: There exists, $\nexists$: There does not exist.

MODULE - IV
Functions

Notes

For example N is the set of natural numbers and we know that 2 is a natural number but -2 is not a natural number. It can be written in the symbolic form as $2 \in N$ and $-2 \notin N$.

15. 2 REPRESENTATION OF A SET

There are two methods to represent a set.

15.2.1 (i) Roster method (Tabular form)

In this method a set is represented by listing all its elements, separating these by commas and enclosing these in curly bracket.

If V be the set of vowels of English alphabet, it can be written in Roster form as :

$V = \{a, e, i, o, u\}$

(ii) If A be the set of natural numbers less than 7.

then $A=\{1, 2, 3, 4, 5, 6\}$, is in the Roster form.

Note : To write a set in Roster form elements are not to be repeated i.e. all elements are taken as distinct. For example if A be the set of letters used in the word mathematics, then

$A = \{m, a, t, h, e, i, c, s\}$

15.2.2 Set-builder form

In this form elements of the set are not listed but these are represented by some common property.

Let V be the set of vowels of English alphabet then V can be written in the set builder form as:

$V = \{x : x \text{ is a vowel of English alphabet}\}$

(ii) Let A be the set of natural numbers less than 7.

then $A = \{x : x \in N \text{ and } 1 \le x < 7\}$

Note : Symbol ':' read as 'such that'

Example: 15.1 Write the following in set -builder form :

(a) $A=\{-3,-2,-1,0,1,2,3\}$ (b) $B = \{3,6,9,12\}$

Solution : (a) $A = \{x : x \in Z \text{ and } -3 \le x \le 3\}$

(b) $B = \{x : x = 3n \text{ and } n \in N, n \le 4\}$

Example 15.2 Write the following in Roster form.

(a) $C = \{x : x \in N \text{ and } 50 \le x \le 60\}$

(b) $D = \left\{x : x \in R \text{ and } x^2 - 5x + 6 = 0\right\}$

Solution : (a) $C = \{50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60\}$

(b) $x^2 - 5x + 6 = 0$

$\Rightarrow \quad (x-3)(x-2) = 0$

$\Rightarrow \quad x = 3, 2.$

$\therefore \quad D = \{2, 3\}$

15.3 CLASSIFICATION OF SETS

15.3.1 Finite and infinite sets

Let A and B be two sets where

A = {x : x is a natural number}

B = {x : x is a student of your school}

As it is clear that the number of elements in set A is not finite (infinite) while number of elements in set B is finite. A is said to be an infinite set and B is said to be is finite set.

A set is said to be finite if its elements can be counted and it is said to be infinite if it is not possible to count upto its last element.

15.3.2 Empty (Null) Set : Consider the following sets.

$$A = \{x : x \in R \text{ and } x^2 + 1 = 0\}$$

B = {x : x is number which is greater than 7 and less than 5}

Set A consists of real numbers but there is no real number whose square is -1. Therefore this set consists of no element. Similiarly there is no such number which is less than 5 and greater than 7. Such a set is said to be a null (empty) set. It is denoted by the symbol void, ϕ or { }

A set which has no element is said to be a null/empty/void set, and is denoted by ϕ.

15.3.3 Singleton Set : Consider the following set :

A = {x : x is an even prime number}

As there is only one even prime number namely 2, so set A will have only one element. Such a set is said to be singleton. Here A = $\{2\}$.

A set which has only one element is known as singleton.

15.3.4 Equal and equivalent sets : Consider the following examples.

(i) $A = \{1, 2, 3\}$, $B = \{2, 1, 3\}$ (ii) $D = \{1, 2, 3\}$, $E = \{a, b, c\}$.

In example (i) Sets A and B have the same elements. Such sets are said to be equal sets and it is written as A = B. In example (ii) set D and E have the same number of elements but elements are different. Such sets are said to be equivalent sets and are written as A $\approx$ B.

Two sets A and B are said to be equivalent sets if they have same number of elements but they are said to be equal if they have not only the same number of elements but elements are also the same.

15.3.5 Disjoint Sets : Two sets are said to be disjoint if they do not have any common element. For example,sets A= { 1,3,5} and B = { 2,4,6 } are disjoint sets.

xample 15.3 Given that

$A = \{2, 4\}$ and $B = \{x : x \text{ is a solution of } x^2 + 6x + 8 = 0\}$

Are A and B disjoint sets ?

Solution : If we solve $x^2 + 6x + 8 = 0$,we get

$x = -4, -2$.

$\therefore$ $B = \{-4, -2\}$ and $A = \{2, 4\}$

Clearly ,A and B are disjoint sets as they do not have any common element.

Example 15.4 If $A = \{x : x \text{ is a vowel of English alphabet}\}$

and $B = \{y : y \in N \text{ and } y \leq 5\}$

Is (i) $A = B$ (ii) $A \approx B$?

Solution : $A = \{a, e, i, o, u\}$, $b = \{1, 2, 3, 4, 5\}$.

Each set is having five elements but elements are different

$\therefore$ $A \neq B$ but $A \approx B$.

Example 15.5 Which of the following sets

$A = \{x : x \text{ is a point on a line}\}$

$B = \{y : y \in N \text{ and } y \leq 50\}$

are finite or infinite ?

Solution : As the number of points on a line is uncountable (cannot be counted) so A is an infinite set while the number of natural numbers upto fifty can be counted so B is a finite set.

Example 15.6 Which of the following sets

$A = \{x : x \text{ is irrational and } x^2 - 1 = 0\}$.

$B = \{x : x \in z \text{ and } -2 \leq x \leq 2\}$ are empty?

Solution : Set A consists of those irrational numbers which satisfy $x^2 - 1 = 0$. If we solve $x^2 - 1 = 0$ we get $x = \pm 1$. Clearly ± 1 are not irrational numbers. Therefore A is an empty set. But $B = \{-2, -1, 0, 1, 2\}$. B is not an empty set as it has five elements.

Example 15.7 Which of the following sets are singleton ?

$A = \{x : x \in Z \text{ and } x - 2 = 0\}$ $\qquad$ $B = \{y : y \in R \text{ and } y^2 - 2 = 0\}$.

Solution : Set A contains those integers which are the solution of $x - 2 = 0$ or $x = 2$. $\therefore$ $A = \{2\}$.

$\Rightarrow$ A is a singleton set.

B is a set of those real numbers which are solutions of $y^2 - 2 = 0$ or $y = \pm\sqrt{2}$

$\therefore$ $B = \{-\sqrt{2}, \sqrt{2}\}$ Thus, B is not a singleton set.

CHECK YOUR PROGRESS 15.1

1. Which of the following collections are sets ?
   - (i) The collection of days in a week starting with S.
   - (ii) The collection of natural numbers upto fifty.
   - (iii) The collection of poems written by Tulsidas.
   - (iv) The collection of fat students of your school.

(2) Insert the appropriate symbol in blank spaces.

If $A = \{1, 2, 3\}$.

(i) 1...........A (ii) 4........A.

3. Write each of the following sets in the Roster form :
   - (i) $A = \{x : x \in z \text{ and } -5 \le x \le 0\}$.
   - (ii) $B = \{x : x \in R \text{ and } x^2 - 1 = 0\}$.
   - (iii) C = {x : x is a letter of the word banana}.
   - (iv) D = {x : x is a prime number and exact divisor of 60}.
4. Write each of the following sets are in the set builder form ?
   - (i) A = {2, 4, 6, 8, 10} (ii) B = {3, 6, 9,...... ∞ }
   - (iii) C = {2, 3, 5, 7} (iv) $D = \{-\sqrt{2}, \sqrt{2}\}$

   Are A and B disjoints sets ?
5. Which of the following sets are finite and which are infinite ?
   - (i) Set of lines which are parallel to a given line.
   - (ii) Set of animals on the earth.
   - (iii) Set of Natural numbers less than or equal to fifty.
   - (iv) Set of points on a circle.
6. Which of the following are null set or singleton ?
   - (i) $A = \{x : x \in R \text{ and } x \text{ is a solution of } x^2 + 2 = 0\}$.
   - (ii) $B = \{x : x \in Z \text{ and } x \text{ is a solution of } x - 3 = 0\}$.
   - (iii) $C = \{x : x \in Z \text{ and } x \text{ is a solution of } x^2 - 2 = 0\}$.
   - (iv) D = {x : x is a student of your school studying in both the classes XI and XII }
7. In the following check whether A = B or $A \approx B$.
   - (i) A = {a}, B = {x : x is an even prime number}.
   - (ii) A = {1, 2, 3, 4}, B = {x : x is a letter of the word guava}.
   - (iii) A = {x : x is a solution of $x^2 - 5x + 6 = 0$ },B = {2 , 3}.

15.4 SUB- SET

Let set A be a set containing all students of your school and B be a set containing all students of class XII of the school. In this example each element of set B is also an element of set A. Such a set B is said to be subset of the set A. It is written as $B \subseteq A$

Consider $D = \{1, 2, 3, 4, ...\}$

$E = \{..... -3 -2, -1, 0, 1, 2, 3, ...\}$

Clearly each element of set D is an element of set E also $\therefore\ D \subseteq E$

If A and B are any two sets such that each element of the set A is an element of the set B also, then A is said to be a subset of B.

Remarks

(i) Each set is a subset of itself i.e. $A \subseteq A$.

(ii) Null set has no element so the condition of becoming a subset is automatically satisfied. Therefore null set is a subset of every set.

(iii) If $A \subseteq B$ and $B \subseteq A$ then A = B.

(iv) If $A \subseteq B$ and $A \neq B$ then A is said to be a proper subset of B and B is said to be a super set of A. i.e. $A \subset B$ or $B \supset A$.

Example 15.8 If A = {x : x is a prime number less than 5} and B = {y : y is an even prime number} then is B a proper subset of A ?

Solution : It is given that

$A = \{2, 3\}$, $B = \{2\}$.

Clearly $B \subseteq A$ and $B \neq A$

We write $B \subset A$

and say that B is a proper subset of A.

Example 15.9 If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5\}$.

is $A \subseteq B$ or $B \subseteq A$?

Solution : Here $1 \in A$ but $1 \notin B \Rightarrow A \not\subseteq B$.

Also $5 \in B$ but $5 \notin A \Rightarrow B \not\subseteq A$.

Hence neither A is a subset of B nor B is a subset of A.

Example 15.10 If A = {a, e, i, o, u}

B = {e, i, o, u, a }

Is $A \subseteq B$ or $B \subseteq A$ or both ?

Solution : Here in the given sets each element of set A is an element of set B also

$\therefore \quad A \subseteq B$ .... (i)

and each element of set B is an element of set A also. $\therefore \; B \subseteq A$ ......(ii)

From (i) and (ii)

$A = B$

15.5 POWER SET

Let $A = \{a, b\}$

Subset of A are ϕ, $\{a\}$, $\{b\}$ and $\{a, b\}$.

If we consider these subsets as elements of a new set B (say) then

$$B = \{\phi, \{a\}, \{b\}, \{a, b\}\}$$

B is said to be the power set of A.

Notation : Power set of a set A is denoted by P(A).
Power set of a set A is the set of all subsets of the given set.

Example 15.11 Write the power set of each of the following sets :

(i) $A = \{x : x \in R \text{ and } x^2 + 7 = 0\}$.

(ii) $B = \{y : y \in N \text{ and} 1 \le y \le 3\}$.

Solution :

(i) Clearly $A = \phi$ (Null set)

$\therefore$ ϕ is the only subset of given set

$\therefore$ $P(A) = \{\phi\}$

(ii) The set B can be written as $\{1, 2, 3\}$

Subsets of B are ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$.

$\therefore$ $P(B) = \{\ \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\ \}$.

15.6 UNIVERSAL SET

Consider the following sets.

A = {x : x is a student of your school}
B = {y : y is a male student of your school}
C = {z : z is a female student of your school}
D = {a : a is a student of class XII in your school}

Clearly the set B, C, D are all subsets of A.

A can be considered as the universal set for this particular example. Universal set is generally denoted by U.

In a particular problem a set U is said to be a universal set if all the sets in that problem are subsets of U.

Remarks

(i) Universal set does not mean a set containing all objects of the universe.

(ii) A set which is a universal set for one problem may not be a universal set for another problem.

Example 15.12 Which of the following set can be considered as a universal set ?

X = {x : x is a real number}

Y = {y : y is a negative integer}

Z = {z : z is a natural number}

Solution : As it is clear that both sets **Y** and **Z** are subset of **X**.

∴ **X** is the universal set for this problem.

15.7 VENN DIAGRAM

British mathematician John Venn (1834 – 1883 AD) introduced the concept of diagrams to represent sets. According to him universal set is represented by the interior of a rectangle and other sets are represented by interior of circles.

For example if U= {1, 2, 3, 4, 5}, A = {2, 4} and B = {1,3}, then these sets can be represented as

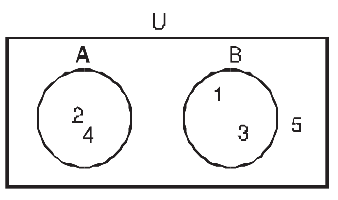


Fig. 15.1

Diagramatical representation of sets is known as a Venn diagram.

15.8 DIFFERENCE OF SETS

Consider the sets

A = {1, 2, 3, 4, 5} and B= {2, 4, 6}.

A new set having those elements which are in A but not B is said to be the difference of sets A and B and it is denoted by A – B.

∴ A – B= {1, 3, 5}

Similiarly a set of those elements which are in B but not in A is said to be the difference of B and A and it is devoted by B – A.

∴ B – A = {6}

In general, if A and B are two sets then

$$A - B = \{ x : x \in A \text{ and } x \notin B \}$$

$$B - A = \{ x : x \in B \text{ and } x \notin A \}$$

Difference of two sets can be represented using Venn diagram as :

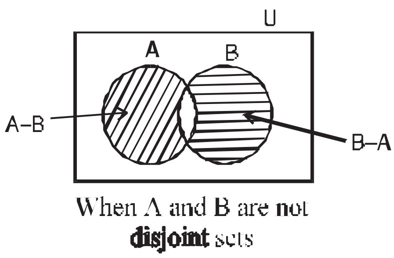


Fig. 15.2

or

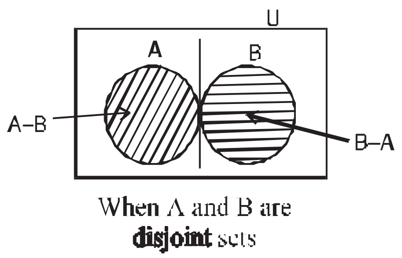


Fig. 15.3

15.9. COMPLEMENT OF A SET

Let **X** denote the universal set and **Y, Z** its sub set where

X = {x : x is any member of the family}

Y = {x : x is a male member of the family}

Z = {x : x is a female member of the family}

X – Y is a set having female members of the family.

X – Z is a set having male members of the family.

X – Y is said to be the complement of Y and is usally denoted by Y' or Y^c.

X – Z is said to be complement of Z and denoted by Z' or Z^c.

If U is the universal set and A is its subset then the complement of A is a set of those elements which are in U which are not in A. It is denoted by A' or A^c.

$$A' = U - A = \{x : x \in U \text{ and } x \notin A\}$$

The complement of a set can be represented using Venn diagram as :

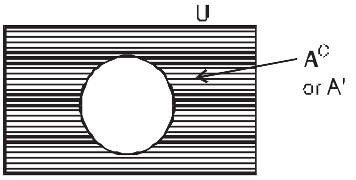


Fig. 15.4

Remarks

(i) Difference of two sets can be found even if none is a subset of the other but complement of a set can be found only when the set is a subset of some universal set.

(ii) $\phi^c = U$. (iii) $U^c = \phi$.

Example 15.13 Given that

A = {x : x is a even natural number less than or equal to 10}

and B = {x : x is an odd natural number less than or equal to 10}

Find (i) $A-B$ (ii) $B-C$ (iii) is $A-B=B-A$ **?**

Solution : It is given that

A = {2, 4, 6, 8, 10}, B = {1, 3, 5, 7, 9}

Therefore,

(i) $A-B=\{2, 4, 6, 8, 10\}$

(ii) $B-A=\{1, 3, 5, 7, 9\}$

(iii) Clearly from (i) and (ii) $A-B \neq B-A$.

Example 15.14 Let U be the universal set and A its subset where

$U=\{x : x \in N$ and $x \leq 10\}$

A = {y : y is a prime number less than 10}

Find (i) A^c (ii) Represent A^c in Venn diagram.

Solution : It is given

U= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

A = { 2, 3, 5, 7}

(i) $A^c = U - A = \{1, 4, 6, 8, 9, 10\}$

(ii)

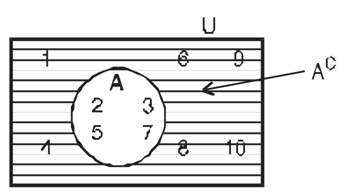


Fig. 15.5

CHECK YOUR PROGRESS 15.2

1. Insert the appropriate symbol in the blank spaces, given that

   A={1, 3, 5, 7, 9}

   (i) ϕ ....A (ii) {2, 3, 9}..........A

   (iii) 3............A (iv) 10...................A

2. Given that A = {a, b}, how many elements P(A) has ?

3. Let A = {ϕ, {1} ,{2}, {1,2}}

   Which of the following is true or false ?

(i) $\{1,2\} \subset A$ (ii) $\phi \in A$.

4. Which of the following statements are true or false ?
   (i) Set of all boys, is contained in the set of all students of your school.
   (ii) Set of all boy students of your school, is contained in the set of all students of your school.
   (iii) Set of all rectangles, is contained in the set of all quadrilaterals.
   (iv) Set of all circles having centre at origin is contained in the set of all ellipses having centre at origin.
5. If A = { 1, 2, 3, 4, 5 } , B= {5, 6, 7} find (i) $A - B$ (ii) $B - A$.
6. Let N be the universal set and A, B, C, D be its subsets given by

   A = {x : x is a even natural number}

   $B = \{x : x \in N$ and x is a multiple of 3}

   $C = \{x : x \in N$ and $x \geq 5\}$

   $D = \{x : x \in N$ and $x \leq 10\}$

   Find complements of A, B, C and D respectively.

15.10. INTERSECTION OF SETS

Consider the sets

A = {1, 2, 3, 4} and B = { 2, 4, 6}

It is clear, that there are some elements which are common to both the sets A and B. Set of these common elements is said to be interesection of A and B and is denoted by $A \cap B$.

Here $A \cap B$ = {2, 4 }

If A and B are two sets then the set of those elements which belong to both the sets is said to be the intersection of A and B. It is devoted by $A \cap B$.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

$A \cap B$ can be represented using Venn diagram as :

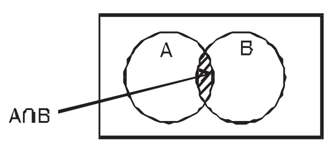


Fig. 15.6

Remarks

If $A \cap B = \phi$ then A and B are said to be **disjoint sets**. In Venn diagram disjoint sets can be represented as

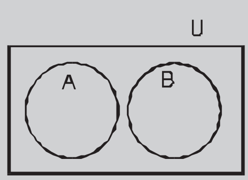


Fig.15.7

Example 15.15 Given that

$A = \{x : x \text{ is a king out of 52 playing cards}\}$

and $B = \{ y : y \text{ is a spade out of 52 playing cards}\}$

Find (i) $A \cap B$ (ii) Represent $A \cap B$ by using Venn diagram .

Solution : (i) As there are only four kings out of 52 playing cards, therefore the set A has only four elements. The set B has 13 elements as there are 13 spade cards but out of these 13 spade cards there is one king also. Therefore there is one common element in A and B. $\therefore A \cap B = \{\text{King of spade}\}$.

(ii)

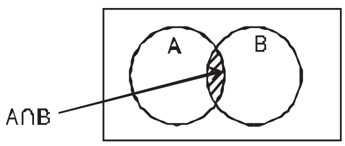


Fig. 15.8

15.11 UNION OF SETS

Consider the following examples :

(i) A is a set having all players of Indian men cricket team and B is a set having all players of Indian women cricket team. Clearly A and B are disjoint sets. Union of these two sets is a set having all players of both teams and it is denoted by $A \cup B$.

(ii) D is a set having all players of cricket team and E is the set having all players of Hockey team, of your school. Suppose three players are common to both the teams then union of D and E is a set of all players of both the teams but three common players to be written once only.

If A and B are only two sets then union of A and B is the set of those elements which belong to A or B.

In set builder form :

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

OR

$$A \cup B = \{x : x \in A - B \text{ or } x \in B - A \text{ or } x \in A \cap B\}$$

$A \cup B$ can be represented using Venn diagram as :

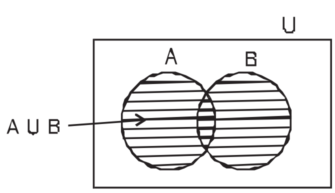


Fig. 15.9

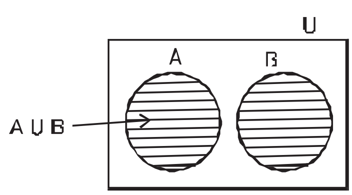


Fig. 15.10

$$n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B).$$

or $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

where n $A \cup B$ stands for number of elements in $A \cup B$ so on.

Example 15.16 $A = \{x : x \in Z^+ \text{ and } \leq 5\}$

$B = \{y : y \text{ is a prime number less than } 10\}$

Find (1) $A \cup B$ (ii) represent $A \cup B$ using Venn diagram.

Solution : We have,

A={1, 2, 3, 4, 5} B = {2, 3, 5, 7}.

$\therefore$ $A \cup B$ = {1, 2, 3, 4, 5, 7}.

(ii)

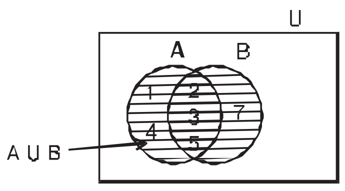


Fig.15.11

CHECK YOUR PROGRESS 15.3

1. Which of the following pairs of sets are disjoint and which are not ?

   (i) {x : x is an even natural number}, {y : y is an odd natural number}

   (ii) {x : x is a prime number and divisor of 12}, $\{y : y \in N \text{ and } 3 \leq y \leq 5\}$

   (iii) {x :x is a king of 52 playing cards}, { y : y is a diamond of 52 playing cards}

   (iv) {1, 2, 3, 4, 5}, {a, e, i, o, u}

2. Find the intersection of A and B in each of the following :

   (i) $A = \{x : x \in Z\}$, B= $\{x : x \in N\}$ (ii) A = {Ram, Rahim, Govind, Gautam}

   B = {Sita, Meera, Fatima, Manprit}

3. Given that A = {1, 2, 3, 4, 5}, B={5, 6, 7, 8, 9, 10}

   find (i) $A \cup B$ (ii) $A \cap B$.

4. If A =$\{x : x \in N\}$, B=$\{y : y \in z \text{ and } -10 \leq y \leq 0\}$ find $A \cup B$ and write your answer in the Roster form as well as set-builder form.

5. If A ={2, 4, 6, 8, 10}, B {8, 10, 12, 14}, C ={14, 16, 18, 20}.

   Find (i) $A \cup (B \cup C)$ (ii) $A \cap (B \cap C)$.

6. Let U = {1, 2, 3,.......10}, A = {2, 4, 6, 8, 10}, B = { 1, 3, 5, 7, 9, 10}

   Find (i) $(A \cup B)'$ (ii) $(A \cap B)'$ (iii) (B')' (iv) $(B - A)'$.

7. Draw Venn diagram for each the following :

(i) $A \cap B$ when $B \subset A$ (ii) $A \cap B$ when A and B are disjoint sets.

(iii) $A \cap B$ when A and B are neither subsets of each other nor disjoint sets.

8. Draw Venn diagram for each the following :

(i) $A \cup B$ when $A \subset B$. (ii) $A \cup B$ when A and B are disjoint sets.

(iii) $A \cup B$ when A and B are neither subsets of each other nor disjoint sets.

9. Draw Venn diagram for each the following :

(i) $A - B$ and $B - A$ when $A \subset B$.

(ii) $A - B$ and $B - A$ when A and B are disjoint sets.

(iii) $A - B$ and $B - A$ when A and B are neither subsets of each other nor disjoint sets.

15.12 CARTESIAN PRODUCT OF TWO SETS

Consider two sets A and B where

$A=\{1, 2\}$, $B= \{3, 4, 5\}$.

Set of all ordered pairs of elements of A and B

is $\{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

This set is denoted by A × B and is called the cartesian product of sets A and B.

i.e. $A\times B =\{(1, 3), (1, 4),(1, 5),(2, 3),(2, 4),(2, 5)\}$

Cartesian product of B sets and A is denoted by B×A.

In the present example, it is given by

$B\times A = \{(3, 1),(3, 2),(4, 1),(4, 2),(5, 1),(5, 2)\}$

Clearly $A\times B \neq B\times A$.

In the set builder form :

$A\times B = \{(a,b) : a\in A \text{ and } b\in B\}$

$B\times A = \{(b,a) : b\in B \text{ and } a\in A\}$

Note : If $A = \phi$ or $B = \phi$ or $A, B = \phi$

then $A\times B = B\times A = \phi$.

Example 15.17

(1) Let $A=\{a,b,c\}$, $B=\{d,e\}$, $C=\{a,d\}$.

Find (i) $A\times B$ (ii) $B\times A$ (iii) $A\times(B \cup C)$ (iv) $(A\cap C)\times B$

(v) $(A\cap B)\times C$ (vi) $A\times(B-C)$.

Solution : (i) $A\times B =\{(a, d),(a, e), (b, d), (b, e), (c, d), (c, e)\}$.

(ii) $B\times A = \{(d, a),(d, b), (d, c), (e, a) (e, b),(e, c)\}$.

(iii) $A = \{a, b, c\}$, $B \cup C=\{a,d,e\}$.

$\therefore$ $A \times (B \cup C)$ ={(a, a),(a, d),(a, e),(b, a),(b, d),(b, e), (c, a),(c, d),(c, e).

(iv) $A \cap C = \{a\}$, $B=\{d, e\}$.

$\therefore$ $(A \cap C)\times B=\{(a, d), (a, e)\}$

(v) $A \cap B = \phi$, $c=\{a,d\}$, $\therefore$ $A \cap B \times c = \phi$

(vi) $A = \{a,b,c\}$, $B - C = \{e\}$.

$\therefore$ $A \times (B - C) = \{(a,e),(b,e),(c,e)\}$.

15.13 RELATIONS

Consider the following example :

A={Mohan, Sohan, David, Karim}

B={Rita, Marry, Fatima}

Suppose Rita has two brothers Mohan and Sohan, Marry has one brother David, and Fatima has one brother Karim. If we define a relation R " is a brother of" between the elements of A and B then clearly.

Mohan R Rita, Sohan R Rita, David R Marry, Karim R Fatima.

After omiting R between two names these can be written in the form of ordered pairs as :

(Mohan, Rita), (Sohan, Rita), (David, Marry), (Karima, Fatima).

The above information can also be written in the form of a set R of ordered pairs as

R= {(Mohan, Rita), (Sohan, Rita), (David, Marry), Karim, Fatima}

Clearly $R \subseteq A \times B$, i.e.$R = \{(a,b): a \in A, b \in B \text{ and } aRb\}$

If A and B are two sets then a relation R from A toB is a sub set of A×B.

If (i) $R = \phi$, R is called a void relation.

(ii) R=A×B, R is called a universal relation.

(iii) If R is a relation defined from A to A, it is called a relation defined on A.

(iv) $R = \{ (a,a) \ \forall a \in A \}$, is called the identity relation.

15.13.1 Domain and Range of a Relation

If R is a relation between two sets then the set of its first elements (components) of all the ordered pairs of R is called Domain and set of 2nd elements of all the ordered pairs of R is called range, of the given relation.

Consider previous example given above.

Domain = {Mohan, Sohan, David, Karim}

Range = {Rita, Marry, Fatima}

Example 15.18 Given that $A = \{2, 4, 5, 6, 7\}$, $B = \{2, 3\}$.

R is a relation from A to B defined by

$R = \{(a, b) : a \in A, \; b \in B$ and a is divisible by b$\}$

find (i) R in the roster form (ii) Domain of R (iii) Range of R

(iv) Repersent R diagramatically.

Solution : (i) R = {(2, 2), (4, 2), (6, 2), (6, 3)}

(ii) Domain of R = {2, 4, 6}

(iii) Range of R = {2, 3}

(iv)

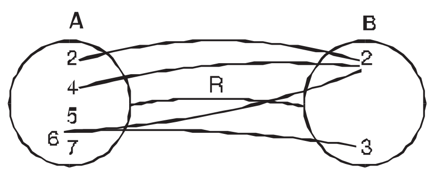


Fig. 15.12

Example 15.19 If R is a relation 'is greater than' from A to B, where

A= {1, 2, 3, 4, 5} and B = {1,2,6}.

Find (i) R in the roster form. (ii) Domain of R (iii) Range of R.

Solution :

(i) R = {(3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)}

(ii) Domain of R = {3, 4, 5}

(iii) Range of R = {1, 2}

CHECK YOUR PROGRESS 15.4

1. Given that A = {4, 5, 6, 7}, B = {8, 9}, C = {10}

   Verify that $A \times (B - C) = (A \times B) - (A \times C)$.

2. If U is a universal set and A, B are its subsets.

   Where U= {1, 2, 3, 4, 5}.

   A = {1,3,5}, B = {x : x is a prime number} find A' × B'

3. If A = {4, 6, 8, 10}, B = {2, 3, 4, 5}

   R is a relation defined from A to B where

   $R = \{(a, b) : a \in A, \; b \in B$ and a is a multiple of b$\}$

   find (i)R in the Roster form (ii) Domain of R (iii) Range of R .

4. If R be a relation from N to N defined by

   $R = \{(x,y) : 4x + y = 12, \; x, y \in N\}$

   find (i) R in the Roster form (ii) Domain of R (iii) Range of R.

5. If R be a relation on N defined by

   $R = \{(x, x^2) : x$ is a prime number less than 15$\}$

Find (i) R in the Roster form (ii) Domain of R (iii) Range of R

6. If R be a relation on set of real numbers defined by $R=\{(x,y) : x^2 + y^2 = 0\}$ Find

(i) R in the Roster form (ii) Domain of R (iii) Range of R.

15.14 DEFINITION OF A FUNCTION

Consider the relation

$f : \{(a, 1), (b, 2), (c, 3), (d, 5)\}$

In this relation we see that each element of A has a unique image in *B*

This relation *f* from set *A* to *B* where every element of *A* has a unique image in *B* is defined as a function from *A* to *B*. So we observe that ***in a function no two ordered pairs have the same first element.***

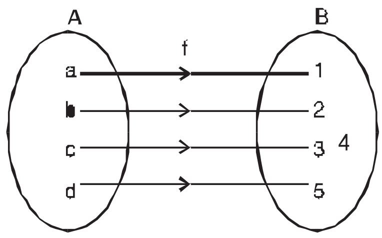


Fig.15.13

We also see that $\exists$ an element $\in$ B, i.e., 4 which does not have its preimage in A. Thus here:

(i) the set B will be termed as co-domain and

(ii) the set $\{1, 2, 3, 5\}$ is called the range.

From the above we can conclude that ***range is a subset of co-domain.***

Symbolically, this function can be written as

$$f : A \to B \qquad \text{or} \qquad A \xrightarrow{f} B$$

Example 15.20 Which of the following relations are functions from A to B. Write their domain and range. If it is not a function give reason ?

(a) $\{(1,-2),(3,7),(4,-6),(8,1)\}$, $A = \{1,3,4,8\}$, $B = \{-2,7,-6,1,2\}$

(b) $\{(1,0),(1-1),(2,3),(4,10)\}$, $A = \{1,2,4\}$, $B = \{0,-1,3,10\}$

(c) $\{(a,b),(b,c),(c,b),(d,c)\}$, $A = \{a,b,c,d,e\}$ $B = \{b,c\}$

(d) $\{(2,4),(3,9),(4,16),(5,25),(6,36\}$, $A = \{2,3,4,5,6\}$, $B = \{4,9,16,25,36\}$

(e) $\{(1,-1),(2,-2),(3,-3),(4,-4),(5,-5)\}$, $A = \{0,1,2,3,4,5\}$,

$B = \{-1,-2,-3,-4,-5\}$

(f) $$\left\{\left(\sin\frac{\pi}{6},\frac{1}{2}\right),\left(\cos\frac{\pi}{6},\frac{\sqrt{3}}{2}\right),\left(\tan\frac{\pi}{6},\frac{1}{\sqrt{3}}\right),\left(\cot\frac{\pi}{6},\sqrt{3}\right)\right\},$$

$$A = \left\{\sin\frac{\pi}{6},\cos\frac{\pi}{6},\tan\frac{\pi}{6},\cot\frac{\pi}{6}\right\} \quad B = \left\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{3}},\sqrt{3},1\right\}$$

(g) $\{(a,b),(a,2),(b,3),(b,4)\}$, $A = \{a,b\}$, $B = \{b,2,3,4\}$.

Solution :

(a) It is a function.

Domain= $\{1,3,4,8\}$, Range $=\{-2,7,-6,1\}$

(b) It is not a function. Because Ist two ordered pairs have same first elements.

(c) It is not a function.

Domain= $\{a,b,c,d\} \neq A$, Range $=\{b,c\}$

(d) It is a function.

Domain $=\{2,3,4,5,6\}$, Range $=\{4,9,16,25,36\}$

(e) It is not a function .

Domain= $\{1,2,3,4,5\} \neq A$, Range $=\{-1,-2,-3,-4,-5\}$

(f) It is a function .

Domain= $\left\{\sin\frac{\pi}{6},\cos\frac{\pi}{6},\tan\frac{\pi}{6},\cot\frac{\pi}{6}\right\}$, Range $=\left\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{3}},\sqrt{3}\right\}$

(g) It is not a function.

First two ordered pairs have same first component and last two ordered pairs have also same first component.

Example 15.21 State whether each of the following relations represent a function or not.

(a)

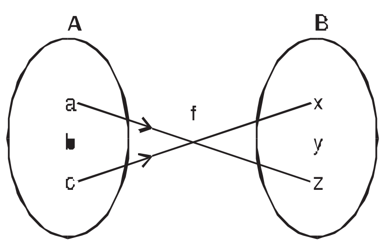


Fig.15.14

(b)

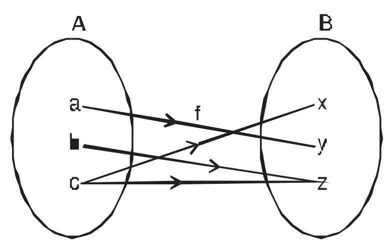


Fig.15.15

(c)

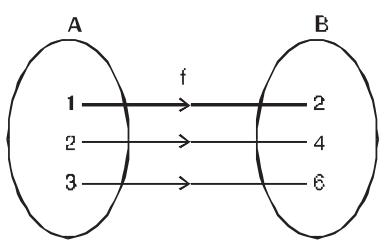


Fig. 15.16

(d)

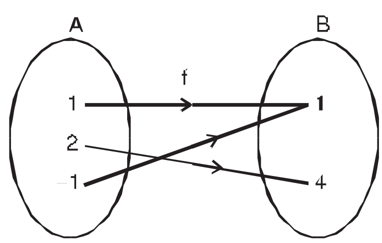


Fig.15.17

Solution :

(a) f is not a function because the element b of A does not have an image in B.

(b) f is not a function because the element c of A does not have a unique image in B.

(c) f is a function because every element of A has a unique image in B.

(d) f is a function because every element in A has a unique image in B.

Example 15.22 Which of the following relations from $R \rightarrow$. R are functions?

(a) $y = 3x + 2$ (b) $y < x + 3$ (c) $y = 2x^2 + 1$

Solution :

(a) $y = 3x + 2$

Here corresponding to every element $x \in R, \exists$ a unique element $y \in R$.

$\therefore$ It is a function.

(b) $y < x + 3$.

For any real value of x we get more than one real value of y.

$\therefore$ It is not a function.

(c) $y = 2x^2 + 1$

For any real value of x, we will get a unique real value of y.

$\therefore$ It is a function.

CHECK YOUR PROGRESS 15.5

1. Which of the following relations are functions from A to B ?

(a) $\{(1,-2),(3,7),(4,-6),(8,11)\}$, $A = \{1,3,4,8\}$, $B = \{-2,7,-6,11\}$

(b) $\{(1,0),(1,-1),(2,3),(4,10)\}$, $A = \{1,2,4\}, B = \{1,0,-1,3,10\}$

(c) $\{(a,2),(b,3),(c,2),(d,3)\}$, $A = \{a,b,c,d\}, B = \{2,3\}$

(d) $\{(1,1),(1,2),(2,3),(-3,4)\}$, $A = \{1,2,-3\}, B = \{1,2,3,4\}$

(e) $\left\{\left(2,\frac{1}{2}\right),\left(3,\frac{1}{3}\right),\ldots,\left(10,\frac{1}{10}\right)\right\}$,

$A = \{1,2,3,4\}, B = \left\{\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{11}\right\}$

(f) $\{(1,1),(-1,1),(2,4),(-2,4)\}$, $A = \{0,1,-1,2,-2\}, B = \{1,4\}$

2. Which of the following relations represent a function ?

(a) (b)

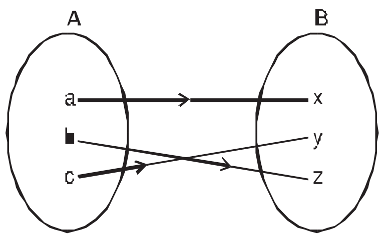


Fig. 15.18

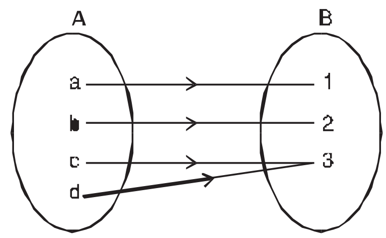


Fig. 15.19

(c) (d)

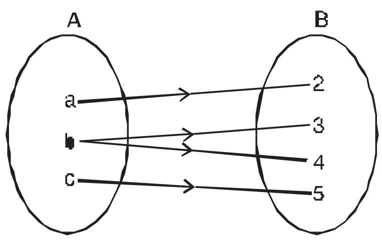


Fig. 15.20

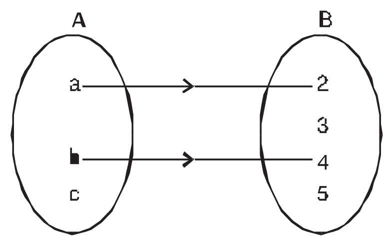


Fig. 15.21

3. Which of the following relations defined from $R \to R$ are functions ?

(a) $y = 2x + 1$ (b) $y > x + 3$ (c) $y < 3x + 1$ (d) $y = x^2 + 1$

4. Write domain and range for each of the following functions :

(a) $\left\{\left(\sqrt{2}, 2\right), \left(\sqrt{5}, -1\right), \left(\sqrt{3}, 5\right)\right\}$

(b) $\left\{\left(-3, \frac{1}{2}\right), \left(-2, \frac{1}{2}\right), \left(-1, \frac{1}{2}\right)\right\}$

(c) $\left\{(1,1), (0,0), (2,2), (-1,-1)\right\}$

(d) $\left\{(\text{Deepak}, 16), (\text{Sandeep}, 28), (\text{Rajan}, 24)\right\}$

5. Write domain and range for each of the following mappings :

(a) (b)

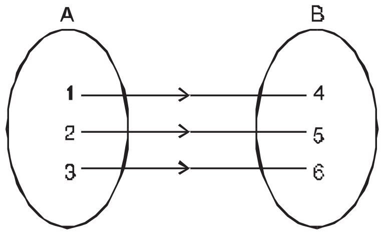


Fig. 15.22

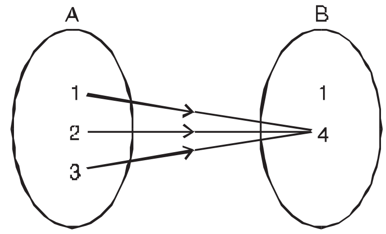


Fig. 15.23

(c)

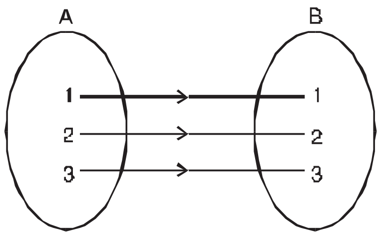


Fig. 15.24

(d)

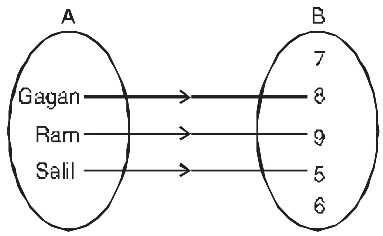


Fig. 15.25

(e)

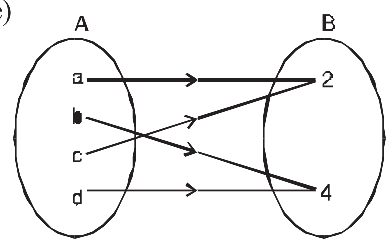


Fig. 15.26

15.14.1 Some More Examples on Domain and Range

Let us consider some functions which are only defined for a certain subset of the set of real numbers.

Example 15.23 Find the domain of each of the following functions :

(a) $y = \dfrac{1}{x}$ (b) $y = \dfrac{1}{x-2}$ (c) $y = \dfrac{1}{(x+2)(x-3)}$

Solution : The function $y = \dfrac{1}{x}$ can be described by the following set of ordered pairs.

$$\left\{ ....,\left(-2,-\frac{1}{2}\right),(-1,-1),(1,1)\left(2,\frac{1}{2}\right),.... \right\}$$

Here we can see that x can take all real values except 0 because the corresponding image, i.e., $\dfrac{1}{0}$ is not defined.

$\therefore$ Domain $= R - \{0\}$ [Set of all real numbers except 0]

Note : Here range $= R - \{0\}$

(b) x can take all real values except 2 because the corresponding image, i.e., $\dfrac{1}{(2-2)}$ does not exist.

$\therefore$ Domain $= R - \{2\}$

(c) Value of y does not exist for $x = -2$ and $x = 3$

$\therefore$ Domain $= R - \{-2, 3\}$

Example 15.24 Find domain of each of the following functions :

(a) $y = +\sqrt{x-2}$ (b) $y = +\sqrt{(2-x)(4+x)}$

Solution :(a) Consider the function $y = +\sqrt{x-2}$

In order to have real values of y, we must have $(x-2) \geq 0$

i.e. $x \geq 2$

$\therefore$ Domain of the function will be all real numbers ≥ 2.

(b) $y = +\sqrt{(2-x)(4+x)}$

In order to have real values of y, we must have $(2-x)(4+x) \geq 0$

We can achieve this in the following two cases :

Case I : $(2-x) \geq 0$ and $(4+x) \geq 0$

$\Rightarrow$ $x \leq 2$ and $x \geq -4$

$\therefore$ Domain consists of all real values of x such that $-4 \leq x \leq 2$

Case II : $2 - x \leq 0$ and $4 + x \leq 0$

$\Rightarrow$ $2 \leq x$ and $x \leq -4$.

But, x cannot take any real value which is greater than or equal to 2 and less than or equal to -4.

$\therefore$ From both the cases, we have

Domain $= -4 \leq x \leq 2 \,\forall\, x \in R$

Example 15.25 For the function

$f(x) = y = 2x + 1$, find the range when domain $= \{-3, -2, -1, 0, 1, 2, 3\}$.

Solution : For the given values of x, we have

$$f(-3) = 2(-3) + 1 = -5$$

$$f(-2) = 2(-2) + 1 = -3$$

$$f(-1) = 2(-1) + 1 = -1$$

$$f(0) = 2(0) + 1 = 1$$

$$f(1) = 2(1) + 1 = 3$$

$$f(2) = 2(2) + 1 = 5$$

$$f(3) = 2(3) + 1 = 7$$

The given function can also be written as a set of ordered pairs.

i.e., $\{(-3,-5),(-2,-3),(-1,-1),(0,1)(1,3),(2,5)(3,7)\}$

$\therefore$ Range $= \{-5,-3,-1,1,3,5,7\}$

Example 15.26 If $f(x) = x + 3,\ 0 \le x \le 4$, find its range.

Solution : Here $0 \le x \le 4$

or $0 + 3 \le x + 3 \le 4 + 3$

or $3 \le f(x) \le 7$

$\therefore$ Range $= \{f(x) : 3 \le f(x) \le 7\}$

Example 15. 27 If $f(x) = x^2,\ -3 \le x \le 3$, find its range.

Solution : Given $-3 \le x \le 3$

or $0 \le x^2 \le 9$ or $0 \le f(x) \le 9$

$\therefore$ Range $= \{f(x) : 0 \le f(x) \le 9\}$

CHECK YOUR PROGRESS 15.6

1. Find the domain of each of the following functions $x \in R$:

(a) (i) $y = 2x$ (ii) $y = 9x + 3$ (iii) $y = x^2 + 5$

(b) (i) $y = \dfrac{1}{3x - 1}$ (ii) $y = \dfrac{1}{(4x + 1)(x - 5)}$

(iii) $y = \dfrac{1}{(x - 3)(x - 5)}$ (iv) $y = \dfrac{1}{(3 - x)(x - 5)}$

(c) (i) $y = \sqrt{6 - x}$ (ii) $y = \sqrt{7 + x}$

(iii) $y = \sqrt{3x + 5}$

(d) (i) $y = \sqrt{(3 - x)(x - 5)}$ (ii) $y = \sqrt{(x - 3)(x + 5)}$

(iii) $y = \dfrac{1}{\sqrt{(3 + x)(7 + x)}}$ (iv) $y = \dfrac{1}{\sqrt{(x - 3)(7 + x)}}$

2. Find the range of the function, given its domain in each of the following cases.

(a) (i) $f(x) = 3x + 10,\quad x \in \{1, 5, 7, -1, -2\}$

(ii) $f(x) = 2x^2 + 1,\quad x \in \{-3, 2, 4, 0\}$

(iii) $f(x) = x^2 - x + 2,\quad x \in \{1, 2, 3, 4, 5\}$

(b) (i) $f(x) = x - 2,\ 0 \le x \le 4$ (ii) $f(x) = 3x + 4,\ -1 \le x \le 2$

(c) (i) $f(x) = x^2,\ -5 \le x \le 5$ (ii) $f(x) = 2x,\ -3 \le x \le 3$

(iii) $f(x) = x^2 + 1,\ -2 \le x \le 2$ (iv) $f(x) = \sqrt{x},\ 0 \le x \le 25$

(d) (i) $f(x) = x + 5,\ x \in R$ (ii) $f(x) = 2x - 3,\ x \in R$

(iii) $f(x) = x^3,\ x \in R$ (iv) $f(x) = \frac{1}{x},\ \{x : x < 0\}$

(v) $f(x) = \frac{1}{x - 2},\ \{x : x \le 1\}$ (vi) $f(x) = \frac{1}{3x - 2},\ \{x : x \le 0\}$

(vii) $f(x) = \frac{2}{x},\ \{x : x > 0\}$ (viii) $f(x) = \frac{x}{x + 5},\ \{x : x \ne -5\}$

15.15 CLASSIFICATION OF FUNCTIONS

Let f be a function from A to B. If every element of the set B is the image of at least one element of the set A i.e. if there is no unpaired element in the set B then we say that the function f maps the set A onto the set B. Otherwise we say that the function maps the set A into the set B.

Functions for which each element of the set A is mapped to a different element of the set B are said to be ***one-to-one.***

One-to-one function

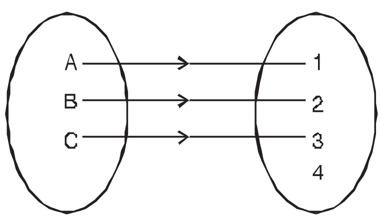


Fig.15.27

The domain is $\{A, B, C\}$

The co-domain is $\{1, 2, 3, 4\}$

The range is $\{1, 2, 3\}$

A function can map more than one element of the set A to the same element of the set B. Such a type of function is said to be ***many-to-one.***

Many-to-one function

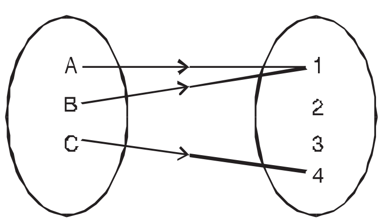


Fig. 15.28

The domain is $\{A, B, C\}$

The co-domain is $\{1, 2, 3, 4\}$

The range is $\{1, 4\}$

A function which is both one-to-one and onto is said to be a bijective function.

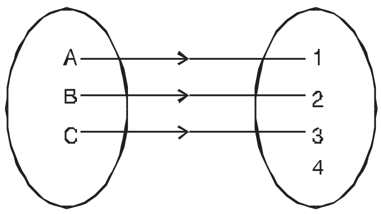


Fig. 15.29

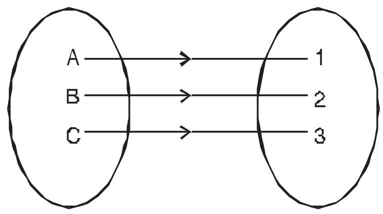


Fig. 15.30

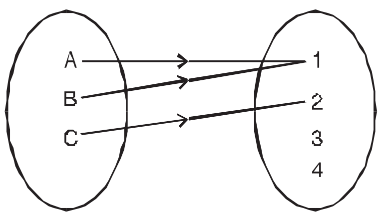


Fig. 15.31

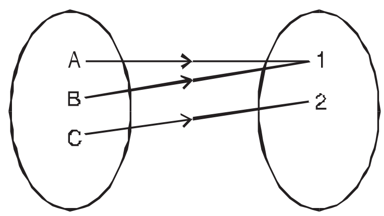


Fig. 15.32

Fig. 15.29 shows a one-to-one function mapping $\{A, B, C\}$ into $\{1, 2, 3, 4\}$.

Fig. 15.30 shows a one-to-one function mapping $\{A, B, C\}$ onto $\{1, 2, 3\}$.

Fig. 15.31shows a many-to-one function mapping $\{A, B, C\}$ into $\{1, 2, 3, 4\}$.

Fig. 15.32 shows a many-to-one function mapping $\{A, B, C\}$ onto $\{1, 2\}$.

Function shown in Fig. 15.30 is also a bijective Function.

Note : Relations which are one-to-many can occur, but they are not functions. The following figure illustrates this fact.

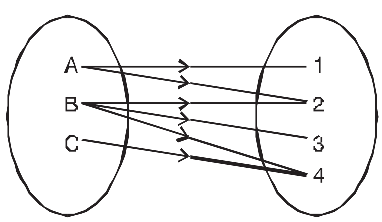


Fig. 15.33

Example 15.28 Without using graph prove that the function

$F : R \to R$ defiend by $f(x) = 4 + 3x$ is ***one-to-one***.

Solution : For a function to be one-one function

$$F(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall \quad x_1, x_2 \in \text{domain}$$

$\therefore$ Now $f(x_1) = f(x_2)$ gives

$4 + 3x_1 = 4 + 3x_2$ or $x_1 = x_2$

$\therefore$ F is a ***one-one function***.

Example 15.29 Prove that

F : R $\to$ R defined by $f(x) = 4x^3 - 5$ is a bijection

Solution : Now $f(x_1) = f(x_2) \quad \forall\ x_1, x_2$ Domain

$\therefore \quad 4x_1^3 - 5 = 4x_2^3 - 5$

$\Rightarrow \quad x_1^3 = x_2^3$

$\Rightarrow \quad x_1^3 - x_2^3 = 0 \Rightarrow \quad (x_2 - x_1)\left(x_1^2 + x_1x_2 + x_2^2\right) = 0$

$\Rightarrow \quad x_1 = x_2$ or

$x_1^2 + x_1x_2 + x_2^2 = 0$ (rejected). It has no real value of x_1 and x_2.

$\therefore$ F is a ***one-one function***.

Again let $y = (x)$ where $y \in$ codomain, $x \in$ domain.

We have $y = 4x^3 - 5$ or $x = \left(\dfrac{y+5}{4}\right)^{1/3}$

$\therefore$ For each $y \in$ codomain $\exists\ x \in$ domain such that $f(x) = y$.

Thus F is ***onto function***.

$\therefore$ F is a bijection.

Example 15.30 Prove that F : R $\to$ R defined by $F(x) = x^2 + 3$ is neither ***one-one*** nor ***onto function***.

Solution : We have $F(x_1) = F(x_2) \quad \forall\ x_1, x_2$ domain giving

$x_1^2 + 3 = x_2^2 + 3 \quad \Rightarrow x_1^2 = x_2^2$

or $\quad x_1^2 - x_2^2 = 0 \quad \Rightarrow x_1 = x_2 \quad$ or $\quad x_1 = -x_2$

or F is not ***one-one function***.

Again let $y = F(x)$ where $y \in$ codomain

$x \in$ domain.

$\Rightarrow \quad y = x^2 + 3 \quad \Rightarrow \quad x = \pm\sqrt{y - 3}$

$\Rightarrow \quad \forall\ y < 3$ no real value of x in the domain.

$\therefore$ F is not an ***onto finction***.

15.16 GRAPHICAL REPRESENTATION OF FUNCTIONS

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. For example, consider $y = x^2$.

$$y = x^2$$

x	0	1	–1	2	–2	3	–3	4	–4
y	0	1	1	4	4	9	9	16	16

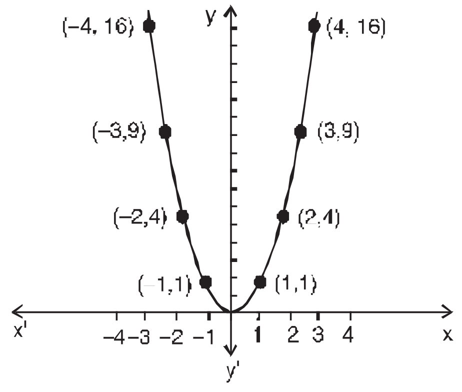


Fig. 15.34

Does this represent a function?

Yes, this represent a function because corresponding to each value of $x\ \exists$ a unique value of y.

Now consider the equation $x^2 + y^2 = 25$

$$x^2 + y^2 = 25$$

x	0	0	3	3	4	4	5	–5	–3	–3	–4	–4
y	5	–5	4	–4	3	–3	0	0	4	–4	3	–3

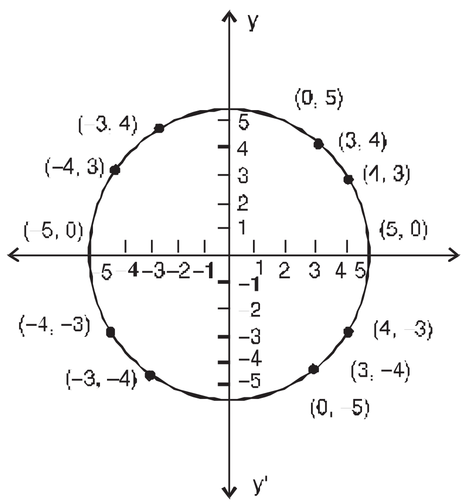


Fig. 15.35

This graph represents a circle.

Does it represent a function ?

No, this does not represent a function because corresponding to the same value of x, there does not exist a unique value of y.

CHECK YOUR PROGRESS 15.7

1. (i) Does the graph represent a function?

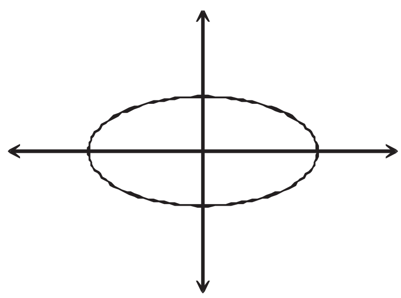


Fig. 15.36

(ii) Does the graph represent a function ?

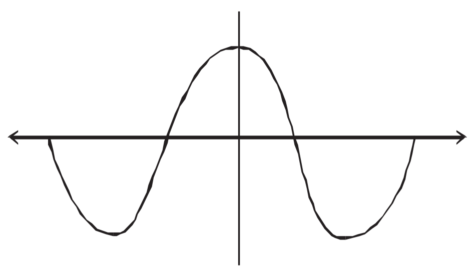


Fig. 15.37

2. Which of the following functions are into function ?

(a)

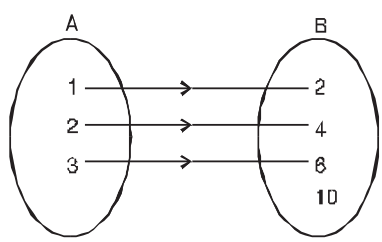


Fig.15.38

(b) $f : N \to N,$ defined as $f(x) = x^2$

Here N represents the set of natural numbers.

(c) $f : N \to N,$ defined as $f(x) = x$

3. Which of the following functions are onto function if $f : R \to R$

(a) $f(x) = 115x + 49$ (b) $f(x) = |x|$

4. Which of the following functions are one-to-one functions ?

(a) $f : \{20,21,22\} \to \{40,42,44\}$ defined as $f(x) = 2x$

(b) $f : \{7,8,9\} \to \{10\}$ defined as $f(x) = 10$

(c) $f : I \to R$ defined as $f(x) = x^3$

(d) $f : R \to R$ defined as $f(x) = 2 + x^4$

(d) $f : N \to N$ defined as $f(x) = x^2 + 2x$

5. Which of the following functions are many-to-one functions ?

(a) $f : \{-2,-1,1,2\} \to \{2,5\}$ defined as $f(x) = x^2 + 1$

(b) $f : \{0,1,2\} \to \{1\}$ defined as $f(x) = 1$

(c)

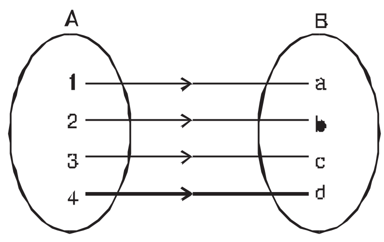


Fig.15.39

(d) $f : N \to N$ defined as $f(x) = 5x + 7$

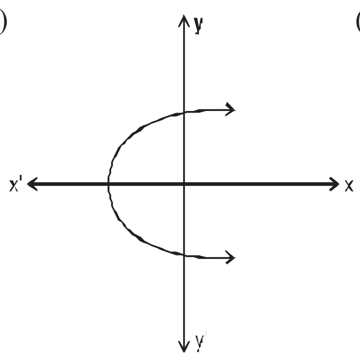
6. Draw the graph of each of the following functions :

(a) $y = 3x^2$ (b) $y = -x^2$ (c) $y = x^2 - 2$

(d) $y = 5 - x^2$ (e) $y = 2x^2 + 1$ (f) $y = 1 - 2x^2$

7. Which of the following graphs represents a function ?

(a)



(b)

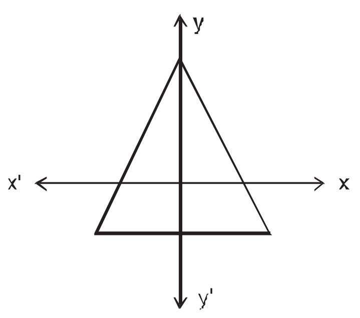


Fig. 15.40 **Fig. 15.41**

MODULE - IV
Functions

Notes

(c)

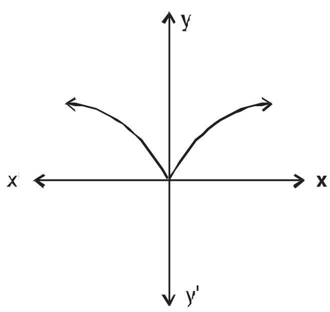


Fig. 15.42

(d)

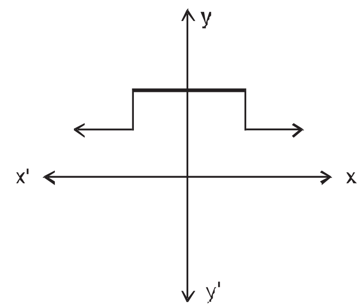


Fig. 15.43

(e)

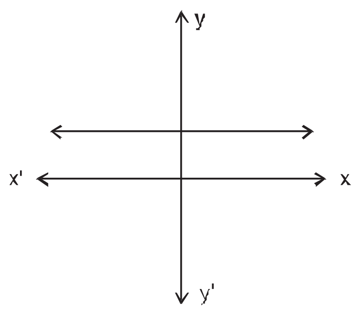


Fig .15.44

Hint : If any line || to y-axis cuts the graph in more than one point, graph does not represent a function.

15.17 SOME SPECIAL FUNCTIONS

15.17.1 Monotonic Function

Let $F : A \to B$ be a function then F is said to be monotonic on an interval (a,b) if it is either increasing or decreasing on that interval.

For function to be increasing on an interval (a,b)

$$x_1 < x_2 \Rightarrow F(x_1) < F(x_2) \quad \forall\, x_1\, x_2 \in (a, b)$$

and for function to be decreasing on (a,b)

$$x_1 < x_2 \Rightarrow F(x_1) > F(x_2) \quad \forall\, x_1\, x_2 \in (a, b)$$

A function may not be monotonic on the whole domain but it can be on different intervals of the domain.

Consider the function $F : R \to R$ defined by $f(x) = x^2$.

Now $\forall\ x_1, x_2 \in [0, \infty]$

$$x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$$

$\Rightarrow$ F is a ***Monotonic Function*** on $[0, \infty]$.

($\because$ It is only increasing function on this interval)

But $\quad \forall\ x_1, x_2 \in (-\infty, 0)$

$$x_1 < x_2 \Rightarrow F(x_1) > F(x_2)$$

$\Rightarrow$ F is a ***Monotonic Function*** on $[-\infty, 0]$

($\because$ It is only a decreasing function on this interval)

Therefore if we talk of the whole domain given function is not monotonic on R but it is monotonic on $(-\infty, 0)$ and $(0, \infty)$.

Again consider the function $F : R \to R$ defined by $f(x) = x^3$.

Clearly $\forall\ x_1\ x_2 \in$ domain

$$x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$$

$\therefore$ Given function is ***monotonic*** on R i.e. on the whole domain.

15.17.2 Even Function

A function is said to be an even function if for each x of domain

$F(-x) = F(x)$

For example, each of the following is an ***even function***.

(i) If $F(x) = x^2$ then $F(-x) = (-x)^2 = x^2 = F(x)$

(ii) If $F(x) = \cos x$ then $F(-x) = \cos(-x) = \cos x = F(x)$

(iii) If $F(x) = |x|$ then $F(-x) = |-x| = |x| = F(x)$

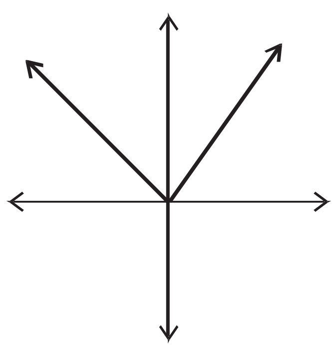


Fig. 15.45

The graph of this even function (modulus function) is shown in the figure above.

Observation

Graph is symmetrical about y-axis.

15.17.3 Odd Function

A function is said to be an odd function if for each x

$$f(-x) = -f(x)$$

For example,

(i) If $f(x) = x^3$

then $f(-x) = (-x)^3 = -x^3 = -f(x)$

(ii) If $f(x) = \sin x$

then $f(-x) = \sin(-x) = -\sin x = -f(x)$

Graph of the odd function y = x is given in Fig.15.46

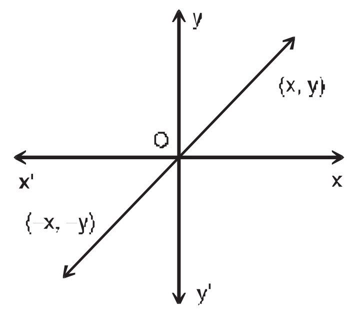


Fig. 15.46

Observation

Graph is symmetrical about origin.

15.17.4 Greatest Integer Function (Step Function)

$f(x) = [x]$ which is the greatest integer less than or equal to x.

$f(x)$ is called Greatest Integer Function or Step Function. Its graph is in the form of steps, as shown in Fig. 16.47.

Let us draw the graph of $y = [x], x \in R$

$[x] = 1, \quad 1 \le x < 2$

$[x] = 2, \quad 2 \le x < 3$

$[x] = 3, \quad 3 \le x < 4$

$[x] = 0, \quad 0 \le x < 1$

$[x] = -1, \quad -1 \le x < -0$

$[x] = -2, \quad -2 \le x < -1$

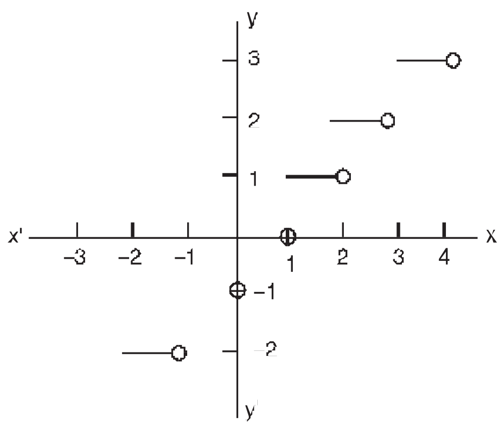


Fig. 15.47

- Domain of the step function is the set of real numbers.
- Range of the step function is the set of integers.

15.17.5 Polynomial Function

Any function defined in the form of a polynomial is called a polynomial function.

For example,

(i) $f(x) = 3x^2 - 4x - 2$

(ii) $f(x) = x^3 - 5x^2 - x + 5$

(iii) $f(x) = 3$

are all polynomial functions.

Note : Functions of the type $f(x) = k$, where k is a constant is also called a constant function.

15.17.6 Rational Function

Function of the type $f(x) = \frac{g(x)}{h(x)}$, where $h(x) \neq 0$ and $g(x)$ and $h(x)$ are polynomial functions are called rational functions.

For example, $f(x) = \frac{x^2 - 4}{x + 1}, x \neq -1$

is a rational function.

15.17.7 Reciprocal Function

Functions of the type $y = \frac{1}{x}, x \neq 0$ is called a reciprocal function.

15.17.8 Exponential Functions

A swiss mathematician Leonhard Euler introduced a number e in the form of an infinite series. In fact

$$e = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots + \frac{1}{\underline{|n}} + \ldots \qquad \ldots..(1)$$

It is well known that the sum of its infinite series tends to a finite limit (i.e., this series is convergent) and hence it is a positive real number denoted by e. This number e is a transcendental irrational number and its value lies between 2 an 3.

Consider now the infinite series

$$1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots + \frac{x^n}{\underline{|n}} + \ldots$$

It can be shown that the sum of its infinite series also tends to a finite limit, which we denote by e^x. Thus,

$$e^x = 1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots + \frac{x^n}{\underline{|n}} + \ldots \qquad \ldots...(2)$$

This is called the **Exponential Theorem** and the infinite series is called the **exponential series**. We easily see that we would get (1) by putting x = 1 in (2).

The function $f(x) = e^x$, where x is any real number is called an **Exponential Function**.

The graph of the exponential function

$$y = e^x$$

MODULE - IV
Functions

Notes

is obtained by considering the following important facts :

(i) As x increases, the y values are increasing very rapidly, whereas as x decreases, the y values are getting closer and closer to zero.

(ii) There is no x-intercept, since $e^x \neq 0$ for any value of x.

(iii) The y intercept is 1, since $e^0 = 1$ and $e \neq 0$.

(iv) The specific points given in the table will serve as guidelines to sketch the graph of e^x (Fig. 15.48).

x	−3	−2	−1	0	1	2	3
$y = e^x$	0.04	0.13	0.36	1.00	2.71	7.38	20.08

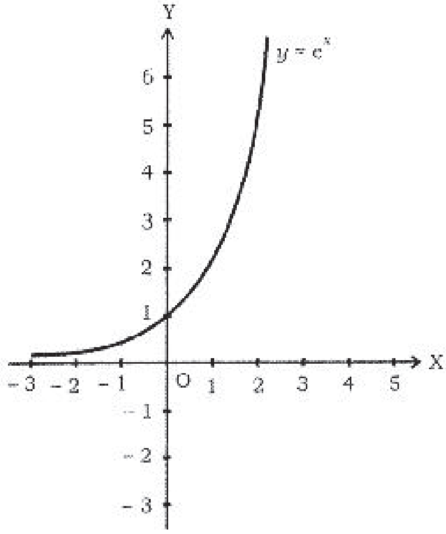


Fig. 15.48

If we take the base different from e, say a, we would get exponential function

$$f(x) = a^x \text{ ,provbided } a > 0, \quad a \neq 1.$$

For example, we may take a = 2 or a = 3 and get the graphs of the functions

$y = 2^x$ (See Fig. 15.49)

and $y = 3^x$ (See Fig. 15.50)

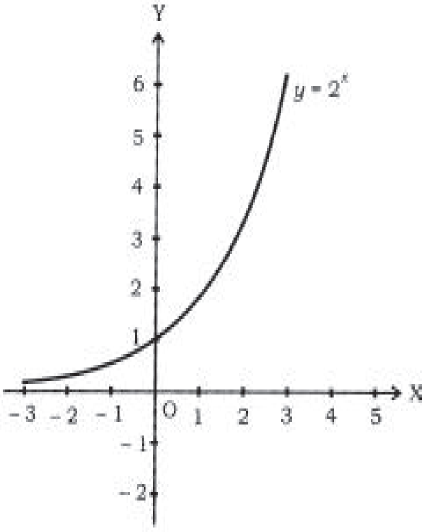


Fig. 15.49

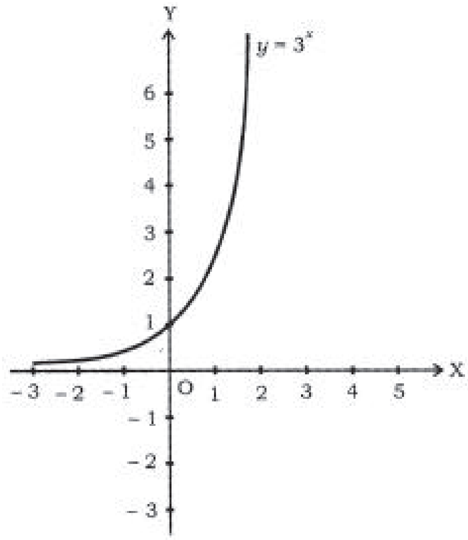


Fig. 15.50

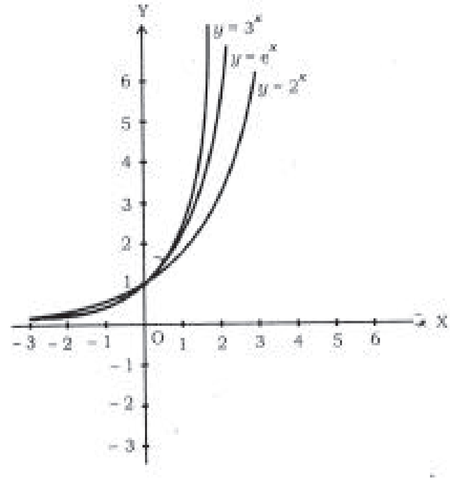


Fig. 15.51

15.17.9 Logarithmic Functions

Consider now the function

$$y = e^x \qquad .....(3)$$

We write it equivalently as

$$x = \log_e y$$

Thus, $\quad y = \log_e x \qquad .....(4)$

is the inverse function of $y = e^x$

The base of the logarithm is not written if it is e and so $\log_e x$ is usually written as log x.

As $y = e^x$ and $y = \log x$ are inverse functions, their graphs are also symmetric w.r.t. the line $y = x$

The graph of the function $y = \log x$ can be obtained from that of $y = e^x$ by reflecting it in the line y = x.

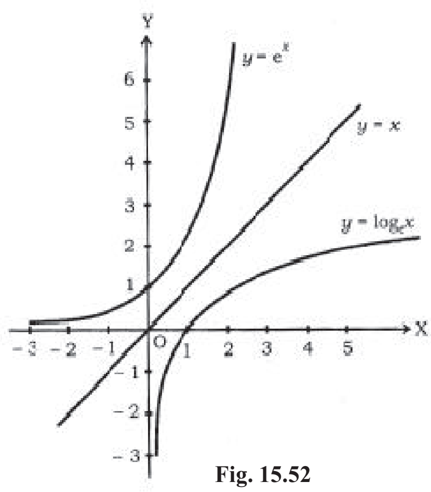


Fig. 15.52

Note

(i) The learner may recall the laws of indices which you have already studied in the Secondary Mathematics :

If a > 0, and m and n are any rational numbers, then

$$a^m \cdot a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$a^0 = 1$$

(ii) The corresponding laws of logarithms are

$$\log_a(mn) = \log_a m + \log_a n$$

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a(m^n) = n\log_a m$$

$$\log_b m = \frac{\log_a m}{\log_a b}$$

or $\log_b m = \log_a m \log_b a$

Here a, b > 0, $a \neq 1$, $b \neq 1$.

CHECK YOUR PROGRESS 15.8

1. Tick mark the correct statement.

   (i) Function $f(x) = 2x^4 + 7x^2 + 9x$ is an even function.

   (ii) Odd function is symmetrical about y-axis.

   (iii) $f(x) = x^{1/2} - x^3 + x^5$ is a polynomial function.

   (iv) $f(x) = \dfrac{x-3}{3+x}$ is a rational function for all $x \in R$.

   (v) $f(x) = \dfrac{\sqrt{5}}{3}$ is a constant function.

   (vi) $f(x) = \dfrac{1}{x}$

   Domain of the function is the set of real numbers except 0.

   (vii) Greatest integer function is neither even nor odd.

2. Which of the following functions are even or odd functions ?

   (a) $f(x) = \dfrac{x^2-1}{x+1}$ (b) $f(x) = \dfrac{x^2}{5+x^2}$ (c) $f(x) = \dfrac{1}{x^2+5}$

   (d) $f(x) = \dfrac{2}{x^3}$ (e) $f(x) = \dfrac{x}{x^2+1}$ (f) $f(x) = \dfrac{5}{x-5}$

(g) $f(x) = \frac{x-3}{3+x}$ (h) $f(x) = x - x^3$

3. Draw the graph of the function $y = [x] - 2$.

4. Specify the following functions as polynomial function, rational function, reciprocal function or constant function.

(a) $y = 3x^8 - 5x^7 + 8x^5$ (b) $y = \frac{x^2 + 2x}{x^3 - 2x + 3}, \; x^3 - 2x + 3 \neq 0$

(c) $y = \frac{3}{x^2}, \; x \neq 0$ (d) $y = 3 + \frac{2x+1}{x}, \; x \neq 0$

(e) $y = 1 - \frac{1}{x}, \; x \neq 0$ (f) $y = \frac{x^2 - 5x + 6}{x - 2}, \; x \neq 2$

(g) $y = \frac{1}{9}$.

15.18 COMPOSITION OF FUNCTIONS

Consider the two functions given below:

$$y = 2x + 1, \quad x \in \{1,2,3\}$$
$$z = y + 1, \quad y \in \{3,5,7\}$$

Then *z* is the composition of two functions *x* and *y* because *z* is defined in terms of *y* and *y* in terms of *x*.

Graphically one can represent this as given below :

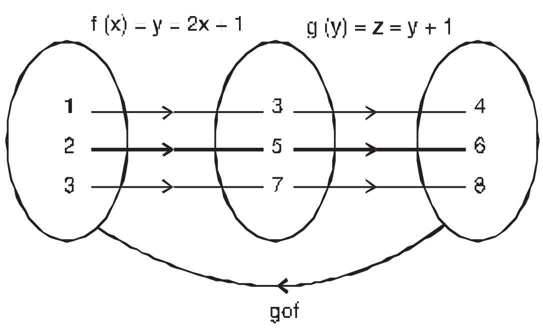


Fig. 15.53

The composition, say, gof of function g and f is defined as function g of function f.

If $f : A \to B$ and $g : B \to C$

then $g \circ f : A$ to C

Let $f(x) = 3x + 1$ and $g(x) = x^2 + 2$

Then $fog(x) = f(g(x))$

$$= f\left(x^2+2\right)$$

$$= 3\left(x^2+2\right)+1 = 3x^2+7 \qquad \text{(i)}$$

and $\quad (gof)(x) = g(f(x))$

$$= g(3x+1)$$

$$= (3x+1)^2+2 = 9x^2+6x+3 \qquad \text{(ii)}$$

Check from (i) and (ii), if

$$fog = gof$$

Evidently, $\quad fog \neq gof$

Similarly, $(fof)(x) = f(f(x)) = f(3x+1)$ [Read as function of function f].

$$= 3(3x+1)+1$$

$$= 9x+3+1 = 9x+4$$

$(gog)(x) = g(g(x)) = g\left(x^2+2\right)$ [Read as function of function g]

$$= \left(x^2+2\right)^2+2$$

$$= x^4+4x^2+4+2$$

$$= x^4+4x^2+6$$

Example 15.31 If $f(x) = \sqrt{x+1}$ and $g(x) = x^2+2$, calculate *fog* and *gof*.

Solution : $\quad fog(x) = f(g(x))$

$$= f\left(x^2+2\right)$$

$$= \sqrt{x^2+2+1}$$

$$= \sqrt{x^2+3}$$

$$(gof)(x) = g(f(x))$$

$$= g\left(\sqrt{x+1}\right)$$

$$= \left(\sqrt{x+1}\right)^2+2$$

$$= x+1+2$$

$$= x+3.$$

Here again, we see that $(fog) \neq gof$

Example 15.32 If $f(x) = x^3,\ f: R \to R$

$$g(x) = \frac{1}{x},\quad g: R-\{0\} \to R-\{0\}$$

Find *fog* and *gof*.

Solution : $(fog)(x) = f(g(x))$

$$= f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 = \frac{1}{x^3}$$

$$(gof)(x) = g(f(x))$$

$$= g(x^3) = \frac{1}{x^3}$$

Here we see that $fog = gof$

Note : We observe from Example 15.31 and Example 15.32 that *fog* and *gof* may or may not be equal.

CHECK YOUR PROGRESS 15.9

1. Find *fog, gof, fof* and *gog* for the following functions :

   $f(x) = x^2 + 2,\ g(x) = 1 - \frac{1}{1-x},\ x \neq 1.$

2. For each of the following functions write *fog, gof, fof* and *gog*.

   (a) $f(x) = x^2 - 4,\ g(x) = 2x + 5$

   (b) $f(x) = x^2,\ g(x) = 3$

   (c) $f(x) = 3x - 7,\ g(x) = \frac{2}{x},\ x \neq 0$

3. Let $f(x) = |x|,\ g(x) = [x]$. Verify that $fog \neq gof$.

4. Let $f(x) = x^2 + 3,\ g(x) = x - 2$

   Prove that $fog \neq gof$ and $f\left(f\left(\frac{3}{2}\right)\right) = g\left(f\left(\frac{3}{2}\right)\right)$

5. If $f(x) = x^2,\ g(x) = \sqrt{x}$. Show that $fog = gof$.

6. Let $f(x) = |x|,\ g(x) = (x)^{\frac{1}{3}},\ h(x) = \frac{1}{x};\ x \neq 0$.

   Find (a) *fog* (b) *goh* (c) *foh* (d) *hog* (e) *fogoh*

   $\left\{ \textbf{Hint : } (fogoh)(x) = f(g(h(x))) = f\left(g\left(\frac{1}{x}\right)\right) \right\}$

MODULE - IV
Functions

Notes

15.19 INVERSE OF A FUNCTION

(A) Consider the relation

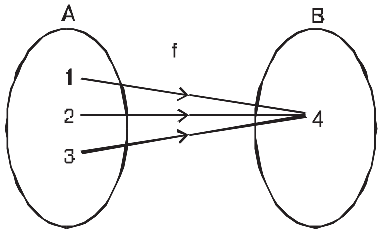


Fig. 15.54

This is a many-to-one function. Now let us find the inverse of this relation. Pictorially, it can be represented as

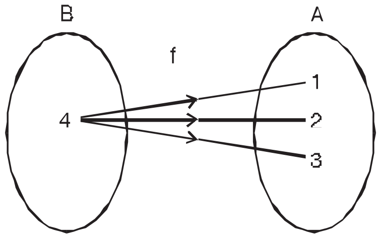


Fig 15.55

Clearly this relation does not represent a function. (Why ?)

(B) Now take another relation

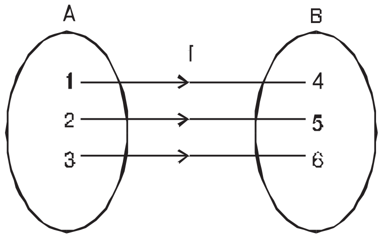


Fig.15.56

It represents one-to-one onto function. Now let us find the inverse of this relation, which is represented pictorially as

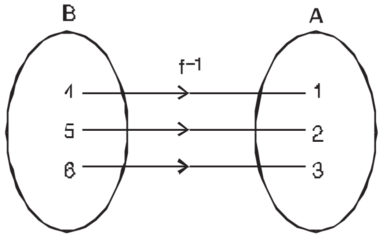


Fig. 15.57

This represents a function.

(C) Consider the relation

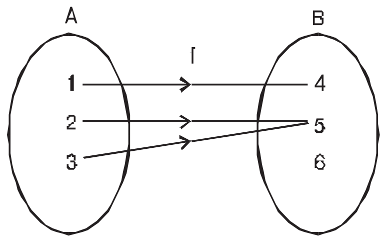


Fig. 15.58

Ir represents many-to-one function. Now find the inverse of the relation.

Pictorially it is represented as

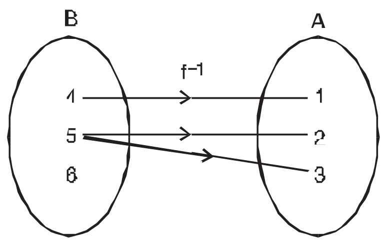


Fig. 15.59

This does not represent a function, because element 6 of set B is not associated with any element of A. Also note that the elements of B does not have a unique image.

(D) Let us take the following relation

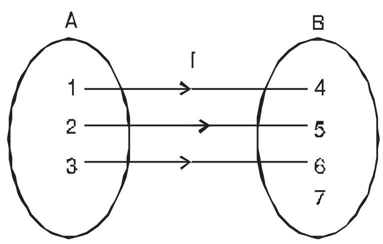


Fig. 15.60

It represent one-to-one into function.

Find the inverse of the relation.

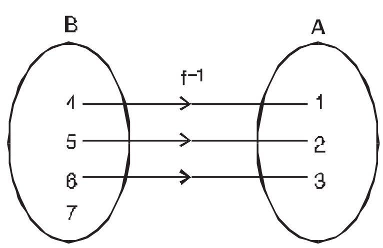


Fig. 15.61

It does not represent a function because the element 7 of *B* is not associated with any element of *A*. From the above relations we see that we may or may not get a relation as a function when we find the inverse of a relation (function).

We see that the inverse of a function exists only if the function is one-to-one onto function i.e. only if it is a bijective function.

CHECK YOUR PROGRESS 15.10

1 (i) Show that the inverse of the function

$$y = 4x - 7 \text{ exists.}$$

(ii) Let f be a one-to-one and onto function with domain A and range B. Write the domain and range of its inverse function.

2. Find the inverse of each of the following functions (if it exists) :

(a) $f(x) = x + 3 \quad \forall x \in R$

(b) $f(x) = 1 - 3x \quad \forall x \in R$

(c) $f(x) = x^2 \quad \forall x \in R$

(d) $f(x) = \dfrac{x+1}{x}, \quad x \neq 0 \quad x \in R$

LET US SUM UP

- Set is a well defined collection of objects.
- To represent a set in Roster form all elements are to be written but in set builder form a set is represented by the common property.
- If the elements of a set can be counted then it is called a finite set and if the elements cannot be counted, it is infinite.
- If each element of set *A* is an element of set *B* also then *A* is called sub set of *B*.
- For two sets *A* and *B*, $A - B$ is a set of those elements which are in *A* but not in *B*.
- Complement of a set *A* is a set of those elements which are in the universal set but not in *A*. i.e. $A^c = U - A$
- Intersection of two sets is a set of those elements when belong to both the sets.
- Union of two sets is a set of those elements which belong to either of the two sets.
- Cartesian product of two sets *A* and *B* is the set of all ordered pairs of the elements of *A* and *B*. It is denoted by $A \times B$. i.e.

  $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.
- Relation is a sub set of A×B where *A* and *B* are sets.

  i.e. $R \subseteq A \times B = \{(a,b) : a \in A \text{ and } b \in B \text{ and } aRb\}$

- Function is a special type of relation.
- Functions $f : A \to B$ is a rule of correspondence from *A* to *B* such that to every element of *A* $\exists$ a unique element in *B*.
- Functions can be described as a set of ordered pairs.
- Let f be a function from *A* to *B*.

  Domain : Set of all first elements of ordered pairs belonging to *f*.

  Range : Set of all second elements of ordered pairs belonging to *f*.
- Functions can be written in the form of equations such as $y = f(x)$

  where x is independent variable, y is dependent variable.

  Domain : Set of independent variable.

  Range : Set of dependent variable.

  Every equation does not represent a function.
- **Vertical line test :** To check whether a graph is a function or not, we draw a line parallel to y-axis. If this line cuts the graph in more than one point, we say that graph does not represent a function.
- Let f be a function from a set A to a set B. Symbolically we can write it as

  $f : A \to B$

  (i) If every element of B is not an image of some element of A then f is said to be into function. In this case, range $\subset$ co-domain.

  (ii) If range = co-domain, then f is said to be onto function.

  (iii) If distinct elements of set A have distinct images in set B then f is called one-to-one function.

  (iv) If many elements in the domain of a function have the same image element in the range, then the function is called many-to-one function.
- **Horizontal Line Test :** To check the one-to-oneness of a function, draw a line parallel, to x-axis. If it cuts the graph at one point, we say that it is one-to-one function.
- A function is said to be monotonic on an interval if it is either increasing or decreasing on that interval.
- A function is called even function if $f(x) = f(-x)$, and odd function if

  $f(-x) = -f(x),\ x - x \in D_f$
- Inverse of a function exists if it is one-to-one and onto.

SUPPORTIVE WESITES

- *http://www.wikipedia.org*
- *http://mathworld.walfram.com*

TERMINAL EXERCISE

1. Which of the following statements are true or false :

(i) $\{1, 2, 3\} = \{ 1, \{2\}, 3\}$ (ii) $\{1, 2, 3 \} = \{3, 1, 2\}$

(iii) $\{a, e, 0 \} = \{a, b, c\}$ (iv) $\{\phi\} = \{ \}$

2. Write the set in Roster form represented by the shaded portion in the following.

(i) $A = \{1, 2, 3, 4, 5 \}$

$B = \{ 5, 6, 7, 8, -9\}$

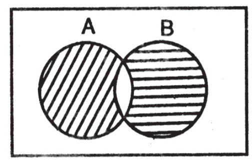


Fig. 15.62

(ii) $A = \{1, 2, 3, 4, 5, 6 \}$

$B = \{ 2, 6, 8, 10, 12 \}$

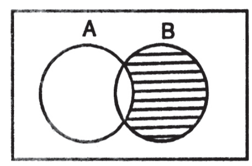


Fig. 15.63

3. Represent the follwoing using Venn diagram.

(i) $(A \cup B)'$ provided A and B are not disjoint sets.

(ii) $(A \cap B)'$ provided A and B are disjoint sets.

(iii) $(A - B)'$ provided A and B are not disjoint sets.

4. Let $U = \{1,2,3,4,5,6, 7,8,9,10\}$, $A\{2,4,6,8\}$, $B = \{1,3,5, 7\}$

Verify that

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

(iii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

5. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

$A = \{1, 3, 5, 7, 9\}$. $B = \{2, 4, 6, 8, 10\}$,

$C = \{1, 2, 3\}$.

Find (i) $A' \cap (B - C)$. (ii) $A \cup (B \cup C)$

(iii) $A' \cap (B \cup C)'$ (iv) $(A \cap B)' \cup C'$

6. What does the shaded ponion represent in each of the following Venn diagrams:

(i)

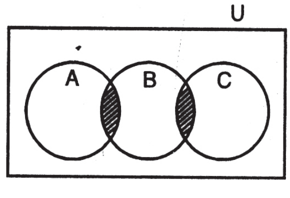


Fig. 15.64

(ii)

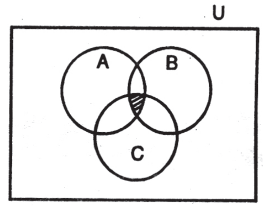


Fig. 15.65

Notes

(iii)

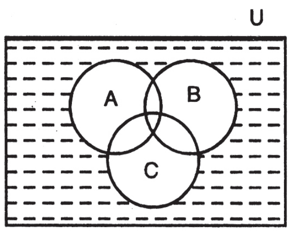


Fig. 15.66

(iv)

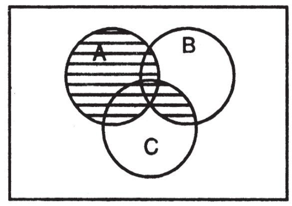


Fig. 15.67

7. Draw Venn diagram for the following:

   (i) $A' \cap (B \cup C)$ (ii) $A' \cap (C - B)$

   Where A, B, C are not disjoint sets and are subjects of the universal set U.

8. Given A = {a, b, c,}, B = { 2, 3 } .

   Find the number of relations from A to B.

9. Given that A = { 7,8,9} , B = {9, 10,11 }, C = {11, 12} verify that

   (i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

   (ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

10. Which of the following equations represent functions?

    In each of the case $x \in R$

    (a) $y = \frac{2x+3}{4-5x}, x \neq \frac{4}{5}$ (b) $y = \frac{3}{x}, x \neq 0$

    (c) $y = \frac{3}{x^2-16}, x \neq 4, -4$ (d) $y = \sqrt{x-1}, x \geq 1$

    (e) $y = \frac{1}{x^2+1}$ (f) $x^2 + y^2 = 25$

11. Write domain and range of the following functions :

    f_t : {(0, 1),(2, 3),(4, 5)(6, 7), ... (100, 101)}

    f_2 : {(−2, 4), (−4, 16), (−6, 36)}

    $f_3 : \left\{(1,1), \left(\frac{1}{2}, -1\right), \left(\frac{1}{3}, 1\right)\left(\frac{1}{4}, -1\right), ...\right\}$

    f_4 : {...... (3, 0), (−1, 2) , (4, −1) }

    f_5 : {...... (−3, 3),(−2, 2),(−1, 1) (0, 0) (1, l), (2, 2), ... }

12. Write domain of the following fiJnctions:

(a) $f(x) = x^3$

(b) $f(x) = \dfrac{1}{x^2 - 1}$

(c) $f(x) = \sqrt{3x+1}$

(d) $f(x) = \dfrac{1}{\sqrt{(x+1)(x+6)}}$

(e) $f(x) = \dfrac{1}{\sqrt{(x-1)(5x-5)}}$

13. Write range of each of the following fimctions :

(a) $y = 3x + 2, x \in R$

(b) $y = \dfrac{1}{x-2}, x \in R - \{2\}$

(c) $y = \dfrac{x-1}{x+1}, x \in \{0, 2, 3, 5, 7, 9\}$

(d) $y = \dfrac{2}{\sqrt{x}}, x \in R^+$(All non-negative real values)

14. Draw the graph of each of the following fimctions :

(a) $y = x^2 + 3, x \in R$

(b) $y = \dfrac{1}{x-2}, x \in R - \{2\}$

(c) $y = \dfrac{x-1}{x+1}, x \in \{0, 2, 3, 5, 7,9\}$

(d) $y = \dfrac{2}{\sqrt{x}}, x \in R^+$.

15. Which of the following graphs represent a fimction ?

(a)

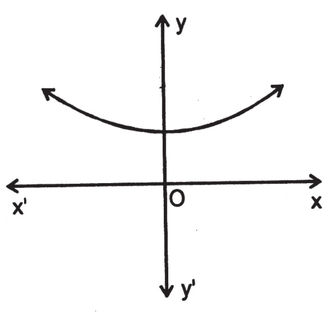


Fig. l5.68

(b)

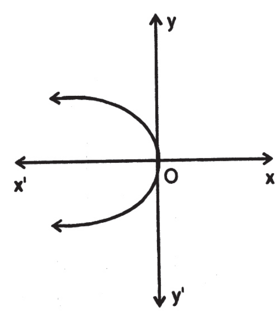


Fig.l5.69

(c)

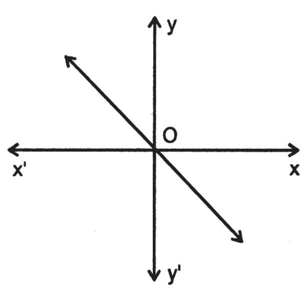


Fig. 15.70

(d)

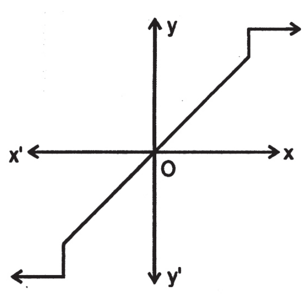


Fig. 15.71

(e)

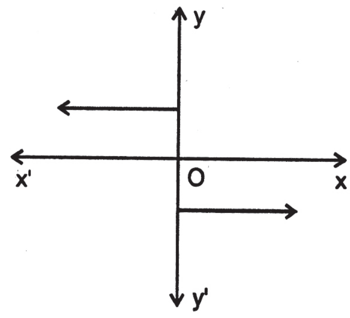


Fig. 15.72

(f)

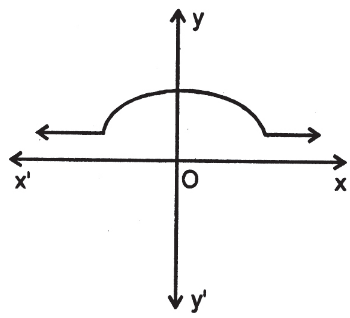


Fig. 15.73

(g)

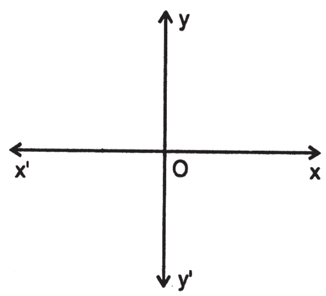


Fig. 15.74

(h)

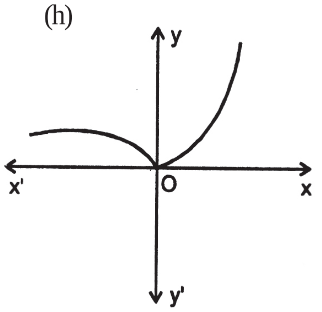


Fig. 15.75

16. Which of the following functions are one-to-one or many-to-one function?

(a) $y = 3x + 5, x \in R$

(b) $y = 1 - 2x^2, x \in R$

(c) $f : \{(1, 2), (2, 2), (3, 2), (4, 2)\}$

(d) $f : \{1, 2, 3\} \to \{4, 5, 6, 8\}$ such that $f(1) = 4, f(2) = 6, f(3) = 8$.

17. Which of the following functions are rational functions ?

(a) $f(x) = \dfrac{2x-3}{x+2}, x \in R - \{-2\}$ (b) $f(x) = \dfrac{x}{\sqrt{x}}, x \in R^4$

(c) $f(x) = \dfrac{x+2}{x^2+4x+4}, x \in R - \{-2\}$ (d) $y = x, x \in R$

1 8. Which of the following functions are polynomial functions ?

(a) $f(x) = x^2 + \sqrt{3}x + 2$ (b) $f(x) = (x+2)^2$

(c) $f(x) = 3 - x + 2x^3 - x^4$ (d) $f(x) = \sqrt{x} + x - 5, x \geq 0$

(e) $f(x) = \sqrt{x^2 - 4}, x \notin (-2, 2)$

19. Which of the following functions are even or odd functions ?

(a) $f(x) = \sqrt{9 - x^2}\ \ x \in [-3, 3]$ (b) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

(c) $f(x) = |x|$ (d) $f(x) = x - x^5$

MODULE -IV
Functions

Notes

(e)

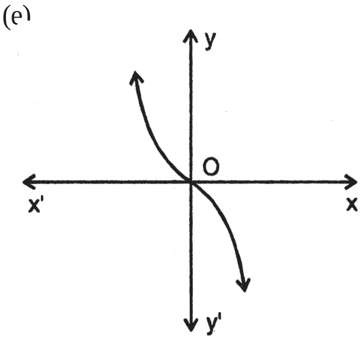


Fig. 15.76

(f)

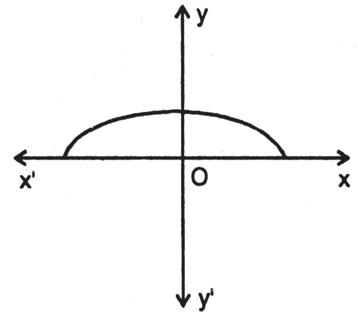


Fig. 15.77

(g)

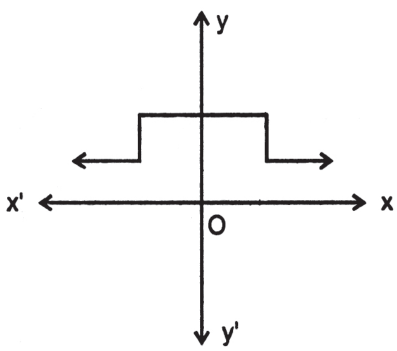


Fig. 15.78

20. Write for each of the following functions fog, gof, fof, gogo

(a) $f(x) = x^3$ $\qquad$ $g(x) = 4x - 1$

(b) $f(x) = \frac{1}{x^2}, x \neq 0$ $\qquad$ $g(x) = x^2 - 2x + 3$

(c) $f(x) = \sqrt{x-4}, x \geq 4$ $\qquad$ $g(x) = x - 4$

(d) $f(x) = x^2 - 1$ $\qquad$ $g(x) = x^2 + 1$

21. (a) Let $f(x) = |x|$, $g(x) = \frac{1}{x}, x \neq 0$, $h(x) = x^{1/3}$. Find fogoh

(b) $f(x) = x^2 + 3$, $g(x) = 2x^2 + 1$. Find fog (3) and gof(3).

22. Which of the following equations describe a function whose inverse exists :

(a) $f(x) = |x|$ $\qquad$ (b) $f(x) = \sqrt{x}, x \geq 0$

(c) $f(x) = x^2 - 1, x \geq 0$ $\qquad$ (d) $f(x) = \frac{3x-5}{4}$

(e) $f(x) = \frac{3x+1}{x-1}$ $x \neq 1$

ANSWERS

CHECK YOUR PROGRESS 15.1

1. (i), (ii), (iii) are sets.
2. (i) $\in$ (ii) $\notin$
3. (i) A = { – 5, – 4, – 3, – 2, – 1, 0}

   (ii) B = { – 1, 1}, (iii) C = { a, b, n} (iv) D = {2, 3, 5} .
4. (i) A = {x : x is a even natural number less than or equal to ten}.

   (ii) B = { x : x $\in$ N and x is a multible of 3}.

   (iii) C = {x : x is a prime number less than 10}.

   (iv) D = { x : x $\in$ R and x is a solution of $x^2 - 2 = 0$}.
5. (i) Infinite (ii) Finite (iii) Finite

   (iv) Infinite
6. (i) Null (ii) Singleton (iii) Null (iv) Null
7. (i) A $\approx$ B (ii) A $\approx$ B (iii) A = B .

CHECK YOUR PROGRESS 15.2

1. (i) $\subset$ (ii) $\not\subset$ (iii) $\in$ (iv) $\notin$
2. 4
3. (i) False (ii)True
4. (i) False (ii) True (iii) True (iv) False
5. (i) {1, 2, 3, 4} (ii) {6, 7} .
6. A e = {x : x is a odd natural number}

   Be = { x : x $\in$ N and x is not a multiple of 3 }

   C^C = {1, 2, 3, 4} .

   De = {11, 12, 13, ... }

CHECK YOUR PROGRESS 15.3

1. (i) Disjoint (ii) Not disjoint (iii) Not disjoint (iv) Disjoint
2. (i) $\{x : x \in N\}$ (ii) 4>
3. (i) {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} (ii) {5}
4. Roster from {-10, -9, -8,0, 1, 2, 3, }

   Set builder from $\{x : x \in Z \text{ and } -10 \leq x \leq oo\}$
5. (i) { x : x is a even natural number less than equal to 20}. (ii) 4>
6. (i) 4> (ii) { 1, 2, 3, 4, 5, 6, 7, 8, 9 }

   (iii) { 1, 3, 5, 7,9,10 } (iv) { 2, 4, 6, 8, 10 }
7. (i)

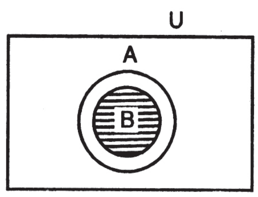


Fig. 15.79

(ii)

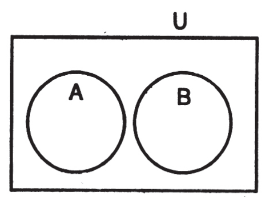


Fig. 15.80

(iii)

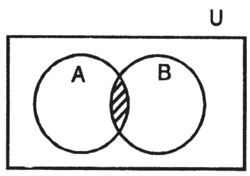


Fig. 15.81

8. (i)

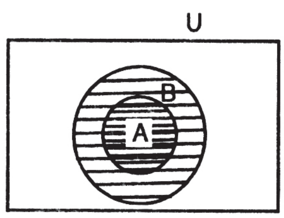


Fig.l5.82

(ii)

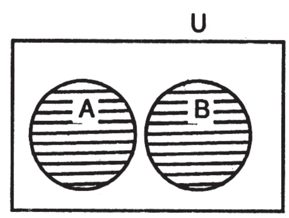


Fig. l5.83

(iii)

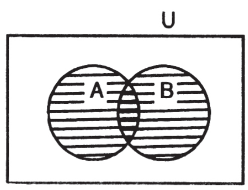
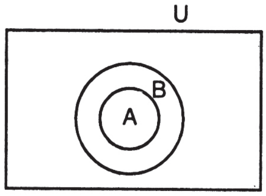
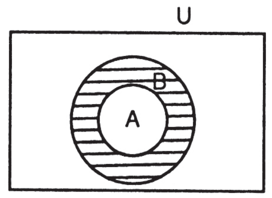


Fig. 15.84

9. (i)



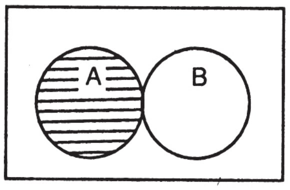
$A - B = \phi$



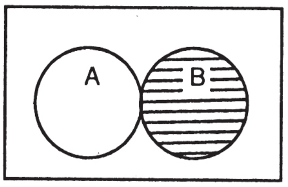
B - A' = Shaded Portion

Fig. 15.85

(ii)



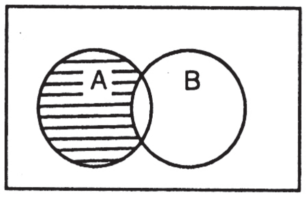
$A - B = A$



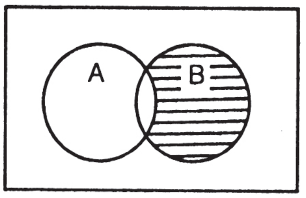
$B - A = B$

Fig. 15.86

(iii)



$A - B$ = Shaded Portion



$B - A$ = Shaded Portion

Fig. 15.87

2. {(2, 1) , (4, 1) , (2,4) , (4, 4)} .

3. (i) R = { (4, 2) ,.(4, 4) , (6, 2), (6, 3), (8,2), (8,4), (10,2), (10,5) } .

   (ii) Domain of R = { 4, 6, 8, 10 } (iii) Rangeof R = {2, 3, 4, 5}.

4. (i) R = {(1, 8), (2, 4)} (ii) Domain of R = {1, 2}.

   (iii) Range of R = {l, 2 }{ 8, 4}

5. (i) R = {(2,4),(3,9),(5, 25), (7,49), (11, 121), (13, 169)}

   Domain of R = {2, 3, 5, 7, 11, 13}, Range of R = { 4, 9, 25, 49, 121, 169}.

6. (i) Domain of R = ϕ (ii) Domain of R = ϕ (iii) Range of R = ϕ

CHECK YOUR PROGRESS 15.5

1. (a), (c), (f) 2. (a), (b) 3. (a), (d)

4. (a) Domain = $\{\sqrt{2}, \sqrt{5}, \sqrt{3}\}$, Range = {2, -1, 5}

   (b) Domain = {-3, -2, -1 },Range = $\left\{\frac{1}{2}\right\}$

   (c) Domain = {1, 0, 2, – 1} , Range = {1, 0, 2, – 1}

   (d) Domain = {Deepak, Sandeep, Rajan} , Range = {16, 28, 24} .

5. (a) Domain = {1, 2, 3}, Range = {4, 5, 6}

   (b) Domain= {1, 2, 3}, Range = {4}

   (c) Domain = {1, 2, 3 } , Range = {1, 2, 3}

   (d) Domain = {Gagan, Ram, Salil}, Range = {8, 9, 5}

   (e) Domain = {a.b, c, d} , Range = {2, 4}

CHECK YOUR PROGRESS 15.6

1. (a) (i) Domain = Set of real numbers. (ii) Domain = Set of real numbers.

   (iii) Domain = Set of a real numbers.

   (b) (i) Domain = $R - \left\{\frac{1}{3}\right\}$ (ii) Domain = $R - \left\{-\frac{1}{4}, 5\right\}$

   (iii) Domain = R – {3, 5 } (iv) Domain = R – {3, 5}

   (c) (i) Domain= $\{x \in R: x \le 6\}$ (ii) Domain = $\{x \in R : x \ge -7\}$

   (iii) Domain = $\left\{x : x \in R, x \ge -\frac{5}{3}\right\}$

   (d) (i) Domain = $\{ x : x \in R \text{ and } 3 \le x \le 5\}$

   (ii) Domain = $\{ x : x \in R\ x \ge 3, x \le -5\}$

   (iii) Domain = $\{x: x \in R\ x \ge -3, x \le -7\}$

   (iv) Domain = $\{ x : x \in R\ x \ge 3, x \le -7\}$

2. (a) (i) Range = {13, 25, 31, 7, 4} (ii) Range = {19, 9, 33, 1}

(iii) Range = { 2, 4, 8, 14, 22 }

(b) (i) Range = $\{f(x) : -2 \le f(x) \le 2\}$ (ii) Range = $\{f(x) : 1 \le f(x) \le 10\}$

(c) (i) Range = $\{f(x) : 1 \le f(x) \le 25\}$ (ii) Range = $\{f(x): -6 \le f(x) \le 6\}$

(iii) Range = $\{f(x) : 1 \le f(x) \le 5\}$ (iv) Range = $\{f(x) : 0 \le f(x) \le 5\}$

(d) (i) Range = R (ii) Range = R (iii) Range = R

(iv) Range = $\{f(x) : f(x) < 0\}$

(v) Range = $\{ f (x) : -1 \le f(x) < 0\}$

(vi) Range= $\{f(x): 0.5 \le f(x) < 0\}$ (vii) Range = $\{f(x) : f(x) > 0\}$

(viii) Range: All values of f(x) except values at $x = -5$.

CHECK YOUR PROGRESS 15.7

1. (i) No. (ii) Yes 2. (a), (b)
3. (a) 4. (a), (c), (e) 5. (a), (b)
6. (a)

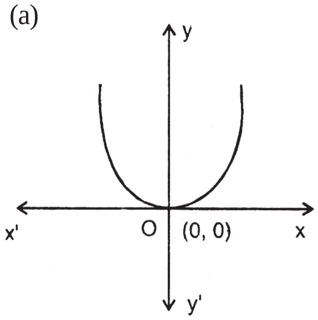


Fig. 15.88

(b)

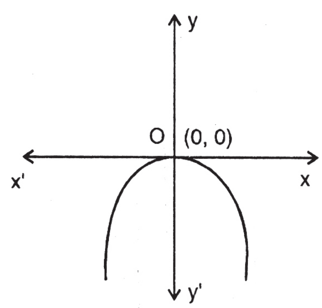


Fig. 15.89

(c)

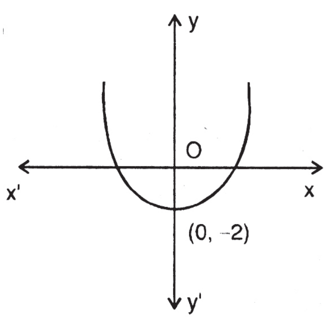


Fig. 15.90

(d)

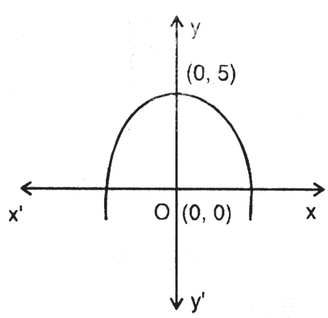


Fig. 15.91

MODULE -IV
Functions

Notes

(e)

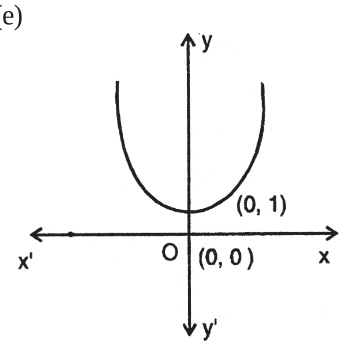


Fig. 15.92

(f)

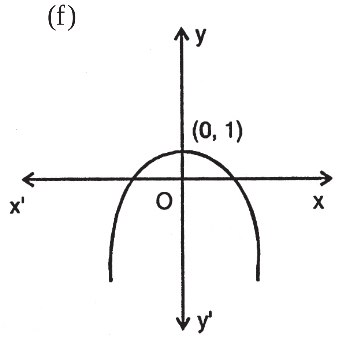


Fig. 15.93

7. (c), (d) and (e).

CHECK YOUR PROGRESS 15.8

1. v, vi, vii are true statements.

   (i), (ii), (iii), (iv) are incorrect statement.

2. (b) (c) are even functions.

   (d) (e) (h) are odd functions.

3.

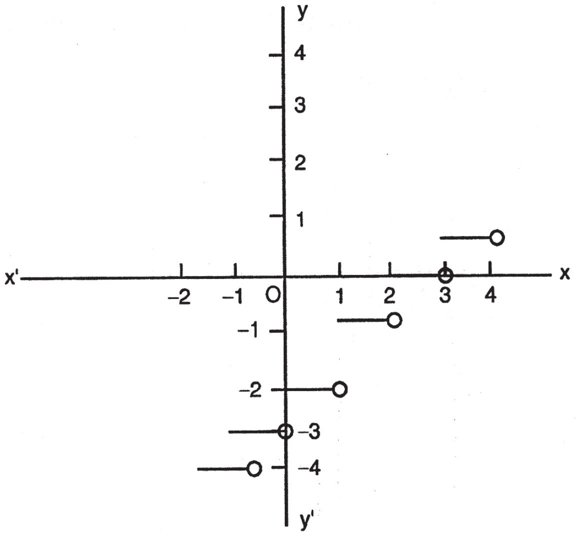


Fig. 15.94

4. (a) Polynomial function (b) Rational ftmction.

   (c) Rational function. (d) Rational function.

   (e) Rational function. (f) Rational function.

(g) Constant function

5. No

CHECK YOUR PROGRESS 15.9

1. (i) fog $= \dfrac{x^2}{(x-1)^2} + 2$ (ii) gof $= \dfrac{x^2+2}{x^2+1}$

   (iii) fof $= x^4 + 4x^2 + 6$ (iv) gog = x

2. (a) fog $= 4x^2 + 20x + 21$

   gof $= 2x^2 - 3$

   fof $= x^4 - 8x^2 + 12$

   gog $= 4x + 15$

   (b) fog = 9, gof = 3, fof $= x^4$, gog = 3

   (c) fog $= \dfrac{6-7x}{x}$, gof $= \dfrac{2}{3x-7}$, fof $= 9x - 28$, gog = x

6. (a) fog $= \left|x^{\frac{1}{3}}\right|$ (b) goh $= x^{\frac{1}{3}}$ (c) foh $= \left|\dfrac{1}{x}\right|$

   (d) hog $= \dfrac{6-7x}{x}$ (e) fogoh(l) = 1

CHECK YOUR PROGRESS 15.10

1. (ii) Domain is B. Range is A.

2. (a) $f^{-1}(x) = x - 3$ (b) $f^{-1}(x) = \dfrac{1-x}{3}$

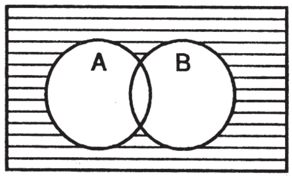
   (c) Inverse does not exist. (d) $f^{-1}(x) = \dfrac{1}{x-1}$

TERMINAL EXERCISE

1. (i) False (ii) True (iii)False (iv) False

2. (i).{ 1, 2, 3, 4, 6, 7, 8, 9} (ii) {8, 10, 12 }

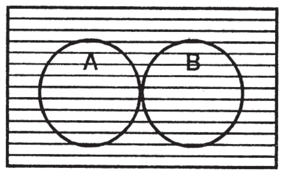
(i)



$(A \cup B)'$ = Shaded Portion

Fig. 15.95

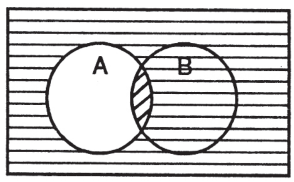
(ii)



$(A \cap B)'$ = Shaded Portion

Fig. 15.96

(iii)



$(A - B)'$ = Shaded Portion

Fig. 15.97

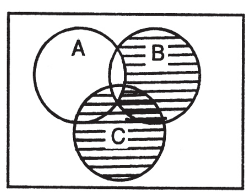
5. (i) { 4, 6, 8, 10 } (ii) {1, 2, 3, 4, 5,6, 7, 8, 9, 10}.

   (iii) ϕ (iv) {1,2,3,4,5,6, 7,8,9,10}.

6. (i) $(A \cap B) \cap (B \cap C)$ (ii) $(A \cap B) \cap C$

   (iii) $[(A \cup B) \cup C]'$ (iv) A u $(B \cap C)$.

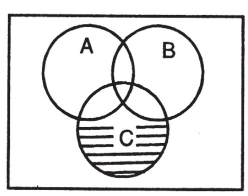
7. (i)



$A' \cap (B \cup C)$ = Shaded Portion

Fig. 15.98

(ii)



$A' \cap (C - B)$ = Shaded Portion

Fig. 15.99

8. 2^6 i.e., 64.

10. (a), (b), (c), (d), (e) are functions.

11. f_1 – Domain = {0, 2, 4, 6, ..., 100}.

    Range = {1, 3, 5, 7, ... 101}.

    f_2 – Domain = { – 2, – 4, – 6, ... } .

    Range = {4, 16, 36, ...}.

    f_3 – Domain = $\left\{-1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$

    Range = {l, – l} .

    f_4 – Domain = {3, – 1, 4}.

    Range = {0, 2, -l}.

    f_5 – Domain = {......., 3, – 2, – 1, 0, 1, 2, 3, ...}.

    Range = {0, 1, 2, 3, ...} .

12. (a) Domain = R . (b) Domain = R – {– 1, 1}.

    (c) Domam = $x \geq -\frac{1}{3}$ $\quad \forall x \in R$

    (d) Domain = $x \geq -1, x \leq -6$. (e) Domam = $x \geq \frac{5}{2}, \; x \leq 1$

13. (a) Range = R (b) Range = All values of y except at x = 2.

    (c) Range = $\left\{-1, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}$

    (d) Range = All values of y for x > 0

15. (a), (c), (e), (t), (h). Use hint given in check your progress 15.7, Q. NO.7 for the solution.

16. One-to-one functions: (a), (d) Many-to-one functions: (b), (c)

17. (a), (c) 18. (a), (b), (c)

19. Even functions: (a), (b), (c), (t), (g)

    Odd functions : (d), (e)

20. (a) fog = $(4x-1)^3$, gof = $4x^3-1$, fof = x^9, gog = $16x-5$

(b) fog = $\dfrac{1}{(x^2-2x+3)^2}$, gof = $\dfrac{3x^4-2x^2+1}{x^4}$

fog = x^4, gog = $x^4-4x^3+4x^2$

(c) fog = $\sqrt{x-8}$ gof = $\sqrt{x-4}-4$

fof = $\sqrt{\sqrt{x-4}-4}$, gog = x - 8

(d) fog = x^4+2x^2, gof = x^4-2x^2+2,

fof = x^4-2x^2, gog = x^4+2x^2+2

21. (a) $\left|\dfrac{1}{x^{\frac{1}{3}}}\right|$ (b) (fog) (3) = 364, (gof)(3) = 289

22. (c), (d), (e)

Functions

15.11 Real Variable Function

1. A function $f: A \to B$ is called a real variable function if $A \subseteq R$.
2. A function $f: A \to B$ is called real valued function if $B \subseteq R$.
3. A function $f: A \to B$ is called real function if $A \subseteq R$, $B \subseteq R$.

Example: a^x $(a > 0)$, $\sin x$, $\log x$ these are all real functions.

15.11.1 n^{th} root of a non-negative real number

Let x be a non-negative real number and n be a positive integer then there exists a unique non-negative real number y such that n such that $y^n = x$. This number y is called the n^{th} root of x and is denoted as $x^{\frac{1}{n}}$ (or) $\sqrt[n]{x}$

Example 1: The domain of the real valued function $f(x) = \sqrt{a^2 - x^2}$ $(a > 0)$ is $[-a, a]$.

[since $\sqrt{a^2 - x^2} \in R \quad (a > 0)$

$\Leftrightarrow a^2 - x^2 \geq 0 \Leftrightarrow x^2 \leq a^2$

$\Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$]

15.11.2 Algebra of real valued functions

If f and g are real valued functions with domains A and B respectively, then both f and g are defined on $A \cap B$ when $A \cap B = \phi$.

i) $(f \pm g)(x) = f(x) \pm g(x)$ defined on $A \cap B$

ii) $(fg)(x) = f(x)\, g(x)$ defined on $A \cap B$

iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \forall\, x \in \text{E, E} = \{x \in \text{A} \cap \text{B} / g(x) \neq 0\}$

iv) Let $f : \text{A} \to \text{R}$ and $n \in \text{N}$, we define $|f|$ and f^n

on A by $|f|(x) = |f(x)| \quad f^n(x) = (f(x))^n \;\forall\, x \in \text{A}$.

v) If $\text{E} = \{x \in \text{A} / f(x) \geq 0\} \neq \phi$, then we define $\sqrt{f}$ on E by $\sqrt{f}(x) = \sqrt{f(x)}$, for all $x \in$ E.

Find the domains of the following real valued functions.

Example 2 : $f(x) = \dfrac{1}{6x - x^2 - 5}$

Solution: $f(x) = \dfrac{1}{6x - x^2 - 5} = \dfrac{1}{(x-1)(5-x)} \in \text{R}$

$\Leftrightarrow (x - 1)(x - 5) \neq 0$

$\Leftrightarrow x \neq 1, 5$

$\therefore$ Domain of f is $\text{R} - \{1, 5\}$.

Example 3: $f(x) = \sqrt{(x+2)(x-3)}$

Solution : $\sqrt{(x+2)(x-3)} \in \text{R}$

$\Leftrightarrow (x + 2)(x - 3) \geq 0$

$\Leftrightarrow x \leq -2 \text{ or } x \geq 3$

$\Leftrightarrow x \in (-\infty, -2] \cup [3, \infty)$

$= \text{R} - (-2, 3)$.

Example 4: $f(x) = \sqrt{x^2 - 1} + \dfrac{1}{\sqrt{x^2 - 3x + 2}}$

Sol : $f(x) = \sqrt{x^2 - 1} + \dfrac{1}{\sqrt{x^2 - 3x + 2}} \in \text{R}$

$\Leftrightarrow x^2 - 1 \geq 0, x^2 - 3x + 2 > 0$

$\Leftrightarrow$ $(x+1)(x-1) \geq 0$ and $(x-1)(x-2) > 0$

$\Leftrightarrow x \in (-\infty, -1) \cup [1, \infty]$ and $x \in (-\infty, 1) \cup (2, \infty)$

$\Leftrightarrow x \in R - (-1, 1) \cap R - (1, 2)$

$\Leftrightarrow x \in R - \{(-1, 1) \cup (1, 2)\}$, $\Leftrightarrow x + R - [-1, 2] = (-\infty, -1)] \cup (2, \infty)$

Domain of f is $= (-\infty, 1) \cup (2, \infty) = R - (-1, 2)$.

Exmaple 5(i) : $f = \{(4, 5)\ (5, 6), (6, -4)\}$ and $g = \{(4, -4), (6, 5), (8, 5)\}$ find (i) $(f - g)$

Sol : $f - g = \{(4, 5+4), (6, -4-5)\}$

$= \{(4, 9), (6, -9)\}$.

Example 5(ii) : Find $2f + 4g$

Sol : Domain of $2f = \{4, 5, 6\}$ Domain of $4g = \{4, 6, 8\}$

But $2f = \{(4, 10)\ (5, 12)\ (6, -8)\}$

$4g = \{(4, -16)\ (6, 20)\ (8, 20)\}$

Domain of $2f + 4g = \{4, 6\}$

$2f + 4g = \{(4, 10 - 16), (6, -8 + 20)\}$

$= \{(4, -6), (6, 12)\}$.

Example 5(iii) : Find fg

Sol : Domain of fg $A \cap B = \{4, 6\}$

$fg = \{(4, (5)\ (-4)), (6, (-4)(-5)\}$

$= \{(4, -20), (6, 20)\}$.

Example 5(iv) : Find $\dfrac{f}{g}$

Sol : Domain of $\dfrac{f}{g} = \{4, 6\}$

$$\therefore \frac{f}{g} = \left\{\left(4, \frac{-5}{4}\right), \left(6, \frac{-4}{5}\right)\right\}$$

Example 5(v) : Find f^2

Sol : Domain of f^2 = A = {4, 5, 6}

$$f^2 = \{(4, 25), (5, 36), (6, 16)\}.$$

Find the Domains and ranges of the following real valued functions.

Example 6: $f(x) = \dfrac{2+x}{2-x}$

Sol : $f(x) = \dfrac{2+x}{2-x} \in \mathrm{R} \Leftrightarrow 2-x \neq 0 \Leftrightarrow x \neq 2$

$\Leftrightarrow x \in \mathrm{R} - \{2\}$

Domain of f is $\mathrm{R} - \{2\}$

Let $f(x) = y$

$\Rightarrow \dfrac{2+x}{2-x} = y \Rightarrow x = \dfrac{2(y-1)}{y+1}$

x is not defined for $y + 1 = 0$ i.e., when $y = -1$

$\therefore$ range of f = $\mathrm{R} - \{-1\}$.

Example 7: $f(x) = \sqrt{9-x^2}$

Sol : $\sqrt{9-x^2} \in \mathrm{R} \Leftrightarrow 9 - x^2 \geq 0$

$\Leftrightarrow (3+x)(3-x) \geq 0$

$\Leftrightarrow x \in [-3, 3]$

$\therefore$ Domain of f = $[-3, 3]$

$$y = f(x) \Rightarrow x = \sqrt{9 - y^2} \in \mathrm{R}$$

$$\Leftrightarrow 9 - y^2 \geq 0 \Leftrightarrow (3 + y)(3 - y) \geq 0$$

$\therefore -3 \leq y \leq 3$ but $f(x)$ attains only non negative values.

$\therefore$ range of f = [0, 3].

15.12 Types of Functions

15.12.1 Implicit function : y is said to be an implicit function of x if it is given in the form of

$$f(x, y) = 0.$$

Ex: $x^2 + xy + y^2 - 2 = 0.$

15.12.2 Polynomial function : If n is a non negative integer, $a_1, a_2, ..., a_n$ are real number (at least one $a_i \neq 0$) then the function f defined on R by $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ for all $x \in \mathrm{R}$ is called a polynomial function.

15.12.3 (i) Rational function :

If $\phi(x)$, $\psi(x)$ are polynomial functions and $\psi(x) \neq 0$ for all $x \in \mathrm{R}$ then the function $\dfrac{\phi(x)}{\psi(x)}$ is called a rational function.

The domain of this function = $\mathrm{R} - \{x / \psi(x) = 0\}$

Ex: $f(x) = \dfrac{5x^2 - 6}{x^2 + 4x + 3}$

15.12.3 (ii) Even and odd functions

Let A be a non empty subset of R such that $-x \in \mathrm{A}$ for all $x \in \mathrm{A}$ and t $f : \mathrm{A} \to \mathrm{R}$

(i) If $f(-x) = f(x)$ for every x in A then f is called an even function.

(ii) If $f(-x) = -f(x)$ for every x in A then f is called an odd function.

Example:

(i) $f(x) = x^2 + 5\sin^2 x$, $g(x) = \cos x$ are even functions

(ii) $f(x) = \sin x$, $g(x) = x^3 + 5x$ are odd functions.

15.12.4 Algebraic functions

Binary operations like addition, subtraction, multiplication, division and extraction of square root etc. are called algebric operations. A function obtained by applying a finite number of algebraic operations on polynomial function is called an algebraic function.

Ex: $f(x) = \sqrt{2x^2 + 3\sqrt{x}}$ is an algebraic function.

15.12.5 Exponential function

The function ax when $1 \neq a > 0$ and x is rational, is called exponential function.

The domain of this functuon (a^x) is R i.e., $(-\infty, \infty)$.

The range is $R^+ = (0, \infty)$

Ex: If $f : R \to R^+$ can be defined as $f(x) = 2^x$ is anexponential function.

Logarithmic function

15.12.6 If $a \neq 1$, $a > 0$ given $x > 0$ then the function $f(x) = \log_a x$ is called lagarithmic function. The domain of f is $(0, \infty)$ and its rangle is R i.e. $(-\infty, \infty)$.

Ex: If $f : R^+ \to R$ then the function $f(x) = \log_2 x$ is called lagarithmic function.

15.12.7 Greatest integer function

The function $f : R \to R$ defined by $f(x) = [x]\ \forall\ x \in R$, for any real number x, we denote by $[x]$ the greatest integer less than or equal to x, is called the greatest integer function.

The domain of this function is R and the range is the set z of all integers.

$0 \leq x \leq 1,\ \forall x$ values $f(x) = 0$

$1 \leq x \leq 2,\ \forall x$ values $f(x) = 1$

$-1 \leq x < 0,\ \forall x$ values $f(x) = -1$

$-2 \leq x \leq 1,\ \forall x$ values $f(x) = -2$

....

Example 8 : If $f(x) = x^2,\ \ g(x) = |x|$ then find the fillowing functions :

1) $f + g$ 2) $f - g$ 3) fg 4) f^2

Example 8.1: Sol: $f(x) = x^2,\ \ g(x) = |x| = \begin{cases} x^2 + x, & x \geq 0 \\ x^2 - x, & x < 0 \end{cases}$, domain f = domain g = R

Hence the domain of all functions (i) through iv is R.

$$(f + g)(x) = f(x) + g(x) = x^2 + |x| = \begin{cases} x^2 + x, & x \geq 0 \\ x^2 - x, & x < 0 \end{cases}.$$

Example 8.2: Sol: $(f - g)(x) = f(x) - g(x) = x^2 - |x| = \begin{cases} x^2 - x, & x \geq 0 \\ x^2 + x, & x < 0 \end{cases}$

Example 8.3: Sol: $(fg)(x) = f(x)g(x) = x^2 |x| = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$

Example 8.4 Sol: $f^2(x) = (f(x))^2 = (x^2)^2 = x^4$

Example 9 : Determine whether the following function is even or odd.

Sol: $f(x) = \log(x + \sqrt{x^2 + 1})$

$$\Rightarrow f(-x) = \log(-x + \sqrt{x^2 + 1})$$

$$f(-x) = \log\left[\frac{(x+\sqrt{x^2+1})(-x+\sqrt{x^2+1})}{(x+\sqrt{x^2+1})}\right]$$

$$= \log\left(\frac{x^2+1-x^2}{x+\sqrt{x^2+1}}\right) = \log\left(x+\sqrt{x^2+1}\right)^{-1}$$

$$f(-x) = -\log\left(x+\sqrt{x^2+1}\right) = -f(x)$$

$\therefore$ f is an odd function

Example 10: Find the domain of the following real valued function $f(x)=\sqrt{x+2}+\dfrac{1}{\log_{10}(1-x)}$

Sol:: $f(x)=\sqrt{x+2}+\dfrac{1}{\log_{10}(1-x)} \in \mathrm{R}$

$\Leftrightarrow$ $x+2 \geq 0, 1-x > 0, 1-x \neq 1$

$\Leftrightarrow$ $x \geq -2, 1 > x, \ x \neq 0$

$\Leftrightarrow$ $x \in [-2, \infty) \cap (-\infty, 1) - \{0\}$

$\Leftrightarrow$ Domain of f is $[-2, 1) - \{0\}$.

Check your progress - 1

1. Find the domain of real valued function $f(x) = \dfrac{2x^2-5x+7}{(x-1)(x-2)(x-3)}$.

2. Find the domain of $f(x) = \sqrt{4x-x^2}$.

3. Find the domain of $f(x) = \sqrt{x^2-25}$.

4. Find the range of the following functions.

   i) $\dfrac{\sin \pi[x]}{1+[x]^2}$ ii) $\dfrac{x^2-4}{x-2}$

5. If $f(x) = 2x - 1,\ g(x) = x^2$ then find the following.

i) $\left[\dfrac{\sqrt{f}}{g}\right](x)$ ii) $(f + g + 2)(x)$

6. If $f = \{(1, 2)\ (2, -3), (3, -1)\}$ then find

i) $2f$ ii) f^2 iii) $\sqrt{f}$

Answers

1. $R - \{1, 2, 3\}$

2. $[0,\ 4]$

3. $R - (-5, 5)$

4. i) $\{0\}$ ii) $R - \{4\}$

5. i) $\dfrac{\sqrt{2x-1}}{x^2}$ ii) $(x + 1)^2$

6. i) $\{(1, 4),\ (2, -6), (3, -2)\}$

ii) $\{(1, 4)\ (2, 9), (3, 1)\}$

iii) $\left\{(1, \sqrt{2})\right\}$

Let us sumup

1. A function $f : A \to B$ is a real valued function if $A \subseteq R,\ \ B \subseteq R$.

2. If $y^n = x$ then y is n^{th} root of x. This can be written as $x^{\frac{1}{n}}$ or $\sqrt[n]{x}$

3. If f, g are real functions with domains A, B two sets then both f, g are defined on $A \cap B$.

4. $(f \pm g)x = f(x) \pm g(x)$ domain $= A \cap B$

5. $(f\ g)x = f(x)\ g(x)$ domain $= A \cap B$

6. $\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}$ – Domain = $\{x/x \in A \cap B,\ g(x) \neq 0\}$

7. Let A be a non empty subset of R such that $-x \in A\ \forall x \in A$ and $f : A \to R$.

   (i) If $f(-x) = f(x)\ \forall x \in A$ then f is called even.

   (ii) If $f(-x) = -f(x)\ \forall x \in A$ then f is called odd.

8. If $f(a) = a^x\ (a > 0)$ then $f(x)$ is Exponential function.

9. If $f(x) = \log_a x\ (a \neq 1,\ x > 0)$ then $f(x)$ is logarithmic function.

Web sites

- http://www.wikipedia.org
- http://mathworld.wolfram.com

16

TRIGONOMETRIC FUNCTIONS - I

We have read about trigonometric ratios in our earlier classes. Recall that we defined the ratios of the sides of a right triangle as follows :

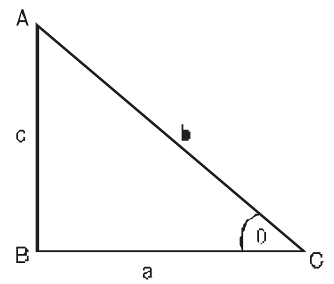


Fig.16.1

$$\sin\theta = \frac{c}{b}, \ \cos\theta = \frac{a}{b}, \ \tan\theta = \frac{c}{a}$$

and cosec $\theta = \frac{b}{c}$, sec $\theta = \frac{b}{a}$, cot $\theta = \frac{a}{c}$

We also developed relationships between these trigonometric ratios as

$$\sin^2\theta + \cos^2\theta = 1, \ \sec^2\theta = 1 + \tan^2\theta, \ \text{cosec}^2\theta = 1 + \cot^2\theta$$

We shall try to describe this knowledge gained so far in terms of functions, and try to develop this lesson using functional approach.

In this lesson, we shall develop the science of trigonometry using functional approach. We shall develop the concept of trigonometric functions using a unit circle. We shall discuss the radian measure of an angle and also define trigonometric functions of the type

y = sin x, y = cos x, y = tan x, y = cot x, y = sec x, y = cosec x, y = a sin x, y = b cos x, etc., where *x, y* are real numbers.

We shall draw the graphs of functions of the type

y = sin x, y = cos x, y = tan x, y = cotx, y = secx, and y = cosecx y = a sin x, y = a cos x.

OBJECTIVES

After studying this lesson, you will be able to :

- define positive and negative angles;
- define degree and radian as a measure of an angle;
- convert measure of an angle from degrees to radians and vice-versa;

- state the formula $\ell = r\,\theta$ where r and θ have their usual meanings;
- solve problems using the relation $\ell = r\,\theta$;
- define trigonometric functions of a real number;
- draw the graphs of trigonometric functions; and
- interpret the graphs of trigonometric functions.

EXPECTED BACKGROUND KNOWLEDGE

- Definition of an angle.
- Concepts of a straight angle, right angle and complete angle.
- Circle and its allied concepts.
- Special products : $(a \pm b)^2 = a^2 + b^2 \pm 2ab$, $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$
- Knowledge of Pythagoras Theorem and Py thagorean numbers.

16.1 CIRCULAR MEASURE OF ANGLE

An angle is a union of two rays with the common end point. An angle is formed by the rotation of a ray as well. Negative and positive angles are formed according as the rotation is clock-wise or anticlock-wise.

16.1.1 A Unit Circle

It can be seen easily that when a line segment makes one complete rotation, its end point describes a circle. In case the length of the rotating line be one unit then the circle described will be a circle of unit radius. Such a circle is termed as ***unit circle.***

16.1.2 A Radian

A radian is another unit of measurement of an angle other than degree.

A radian is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius (r) of the circle. In a unit circle one radian will be the angle subtended at the centre of the circle by an arc of unit length.

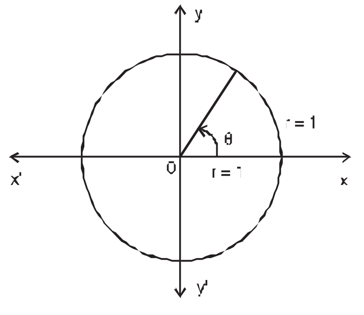


Fig. 16.2

Note : A radian is a constant angle; implying that the measure of the angle subtended by an are of a circle, with length equal to the radius is always the same irrespective of the radius of the circle.

16.1.3 Relation between Degree and Radian

An arc of unit length subtends an angle of 1 radian. The circumference 2π ($\because$ r = 1) subtend an angle of 2π radians.

Hence 2π radians = 360°

$$\pi \text{ radians} = 180°$$

$$\frac{\pi}{2} \text{ radians} = 90°$$

$$\frac{\pi}{4} \text{ radians} = 45°$$

$$1 \text{ radian} = \left(\frac{360}{2\pi}\right)^{\circ} = \left(\frac{180}{\pi}\right)^{\circ}$$

$$\text{or } 1° = \frac{2\pi}{360} \text{ radians} = \frac{\pi}{180} \text{ radians}$$

Example 16.1 Convert

(i) 90° into radians (ii) 15° into radians

(iii) $\frac{\pi}{6}$ radians into degrees. (iv) $\frac{\pi}{10}$ radians into degrees.

Solution :

(i) $1° = \frac{2\pi}{360}$ radians

$\Rightarrow$ $90° = \frac{2\pi}{360} \times 90$ radians or $90° = \frac{\pi}{2}$ radians

(ii) $15° = \frac{2\pi}{360} \times 15$ radians or $15° = \frac{\pi}{12}$ radians

(iii) $1 \text{ radian} = \left(\frac{360}{2\pi}\right)^{\circ}$

$$\frac{\pi}{6} \text{ radians} = \left(\frac{360}{2\pi}\times\frac{\pi}{6}\right)^{\circ}$$

$$\frac{\pi}{6} \text{ radians} = 30°$$

(iv) $\frac{\pi}{10}$ radians $= \left(\frac{360}{2\pi}\times\frac{\pi}{10}\right)^{\circ}$

$$\frac{\pi}{10} \text{ radians} = 18°$$

CHECK YOUR PROGRESS 16.1

1. Convert the following angles (in degrees) into radians :

   (i) 60° (ii) 15° (iii) 75° (iv) 105° (v) 270°

2. Convert the following angles into degrees:

   (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{12}$ (iii) $\frac{\pi}{20}$ (iv) $\frac{\pi}{60}$ (v) $\frac{2\pi}{3}$

3. The angles of a triangle are 45°, 65° and 70°. Express these angles in radians

4. The three angles of a quadrilateral are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$. Find the fourth angle in radians.

5. Find the angle complementary to $\frac{\pi}{6}$.

16.1.4 Relation Between Length of an Arc and Radius of the Circle

An angle of 1 radian is subtended by an arc whose length is equal to the radius of the circle. An angle of 2 radians will be substened if arc is double the radius.

An angle of 2½ radians willbe subtended if arc is 2½ times the radius.

All this can be read from the following table :

Length of the arc (l)	**Angle subtended at the centre of the circle θ (in radians)**
r	1
2r	2
(2½)r	2½
4r	4

Therefore, $\theta = \frac{\ell}{r}$ or $\ell = r\,\theta$

where r = radius of the circle,

θ = angle substended at the centre in radians

and ℓ = length of the arc.

The angle subtended by an arc of a circle at the centre of the circle is given by the ratio of the length of the arc and the radius of the circle.

Note : In arriving at the above relation, we have used the radian measure of the angle and not the degree measure. Thus the relation $\theta = \frac{\ell}{r}$ is valid only when the angle is measured in radians.

Example 16.2 Find the angle in radians subtended by an arc of length 10 cm at the centre of a circle of radius 35 cm.

Solution : $\ell = 10\text{cm}$ and $r = 35\text{cm}.$

$$\theta = \frac{\ell}{r} \text{ radians} \qquad \text{or} \qquad \theta = \frac{10}{35} \text{ radians}$$

or $$\theta = \frac{2}{7} \text{ radians}$$

Example 16.3 If *D* and *C* represent the number of degrees and radians in an angle prove that

$$\frac{D}{180} = \frac{C}{\pi}$$

Solution : $$1 \text{ radian} = \left(\frac{360}{2\pi}\right)^\circ \text{ or} \left(\frac{180}{\pi}\right)^\circ$$

$$\therefore \quad C \text{ radians} = \left(C \times \frac{180}{\pi}\right)^\circ$$

Since *D* is the degree measure of the same angle, therefore,

$$D = C \times \frac{180}{\pi}$$

which implies $$\frac{D}{180} = \frac{C}{\pi}$$

Example 16.4 A railroad curve is to be laid out on a circle. What should be the radius of a circular track if the railroad is to turn through an angle of 45° in a distance of 500m?

Solution : Angle θ is given in degrees. To apply the formula $\ell = r\theta, \theta$ must be changed to radians.

$$\theta = 45^\circ = 45 \times \frac{\pi}{180} \text{ radians} \qquad ....(1)$$

$$= \frac{\pi}{4} \text{ radians}$$

$$\ell = 500 \text{ m} \qquad ....(2)$$

$$\ell = r\,\theta \text{ gives } r = \frac{\ell}{\theta}$$

$$\therefore \quad r = \frac{500}{\frac{\pi}{4}} \text{ m} \qquad \text{[using (1) and (2)]}$$

$$= 500 \times \frac{4}{\pi} \text{ m}$$

$= 2000 \times 0.32$ m $\qquad \left(\frac{1}{\pi} = 0.32\right)$

$= 640$ m

Example 16.5 A train is travelling at the rate of 60 km per hour on a circular track. Through what angle will it turn in 15 seconds if the radius of the track is $\frac{5}{6}$ km.

Solution : The speed of the train is 60 km per hour. In 15 seconds, it will cover

$$\frac{60 \times 15}{60 \times 60} \text{ km}$$

$$= \frac{1}{4} \text{ km}$$

$\therefore$ We have, $\ell = \frac{1}{4}$ km and $r = \frac{5}{6}$ km

$$\therefore \quad \theta = \frac{\ell}{r} = \frac{\frac{1}{4}}{\frac{5}{6}} \text{ radians}$$

$$= \frac{3}{10} \text{ radians}$$

CHECK YOUR PROGRESS 16.2

1. Express the following angles in radians :

   (a) 30° (b) 60° (c) 150°

2. Express the following angles in degrees :

   (a) $\frac{\pi}{5}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{9}$

3. Find the angle in radians and in degrees subtended by an arc of length 2.5 cm at the centre of a circle of radius 15 cm.

4. A train is travelling at the rate of 20 km per hour on a circular track. Through what angle will it turn in 3 seconds if the radius of the track is $\frac{1}{12}$ of a km?.

5. A railroad curve is to be laid out on a circle. What should be the radius of the circular track if the railroad is to turn through an angle of 60° in a distance of 100 m?

6. Complete the following table for l, r, θ having their usual meanings.

	l	r	θ
(a)	1.25m	....	135°
(b)	30 cm	....	$\frac{\pi}{4}$
(c)	0.5 cm	2.5 m	....
(d)	....	6 m	120°
(e)	....	150 cm	$\frac{\pi}{15}$
(f)	150 cm	40 m	....
(g)	....	12 m	$\frac{\pi}{6}$
(h)	1.5 m	0.75 m	....
(i)	25 m	....	75°

16.2 TRIGONOMETRIC FUNCTIONS

While considering, a unit circle you must have noticed that for every real number between *0* and 2π, there exists a ordered pair of numbers x and y. This ordered pair (x, y) represents the coordinates of the point *P*.

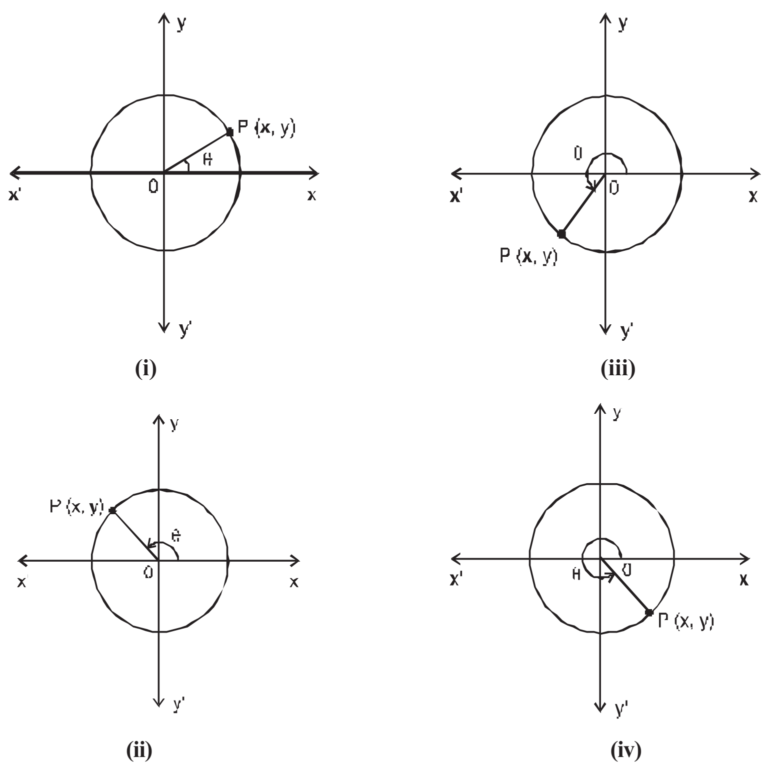


Fig. 16.3

If we consider $\theta = 0$ on the unit circle, we will have a point whose coordinates are (1,0).

If $\theta = \frac{\pi}{2}$, then the corresponding point on the unit circle will have its coordinates (0,1).

In the above figures you can easily observe that no matter what the position of the point, corresponding to every real number θ we have a unique set of coordinates (x, y). The values of x and y will be negative or positive depending on the quadrant in which we are considering the point.

Considering a point P (on the unit circle) and the corresponding coordinates (x, y), we define trigonometric functions as :

$$\sin\theta = y, \ \cos\theta = x$$

$$\tan\theta = \frac{y}{x}\ (\text{for } x \neq 0), \ \cot\theta = \frac{x}{y}\ (\text{for } y \neq 0)$$

$$\sec\theta = \frac{1}{x}\ (\text{for } x \neq 0), \ \text{cosec}\,\theta = \frac{1}{y}\ (\text{for } y \neq 0)$$

Now let the point P move from its original position in anti-clockwise direction. For various positions of this point in the four quadrants, various real numbers θ will be generated. We summarise, the above discussion as follows. For values of θ in the :

I quadrant, both x and y are positve.

II quadrant, x will be negative and y will be positive.

III quadrant, x as well as y will be negative.

IV quadrant, x will be positive and y will be negative.

or	I quadrant	II quadrant	III quadrant	IV quadrant
	All positive	sin positive	tan positive	cos positive
		cosec positive	cot positive	sec positive

Where what is positive can be rememebred by :

	All	sin	tan	cos
Quardrant	I	II	III	IV

If (x, y) are the coordinates of a point P on a unit circle and θ, the real number generated by the position of the point, then $\sin\theta = y$ and $\cos\theta = x$. This means the coordinates of the point P can also be written as $(\cos\theta, \sin\theta)$

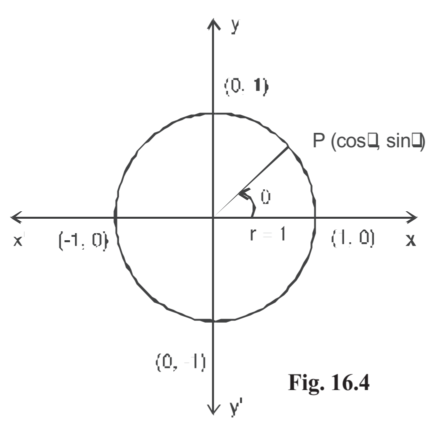


Fig. 16.4

From Fig. 16.5, you can easily see that the values of x will be between -1 and $+1$ as P moves on the unit circle. Same will be true for y also.

Thus, for all P on the unit circle

$-1 \le x \le 1$ and $-1 \le y \le 1$

Thereby, we conclude that for all real numbers θ

$-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$

In other words, $\sin\theta$ and $\cos\theta$ can not be numerically greater than 1

Example 16.6 What will be sign of the following ?

(i) $\sin\frac{7\pi}{18}$ (ii) $\cos\frac{4\pi}{9}$ (iii) $\tan\frac{5\pi}{9}$

Solution :

(i) Since $\frac{7\pi}{18}$ lies in the first quadrant, the sign of $\sin\frac{7\pi}{18}$ will be posilive.

(ii) Since $\frac{4\pi}{9}$ lies in the first quadrant, the sign of $\cos\frac{4\pi}{9}$ will be positive.

(iii) Since $\frac{5\pi}{9}$ lies in the second quadrant, the sign of $\tan\frac{5\pi}{9}$ will be negative.

Example 16.7 Write the values of (i) $\sin\frac{\pi}{2}$ (ii) $\cos 0$ (iii) $\tan\frac{\pi}{2}$

Solution : (i) From Fig.16.5, we can see that the coordinates of the point *A* are (0,1)

$\therefore\ \sin\frac{\pi}{2} = 1$, as $\sin\theta = y$

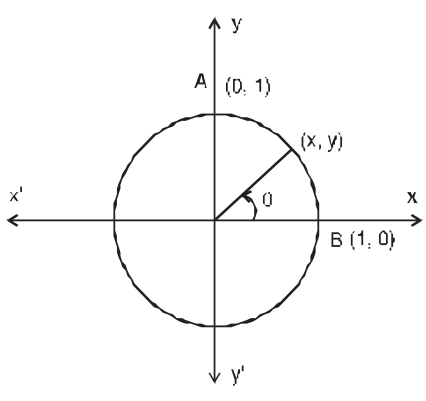


Fig.16.5

(ii) Coordinates of the point B are (1, 0)

$\therefore\ \cos 0 = 1$, as $\cos\theta = x$

(iii) $\tan\frac{\pi}{2} = \dfrac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \frac{1}{0}$ which is not defined

Thus $\tan\frac{\pi}{2}$ is not defined.

MODULE - IV
Functions

Notes

Example 16.8 Write the minimum and maximum values of $\cos\theta$.

Solution : We know that $-1 \le \cos\theta \le 1$

$\therefore$ The maximum value of $\cos\theta$ is 1 and the minimum value of $\cos\theta$ is -1.

CHECK YOUR PROGRESS 16.3

1. What will be the sign of the following ?

(i)	$\cos\frac{2\pi}{3}$	(ii)	$\tan\frac{5\pi}{6}$	(iii)	$\sec\frac{2\pi}{3}$
(iv)	$\sec\frac{35\pi}{18}$	(v)	$\tan\frac{25\pi}{18}$	(vi)	$\cot\frac{3\pi}{4}$
(vii)	$\operatorname{cosec}\frac{8\pi}{3}$	(viii)	$\cot\frac{7\pi}{8}$		

2. Write the value of each of the following :

(i)	$\cos\frac{\pi}{2}$	(ii)	$\sin 0$	(iii)	$\cos\frac{2\pi}{3}$
(iv)	$\tan\frac{3\pi}{4}$	(v)	$\sec 0$	(vi)	$\tan\frac{\pi}{2}$
(vii)	$\tan\frac{3\pi}{2}$	(viii)	$\cos 2\pi$		

16.2.1 Relation Between Trigonometric Functions

By definition $x = \cos\theta$

$y = \sin\theta$

As $\tan\theta = \frac{y}{x}, \ (x \ne 0)$

$= \frac{\sin\theta}{\cos\theta}, \ \theta \ne \frac{n\pi}{2}$

and $\cot\theta = \frac{x}{y}, \ (y \ne 0)$

i.e., $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} \qquad (\theta \ne n\pi)$

Similarly, $\sec\theta = \frac{1}{\cos\theta} \qquad \left(\theta \ne \frac{n\pi}{2}\right)$

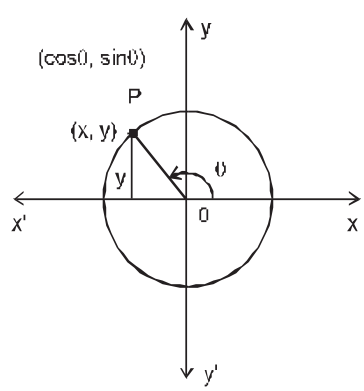


Fig. 16.6

and $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta} \qquad (\theta \neq n\pi)$

Using Pythagoras theorem we have, $x^2 + y^2 = 1$

i.e., $(\cos \theta)^2 + (\sin \theta)^2 = 1$

or, $\mathbf{\cos^2 \theta + \sin^2 \theta = 1}$

Note : $(\cos \theta)^2$ is written as $\cos^2 \theta$ and $(\sin \theta)^2$ as $\sin^2 \theta$

Again $x^2 + y^2 = 1$

or $1 + \left(\dfrac{y}{x}\right)^2 = \left(\dfrac{1}{x}\right)^2$, for $x \neq 0$ or, $1 + (\tan \theta)^2 = (\sec \theta)^2$

i.e. $\mathbf{\sec^2 \theta = 1 + \tan^2 \theta}$

Similarly, $\mathbf{\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta}$

Example 16.9 Prove that $\sin^4 \theta + \cos^4 \theta = 1 - 2\sin^2 \theta \cos^2 \theta$

Solution : L.H.S. $= \sin^4 \theta + \cos^4 \theta$

$$= \sin^4 \theta + \cos^4 \theta + 2\sin^2 \theta \cos^2 \theta - 2\sin^2 \theta \cos^2 \theta$$

$$= \left(\sin^2 \theta + \cos^2 \theta\right)^2 - 2\sin^2 \theta \cos^2 \theta$$

$$= 1 - 2\sin^2 \theta \cos^2 \theta \quad \left(\because \sin^2 \theta + \cos^2 \theta = 1\right)$$

$$= \text{R.H.S.}$$

Example 16.10 Prove that $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec \theta - \tan \theta$

Solution : L.H.S. $= \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

$$= \sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta - \tan\theta = \text{R.H.S.}$$

Example 16.11 If $\sin\theta = \frac{21}{29}$, prove that $\sec\theta + \tan\theta = 2\frac{1}{2}$, given that θ lies in the first quadrant.

Solution : $\sin\theta = \frac{21}{29}$

Also, $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \cos^2\theta = 1 - \sin^2\theta = 1 - \frac{441}{841} = \frac{400}{841} = \left(\frac{20}{29}\right)^2$$

$\Rightarrow \cos\theta = \frac{20}{29}$ ($\cos\theta$ is positive as θ lies in the first quardrant)

$$\therefore \tan\theta = \frac{21}{20}$$

$$\therefore \sec\theta + \tan\theta = \frac{29}{20} + \frac{21}{20} = \frac{29+21}{20}$$

$$= \frac{5}{2} = 2\frac{1}{2} = \text{R.H.S.}$$

CHECK YOUR PROGRESS 16.4

1. Prove that $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

2. If $\tan\theta = \frac{1}{2}$, find the other five trigonometric functions.

3. If $\text{cosec}\,\theta = \frac{b}{a}$, find the other five trigonometric functions, if θ lies in the first quardrant.

4. Prove that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \text{cosec}\,\theta + \cot\theta$

5. If $\cot\theta + \operatorname{cosec}\theta = 1.5$, show that $\cos\theta = \frac{5}{13}$

6. If $\tan\theta + \sec\theta = m$, find the value of $\cos\theta$

7. Prove that $(\tan A + 2)(2\tan A + 1) = 5\tan A + 2\sec^2 A$

8. Prove that $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

9. Prove that $\dfrac{\cos\theta}{1-\tan\theta} + \dfrac{\sin\theta}{1-\cot\theta} = \cos\theta + \sin\theta$

10. Prove that $\dfrac{\tan\theta}{1+\cos\theta} + \dfrac{\sin\theta}{1-\cos\theta} = \cot\theta + \operatorname{cosec}\theta\sec\theta$

16.3 TRIGONOMETRIC FUNCTIONS OF SOME SPECIFIC REAL NUMBERS

The values of the trigonometric functions of 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$ are summarised below in the form of a table :

Real Numbers → / Function ↓	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

As an aid to memory, we may think of the following pattern for above mentioned values of sin function :

$$\sqrt{\frac{0}{4}}, \sqrt{\frac{1}{4}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{4}}$$

On simplification, we get the values as given in the table. The values for cosines occur in the reverse order.

Example 16.12 Find the value of the following :

(a) $\sin\frac{\pi}{4}\sin\frac{\pi}{3} - \cos\frac{\pi}{4}\cos\frac{\pi}{3}$ (b) $4\tan^2\frac{\pi}{4} - \operatorname{cosec}^2\frac{\pi}{6} - \cos^2\frac{\pi}{3}$

Solution :

(a) $\sin\frac{\pi}{4}\sin\frac{\pi}{3} - \cos\frac{\pi}{4}\cos\frac{\pi}{3}$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(b) $4\tan^2\frac{\pi}{4} - \operatorname{cosec}^2\frac{\pi}{6} - \cos^2\frac{\pi}{3}$

$$= 4\,(1)^2 - (2)^2 - \left(\frac{1}{2}\right)^2$$

$$= 4 - 4 - \frac{1}{4} = -\frac{1}{4}$$

Example 16.13 If $A = \frac{\pi}{3}$ and $B = \frac{\pi}{6}$, verify that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Solution : L.H.S. $= \cos(A+B)$

$$= \cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \cos\frac{\pi}{2} = 0$$

R.H.S. $= \cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6}$

$$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$$

$\therefore$ L.H.S. = 0 = R.H.S.

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

CHECK YOUR PROGRESS 16.5

1. Find the value of

(i) $\sin^2\dfrac{\pi}{6}+\tan^2\dfrac{\pi}{4}+\tan^2\dfrac{\pi}{3}$ (ii) $\sin^2\dfrac{\pi}{3}+\operatorname{cosec}^2\dfrac{\pi}{6}+\sec^2\dfrac{\pi}{4}-\cos^2\dfrac{\pi}{3}$

(iii) $\cos\dfrac{2\pi}{3}\cos\dfrac{\pi}{3}-\sin\dfrac{2\pi}{3}\sin\dfrac{\pi}{3}$ (iv) $4\cot^2\dfrac{\pi}{3}+\operatorname{cosec}^2\dfrac{\pi}{4}+\sec^2\dfrac{\pi}{3}\tan^2\dfrac{\pi}{4}$

(v) $\left(\sin\dfrac{\pi}{6}+\sin\dfrac{\pi}{4}\right)\left(\cos\dfrac{\pi}{3}-\cos\dfrac{\pi}{4}\right)+\dfrac{1}{4}$

2. Show that

$$\left(1+\tan\frac{\pi}{6}\tan\frac{\pi}{3}\right)+\left(\tan\frac{\pi}{6}-\tan\frac{\pi}{3}\right)=\sec^2\frac{\pi}{6}\sec^2\frac{\pi}{3}$$

3. Taking $A=\dfrac{\pi}{3}$, $B=\dfrac{\pi}{6}$, verify that

(i) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ (ii) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

4. If $\theta=\dfrac{\pi}{4}$, verify the following :

(i) $\sin 2\theta=2\sin\theta\cos\theta$ (ii) $\cos 2\theta=\cos^2\theta-\sin^2\theta$

$$=2\cos^2\theta-1$$

$$=1-2\sin^2\theta$$

5. If $A=\dfrac{\pi}{6}$, verify that

(i) $\cos 2A=2\cos^2 A-1$ (ii) $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$

(iii) $\sin 2A=2\sin A\cos A$

16.4 GRAPHS OF TRIGONOMETRIC FUNCTIONS

Given any function, a pictorial or a graphical representation makes a lasting impression on the minds of learners and viewers. The importance of the graph of functions stems from the fact that this is a convenient way of presenting many properties of the functions. By observing the graph we can examine several characteristic properties of the functions such as (i) periodicity, (ii) intervals in which the function is increasing or decreasing (iii) symmetry about axes, (iv) maximum and minimum points of the graph in the given interval. It also helps to compute the areas enclosed by the curves of the graph.

16.4.1 Variations of sinθ as θ Varies Continuously From 0 to 2π.

Let $X'OX$ and $Y'OY$ be the axes of coordinates. With centre O and radius OP = unity, draw a circle. Let OP starting from OX and moving in anticlockwise direction make an angle θ with the x-axis, i.e. $\angle XOP = \theta$. Draw $PM \perp X'OX$, then $\sin\theta = MP$ as $OP = 1$.

$\therefore$ The variations of sin θ are the same as those of MP.

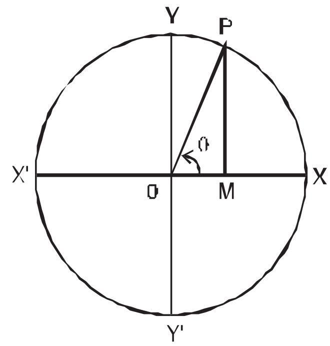


Fig. 16.7

I Quadrant :

As θ increases continuously from 0 to $\frac{\pi}{2}$

PM is positive and increases from 0 to 1.

$\therefore$ sin θ is positive.

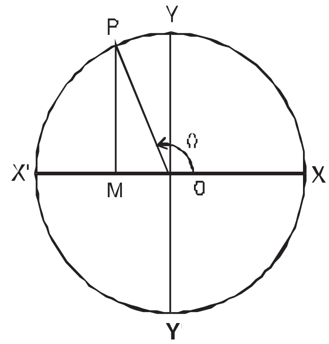


Fig. 16.8

II Quadrant $\left[\frac{\pi}{2}, \pi\right]$

In this interval, θ lies in the second quadrant.

Therefore, point P is in the second quadrant. Here PM = y is positive, but decreases from 1 to 0 as θ varies from $\frac{\pi}{2}$ to π. Thus sin θ is positive.

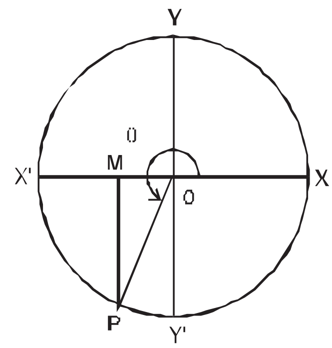


Fig. 16.9

III Quadrant $\left[\pi, \frac{3\pi}{2}\right]$

In this interval, θ lies in the third quandrant. Therefore, point P can move in the third quadrant only. Hence $PM = y$ is negative and decreases from 0 to -1 as θ varies from π to $\frac{3\pi}{2}$. In this interval $\sin\theta$ decreases from 0 to -1. In this interval sin θ is negative.

IV Quadrant $\left[\frac{3\pi}{2}, 2\pi\right]$

In this interval, θ lies in the fourth quadrant. Therefore, point P can move in the fourth quadrant only. Here again PM = y is negative but increases from -1 to 0 as θ varies from $\frac{3\pi}{2}$ to 2π. Thus sin θ is negative in this interval.

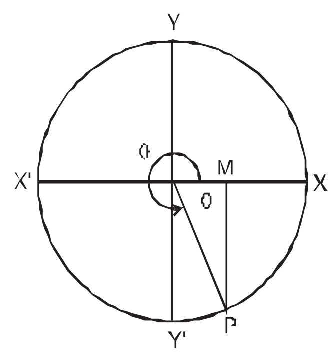


Fig. 16.10

16.4.2 Graph of sin θ as θ varies from 0 to 2π.

Let *X'OX* and *Y'OY* be the two coordinate axes of reference. The values of θ are to be measured along x-axis and the values of sine θ are to be measured along y-axis.

(Approximate value of $\sqrt{2} = 1.41, \frac{1}{\sqrt{2}} \approx .707, \frac{\sqrt{3}}{2} \approx .87$)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin θ	0	.5	.87	1	.87	.5	0	−.5	−.87	−1	−.87	−.5	0

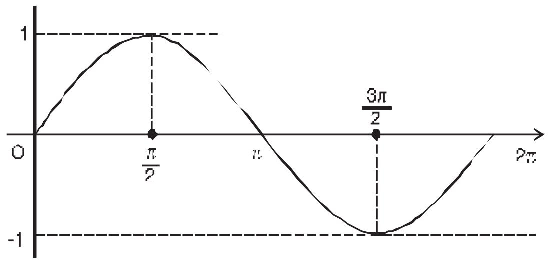


Fig. 16.11

Some Observations

(i) Maximum value of sin θ is 1.

(ii) Minimum value of sin θ is −1.

(iii) It is continuous everywhere.

(iv) It is increasing from 0 to $\frac{\pi}{2}$ and from $\frac{3\pi}{2}$ to 2π. It is decreasing from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.
With the help of the graph drawn in Fig. 16.12 we can always draw another graph. y = sin θ in the interval of $[2\pi, 4\pi]$ (see Fig. 16.11)

What do you observe ?

The graph of y = sin θ in the interval $[2\pi, 4\pi]$ is the same as that in 0 to 2π. Therefore, this graph can be drawn by using the property sin $(2\pi + \theta) = \sin\theta$. Thus, sin θ repeats itself when θ is increased by 2π. This is known as the periodicity of sin θ.

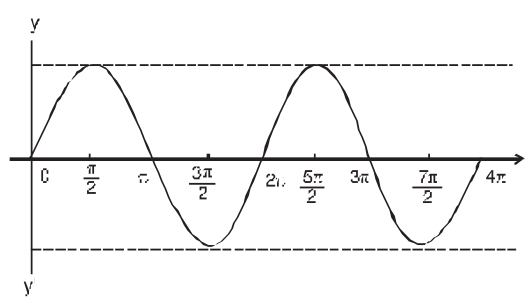


Fig. 16.12

We shall discuss in details the periodicity later in this lesson.

Example 16.14 Draw the graph of $y = \sin 2\theta$.

Solution :

θ :	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
2θ:	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\sin 2\theta$:	0	.87	1	.87	0	$-.87$	-1	$-.87$	0

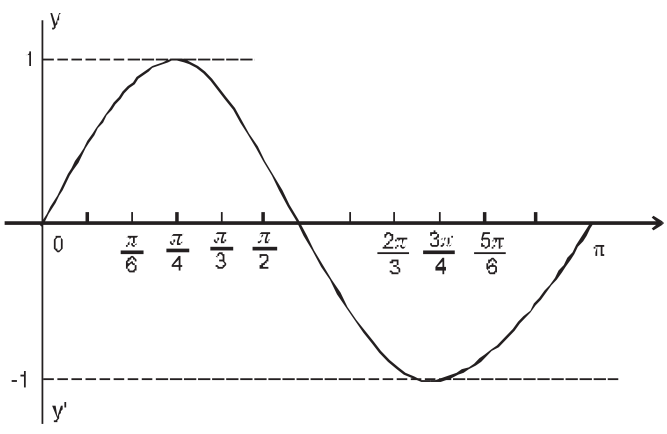


Fig. 16.13

The graph is similar to that of $y = \sin \theta$

Some Observations

1. The other graphs of $\sin \theta$, like a $\sin \theta$, $3 \sin 2\theta$ can be drawn applying the same method.

2. Graph of $\sin \theta$, in other intervals namely $[4\pi, 6\pi]$, $[-2\pi, 0]$, $[-4\pi, -2\pi]$, can also be drawn easily. This can be done with the help of properties of allied angles: $\sin(\theta + 2\pi) = \sin\theta$ $\sin(\theta - 2\pi) = \sin\theta$ i.e., θ repeats itself when increased or decreased by 2π.

CHECK YOUR PROGRESS 16.6

1. What are the maximum and minimum values of $\sin\theta$ in $[0, 2\pi]$?

2. Explain the symmetry in the graph of $\sin\theta$ in $[0, 2\pi]$

3. Sketch the graph of $y = 2\sin\theta$, in the interval $[0, \pi]$

4. For what values of θ in $[\pi, 2\pi]$, $\sin\theta$ becomes

(a) $\frac{-1}{2}$ (b) $\frac{-\sqrt{3}}{2}$

5. Sketch the graph of $y = \sin x$ in the interval of $[-\pi, \pi]$

16.4.3 Graph of cosθ as θ Varies From 0 to 2π

As in the case of $\sin\theta$, we shall also discuss the changes in the values of $\cos\theta$ when θ assumes values in the intervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$.

I Quadrant : In the interval $\left[0, \frac{\pi}{2}\right]$, point P lies in the first quadrant, therefore, $OM = x$ is positive but decreases from 1 to 0 as θ increases from 0 to $\frac{\pi}{2}$.

Thus in this interval $\cos\theta$ decreases from 1 to 0.

$\therefore$ $\cos\theta$ is positive in this quadrant.

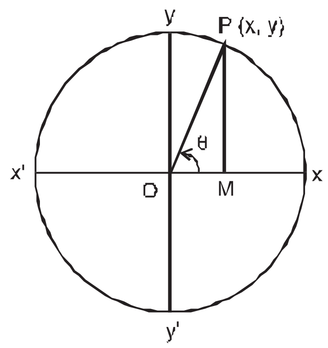


Fig.16.14

II Quadrant : In the interval $\left[\frac{\pi}{2}, \pi\right]$, point P lies in the second quadrant and therefore point M lies on the negative side of x-axis. So in this case $OM = x$ is negative and decreases from 0 to -1 as θ increases from $\frac{\pi}{2}$ to π. Hence in this inverval $\cos\theta$ decreases from 0 to -1.

$\therefore$ $\cos\theta$ is negative.

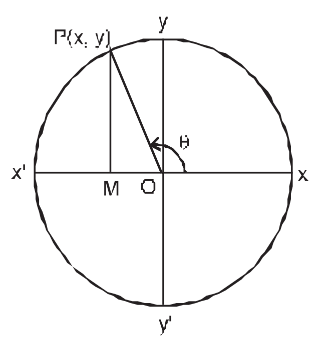


Fig. 16.15

III Quadrant : In the interval $\left[\pi, \frac{3\pi}{2}\right]$, point P lies in the third quadrant and therefore, $OM = x$ remains negative as it is on the negative side of x-axis. Therefore $OM = x$ is negative but increases from -1 to 0 as θ increases from π to $\frac{3\pi}{2}$. Hence in this interval $\cos\theta$ increases from -1 to 0.

$\therefore$ $\cos\theta$ is negative.

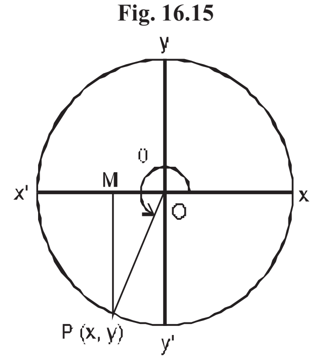


Fig. 16.16

IV Quadrant : In the interval $\left[\frac{3\pi}{2}, 2\pi\right]$, point P lies in the fourth quadrant and M moves on the positive side of x-axis.Therefore $OM = x$ is positive. Also it increases from 0 to 1 as θ increases from $\frac{3\pi}{2}$ to 2π.

Thus in this interval $\cos\theta$ increases from 0 to 1.

$\therefore$ $\cos\theta$ is positive.

Let us tabulate the values of cosines of some suitable values of θ.

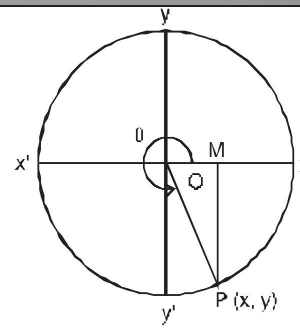


Fig. 16.17

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	.87	.5	0	0.5	−.87	−1	−.87	−.5	0	0.5	.87	1

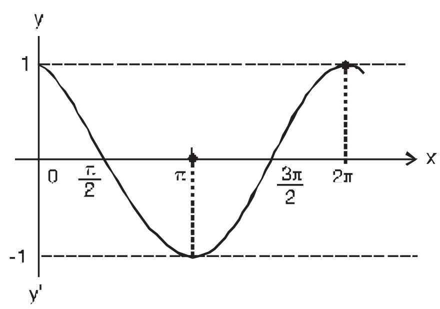


Fig. 16.18

Let $X'OX$ and $Y'OY$ be the axes. Values of θ are measured along x-axis and those of $\cos\theta$ along y-axis.

Some observations

(i) Maximum value of $\cos\theta = 1$.

(ii) Minimum value of $\cos\theta = -1$.

(iii) It is continuous everywhere.

(iv) $\cos(\theta + 2\pi) = \cos$ £ Also $(\theta - 2\pi) = \cos\theta$Cos θ repeats itself when θ is increased or decreased by 2π. It is called periodicity of $\cos\theta$. We shall discuss in details about this in the later part of this lesson.

(v) Graph of $\cos\theta$ in the intervals $[2\pi, 4\pi]$ $[4\pi, 6\pi]$ $[-2\pi, 0]$, will be the same as in $[0, 2\pi]$.

Example 16.15 Draw the graph of $\cos\theta$ as θ varies from $-\pi$ to π. From the graph read the values of θ when $\cos\theta = \pm 0.5$.

Solution :

θ :	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\cos\theta$:	–1.0	–0.87	–0.5	0	.50	–.87	1.0	0.87	0.5	0	–0.5	–0.87	–1

$\cos\theta = 0.5$

when $\theta = \frac{\pi}{3}, \frac{-\pi}{3}$

$\cos\theta = -0.5$

when $\theta = \frac{2\pi}{3}, \frac{-2\pi}{3}$

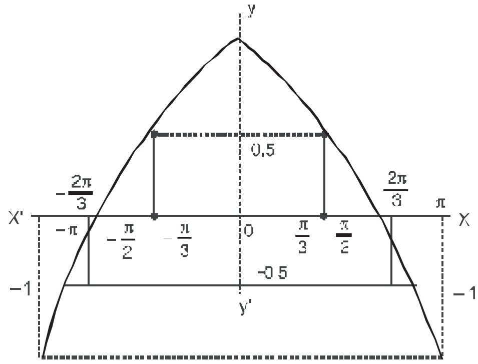


Fig. 16.19

Example 16.16 Draw the graph of $\cos 2\theta$ in the interval 0 to π.

Solution :

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
2θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\cos 2\theta$	1	0.5	0	–0.5	–1	–0.5	0	0.5	1

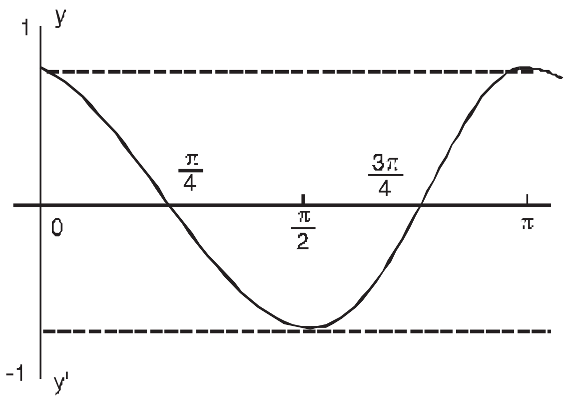


Fig. 16.20

CHECK YOUR PROGRESS 16.7

1. (a) Sketch the graph of $y = \cos\theta$ as θ varies from $\frac{-\pi}{4}$ to $\frac{\pi}{4}$.

   (b) Draw the graph of $y = 3\cos\theta$ as θ varies from 0 to 2π.

   (c) Draw the graph of $y = \cos 3\theta$ from $-\pi$ to π and read the values of θ when $\cos\theta = 0.87$ and $\cos\theta = -0.87$.

   (d) Does the graph of $y = \cos\theta$ in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ lie above x-axis or below x-axis?

   (e) Draw the graph of $y = \cos\theta$ in $[2\pi, 4\pi]$

16.4.4 Graph of tan θ as θ Varies from 0 to 2π

In I Quadrant : $\tan\theta$ can be written as $\frac{\sin\theta}{\cos\theta}$

Behaviour of $\tan\theta$ depends upon the behaviour of $\sin\theta$ and $\frac{1}{\cos\theta}$

In I quadrant, $\sin\theta$ increases from 0 to 1, $\cos\theta$ decreases from 1 to 0

But $\frac{1}{\cos\theta}$ increases from 1 indefintely (and write it as increases from 1 to ∞) $\tan\theta > 0$

$\therefore$ $\tan\theta$ increases from 0 to ∞. (See the table and graph of $\tan\theta$).

In II Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta$ decreases from 1 to 0.

$\cos\theta$ decreases from 0 to -1.

$\tan\theta$ is negative and increases from $-\infty$ to 0

In III Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta$ decreases from 0 to -1

$\cos\theta$ increases from -1 to 0

$\therefore$ $\tan\theta$ is positive and increases from 0 to ∞

In IV Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta$ increases from -1 to 0

$\cos\theta$ increases from 0 to 1

$\tan\theta$ is negative and increases form $-\infty$ to 0

Graph of tan θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0^\circ$	$\frac{\pi}{2}+0^\circ$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0^\circ$	$\frac{3\pi}{2}+0^\circ$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\tan\theta$	0	.58	1.73	$+\infty$	-1.73	-.58	0	.58	1.73	$+\infty$	$-\infty$	-1.73	-.58	0	0

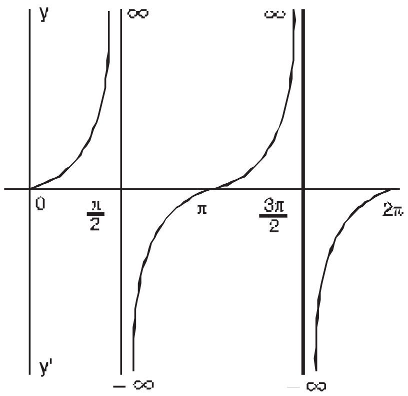


Fig. 16.21

Observations

(i) $\tan(180° + \theta) = \tan\theta$. Therefore, the complete graph of $\tan\theta$ consists of infinitely many repetitions of the same to the left as well as to the right.

(ii) Since $\tan(-\theta) = -\tan\theta$, therefore, if $(\theta, \tan\theta)$ is any point on the graph then $(-\theta, -\tan\theta)$ will also be a point on the graph.

(iii) By above results, it can be said that the graph of $y = \tan\theta$ is symmetrical in opposite quadrants.

(iv) $\tan\theta$ may have any numerical value, positve or negative.

(v) The graph of $\tan\theta$ is discontinuous (has a break) at the points $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

(vi) As θ passes through these values, $\tan\theta$ suddenly changes from $+\infty$ to $-\infty$.

16.4.5 Graph of cot θ as θ Varies From 0 to 2π

The behaviour of $\cot\theta$ depends upon the behaviour of $\cos\theta$ and $\frac{1}{\sin\theta}$ as $\cot\theta = \cos\theta \frac{1}{\sin\theta}$

We discuss it in each quadrant.

I Quadrant : $\cot\theta = \cos\theta \times \frac{1}{\sin\theta}$

$\cos\theta$ decreases from 1 to 0

$\sin\theta$ increases from 0 to 1

$\therefore$ $\cot\theta$ also decreases from $+\infty$ to 0 but $\cot\theta > 0$.

II Quadrant : $\cot\theta = \cos\theta \times \frac{1}{\sin\theta}$

$\cos\theta$ decreases from 0 to -1

$\sin\theta$ decreases from 1 to 0

$\Rightarrow \cot\theta < 0$ or $\cot\theta$ decreases from 0 to $-\infty$

III Quadrant : $\cot\theta = \cos\theta \times \dfrac{1}{\sin\theta}$

$\cos\theta$ increases from -1 to 0

$\sin\theta$ decreases from 0 to -1

$\therefore$ $\cot\theta$ decreases from $+\infty$ to 0.

IV Quadrant : $\cot\theta = \cos\theta \times \dfrac{1}{\sin\theta}$

$\cos\theta$ increases from 0 to 1

$\sin\theta$ increases from -1 to 0

$\therefore \cot\theta < 0$

$\cot\theta$ decreases from 0 to $-\infty$

Graph of $\cot\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi - 0$	$\pi + 0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cot\theta$	∞	1.73	.58	0	-.58	-1.73	$-\infty$	$+\infty$	1.73	.58	0	-.58	-1.73	$-\infty$

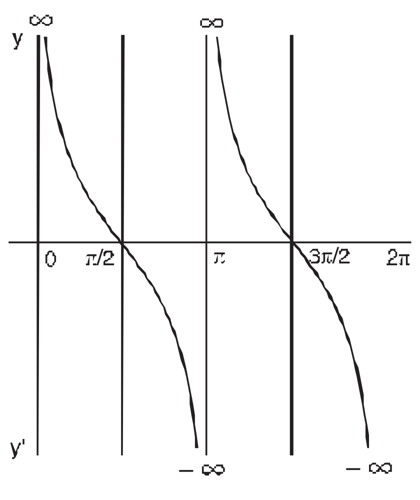


Fig 16.22

Observations

(i) Since $\cot(\pi + \theta) = \cot\theta$, the complete graph of $\cot\theta$ consists of the portion from $\theta = 0$ to $\theta = \pi$ or $\theta = \dfrac{\pi}{2}$ to $\theta = \dfrac{3\pi}{2}$.

(ii) $\cot\theta$ can have any numerical value - positive or negative.

(iii) The graph of $\cot\theta$ is discontinuous, i.e. it breaks at 0, π, 2π, .

(iv) As θ takes values $0, \pi, 2\pi$, $\cot\theta$ suddently changes from $-\infty$ to $+\infty$

CHECK YOUR PROGRESS 16.8

1. (a) What is the maximum value of $\tan\theta$?

   (b) What changes do you observe in $\tan\theta$ at $\frac{\pi}{2}, \frac{3\pi}{2}$?

   (c) Draw the graph of $y = \tan\theta$ from $-\pi$ to π. Find from the graph the value of θ for which $\tan\theta = 1.7$.

2. (a) What is the maximum value of $\cot\theta$?

   (b) Find the value of θ when $\cot\theta = -1$, from the graph.

16.4.6 To Find the Variations And Draw The Graph of sec θ As θ Varies From 0 to 2π.

Let *X'OX* and *Y'OY* be the axes of coordinates. With centre *O*, draw a circle of unit radius.

Let *P* be any point on the circle. Join *OP* and draw $PM \perp X'OX$.

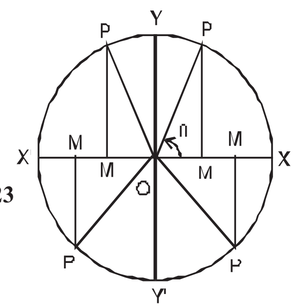


Fig. 16.23

$$\sec\theta = \frac{OP}{OM} = \frac{1}{OM}$$

$\therefore$ Variations will depend upon *OM*.

I Quadrant : $\sec\theta$ is positive as *OM* is positive.

Also $\sec 0 = 1$ and $\sec\frac{\pi}{2} = \infty$ when we approach $\frac{\pi}{2}$ from the right.

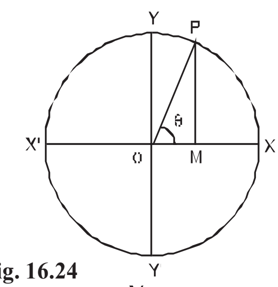


Fig. 16.24

$\therefore$ As θ varies from 0 to $\frac{\pi}{2}$, $\sec\theta$ increases from 1 to ∞.

II Quadrant : $\sec\theta$ is negative as *OM* is negative. $\sec\frac{\pi}{2} = -\infty$ when we approach $\frac{\pi}{2}$ from the left. Also $\sec\pi = -1$.

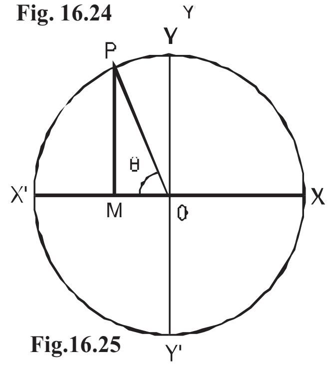


Fig.16.25

$\therefore$ As θ varies from $\frac{\pi}{2}$ to π, $\sec\theta$ changes from

$-\infty$ to -1.

It is observed that as θ passes through $\frac{\pi}{2}$, $\sec\theta$ changes from $+\infty$ to $-\infty$.

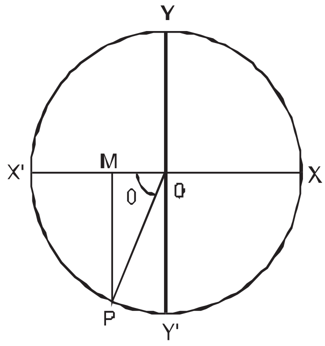


Fig.16.26

III Quadrant : $\sec\theta$ is negative as *OM* is negative. $\sec\pi = -1$ and $\sec\frac{3\pi}{2} = -\infty$ when the angle approaches $\frac{3\pi}{2}$ in the counter clockwise direction. As θ varies from π to $\frac{3\pi}{2}$, $\sec\theta$ decreases from -1 to $-\infty$.

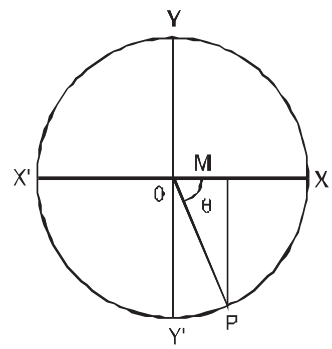


Fig.16.27

IV Quadrant : $\sec\theta$ is positive as OM is positive. when θ is slightly greater than $\frac{3\pi}{2}$, $\sec\theta$ is positive and very large. Also $\sec 2\pi = 1$. Hence $\sec\theta$ decreases from ∞ to 1 as θ varies from $\frac{3\pi}{2}$ to 2π.

It may be observed that as θ passes through $\frac{3\pi}{2}$; $\sec\theta$ changes from $-\infty$ to $+\infty$.

Graph of sec θ as θ varies from 0 to 2π

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0$	$\frac{\pi}{2}+0$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0$	$\frac{3\pi}{2}+0$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cot\theta$	1	1.15	2	$+\infty$	$-\infty$	-2	-1.15	-1	-1.15	-2	$-\infty$	$+\infty$	2	1.15	

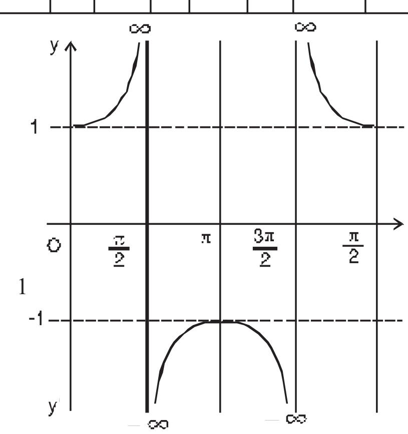


Fig. 16.28

Observations

(a) $\sec\theta$ cannot be numerically less than 1.

(b) Graph of $\sec\theta$ is discontinuous, discontinuties (breaks) occuring at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

(c) As θ passes through $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, see θ changes abruptly from $+\infty$ to $-\infty$ and then from $-\infty$ to $+\infty$ respectively.

16.4.7 Graph of cosec θ as θ Varies From 0 to 2π

Let $X'OX$ and $Y'OY$ be the axes of coordinates. With centre O draw a circle of unit radius. Let P be any point on the circle. Join OP and draw PM perpendicular to $X'OX$.

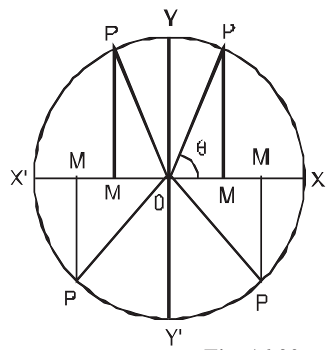


Fig. 16.29

$$\text{cosec}\,\theta = \frac{OP}{MP} = \frac{1}{MP}$$

$\therefore$ The variation ofcosec θ will depend upon MP.

I Quadrant : cosec θ is positive as MP is positive.

cosec $\frac{\pi}{2} = 1$ when θ is very small, MP is also small and therefore, the value of cosec θ is very large.

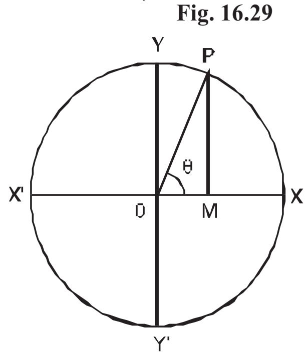


Fig. 16.30

$\therefore$ As θ varies from 0 to $\frac{\pi}{2}$, cosec θ decreases from ∞ to 1.

II Quadrant : PM is positive. Therefore, cosec θ is positive. cosec $\frac{\pi}{2} = 1$ and cosec $\pi = \infty$ when the revolving line approaches π in the counter clockwise direction.

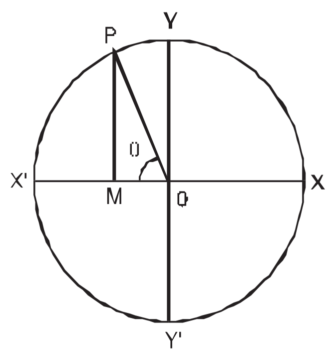


Fig. 16.31

$\therefore$ As θ varies from $\frac{\pi}{2}$ to π, cosec θ increases from 1 to ∞.

III Quadrant : PM is negative

$\therefore$ cosec θ is negative. When θ is slightly greater than π, cosec θ is very large and negative.

Also cosec $\frac{3\pi}{2} = -1$.

$\therefore$ As θ varies from π to $\frac{3\pi}{2}$, cosecθ changes from $-\infty$ to -1.

It may be observed that as θ passes through π, cosec θ changes from $+\infty$ to $-\infty$.

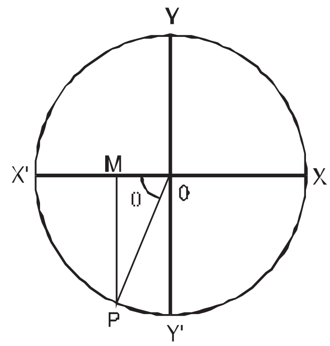


Fig. 16.32

IV Quadrant :

PM is negative.

Therefore, cosec $\theta = -\infty$ as θ approaches 2π.

$\therefore$ as θ varies from $\frac{3\pi}{2}$ to 2π, cosecθ varies from -1 to $-\infty$.

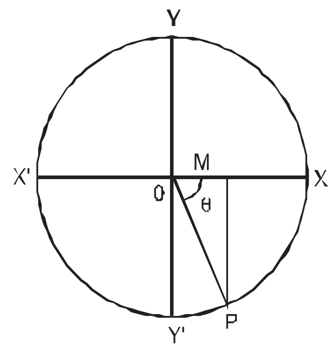


Fig. 16.33

Graph of cosec θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi-0$	$\pi+0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
cosec θ	∞	2	1.15	1	1.15	2	$+\infty$	$-\infty$	-2	-1.15	-1	-1.15	-2	$-\infty$

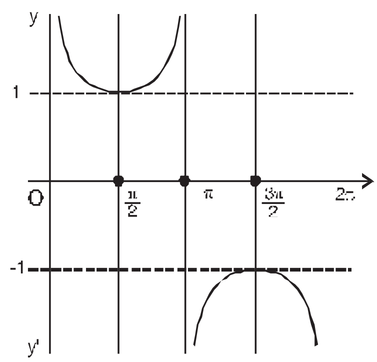


Fig. 16.34

Observations

(a) cosec θ cannot be numerically less than 1.

(b) Graph of cosec θ is discountinous and it has breaks at $\theta = 0$, π, 2π.

(c) As θ passes through π, coses θ changes from $+\infty$ to $-\infty$. The values at 0 and 2π are $+\infty$ and $-\infty$ respectively.

Example 16.17 Trace the changes in the values of $\sec\theta$ as θ lies in $-\pi$ to π.

Soluton :

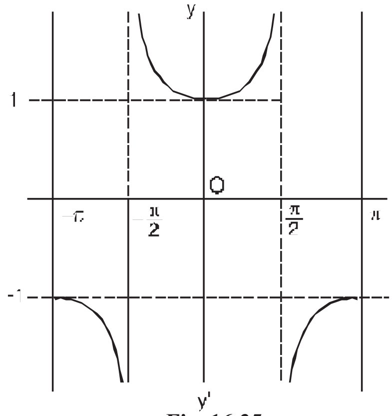


Fig. 16.35

CHECK YOUR PROGRESS 16.9

1. (a) Trace the changes in the values of $\sec\theta$ when θ lies between -2π and 2π and draw the graph between these limits.

   (b) Trace the graph of cosec θ ,when θ lies between -2π and 2π.

16.5 PERIODICITY OF THE TRIGONOMETRIC FUNCTIONS

From your daily experience you must have observed things repeating themselves after regular intervals of time. For example, days of a week are repeated regularly after 7 days and months of a year are repeated regularly after 12 months. Position of a particle on a moving wheel is another example of the type. The property of repeated occurence of things over regular intervals is known as ***periodicity.***

Definition : A function $f(x)$ is said to be periodic if its value is unchanged when the value of the variable in increased by a constant, that is if $f(x+p)=f(x)$ for all x.

If p is smallest positive constant of this type, then p is called the period of the function $f(x)$.

If $f(x)$ is a periodic function with period p, then $\frac{1}{f(x)}$ is also a periodic function with period p.

16.5.1 Periods of Trigonometric Functions

(i) $\sin x = \sin(x+2n\pi)$; $n = 0, \pm 1, \pm 2, ...$

(ii) $\cos x = \cos(x+2n\pi)$; $n = 0, \pm 1, \pm 2, ...$

Also there is no p, lying in 0 to 2π, for which

$\sin x = \sin(x+p)$

$\cos x = \cos(x+p)$, for all x

$\therefore$ 2π is the smallest positive value for which

$\sin(x+2\pi) = \sin x$ and $\cos(x+2\pi) = \cos x$

$\Rightarrow$ sin x and cos x each have the period 2π.

(iii) The period of cosec x is also 2π because cosec x $= \dfrac{1}{\sin x}$.

(iv) The period of sec x is also 2π as sec x $= \dfrac{1}{\cos x}$.

(v) Also $\tan(x+\pi) = \tan x$.

Suppose $p\ (0 < p < \pi)$ is the period of tan x, then

$\tan(x+p) = \tan x$, for all x.

Put x = 0, then tan p = 0, i.e., p = 0 or π.

$\Rightarrow$ the period of tan x is π.

$\therefore$ p can not values between 0 and π for which $\tan x = \tan(x+p)$

$\therefore$ The period of $\tan x$ is π

(vi) Since $\cot x = \dfrac{1}{\tan x}$, therefore, the period of cot x is also π.

Example 16.18 Find the period of each the following functions :

(a) y = 3 sin 2x (b) $y = \cos\dfrac{x}{2}$ (c) $y = \tan\dfrac{x}{4}$

Solution :

(a) Period is $\dfrac{2\pi}{2}$, i.e., π.

(b) $y = \cos\dfrac{1}{2}x$, therefore period $= \dfrac{2\pi}{\frac{1}{2}} = 4\pi$

(c) Period of y = $\tan\dfrac{x}{4} = \dfrac{\pi}{\frac{1}{4}} = 4\pi$

CHECK YOUR PROGRESS 16.10

1. Find the period of each of the following functions :

(a) y = 2 sin 3x (b) y = 3 cos 2x

(c) y = tan 3x (d) $y = \sin^2 2x$

LET US SUM UP

- An angle is generated by the rotation of a ray.
- The angle can be negative or positive according as rotation of the ray is clockwise or anticlockwise.
- A degree is one of the measures of an angle and one complete rotation generates an angle of 360°.
- An angle can be measured in radians, 360° being equivalent to 2π radians.
- If an arc of length l subtends an angle of θ radians at the centre of the circle with radius r, we have $l = r\theta$.
- If the coordinates of a point P of a unit circle are (x, y) then the six trigonometric functions are defined as $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$, $\sec\theta = \frac{1}{x}$ and $\operatorname{cosec}\theta = \frac{1}{y}$.

  The coordinates (x, y) of a point P can also be written as $(\cos\theta, \sin\theta)$.

  Here θ is the angle which the line joining centre to the point P makes with the positive direction of x-axis.
- The values of the trigonometric functions $\sin\theta$ and $\cos\theta$ when θ takes values 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ are given by

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

- Graphs of $\sin\theta$, $\cos\theta$ are continous every where
  - — Maximum value of both $\sin\theta$ and $\cos\theta$ is 1.
  - — Minimum value of both $\sin\theta$ and $\cos\theta$ is -1.
  - — Period of these functions is 2π.

- $\tan\theta$ and $\cot\theta$ can have any value between $-\infty$ and $+\infty$.

— The function $\tan\theta$ has discontinuities (breaks) at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ in $(0,\ 2\pi)$.

— Its period is π.

— The graph of $\cot\theta$ has discontinuities (breaks) at 0, π, 2π. Its period is π.

- $\sec\theta$ cannot have any value numerically less than 1.

(i) It has breaks at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. It repeats itself after 2π.

(ii) $\operatorname{cosec}\theta$ cannot have any value between -1 and $+1$.

It has discontinuities (breaks) at 0, π, 2π.

It repeats itself after 2π.

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. A train is moving at the rate of 75 km/hour along a circular path of radius 2500 m. Through how many radians does it turn in one minute ?

2. Find the number of degrees subtended at the centre of the circle by an arc whose length is 0.357 times the radius.

3. The minute hand of a clock is 30 cm long. Find the distance covered by the tip of the hand in 15 minutes.

4. Prove that

(a) $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$ (b) $\dfrac{1}{\sec\theta + \tan\theta} = \sec\theta - \tan\theta$

(c) $\dfrac{\tan\theta}{1+\tan^2\theta} - \dfrac{\cot\theta}{1+\cot^2\theta} = 2\sin\theta\cos\theta$ (d) $\dfrac{1+\sin\theta}{1-\sin\theta} = (\tan\theta + \sec\theta)^2$

(e) $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

(f) $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$

5. If $\theta = \frac{\pi}{4}$, verify that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

6. Evaluate :

(a) $\sin\frac{25\pi}{6}$ (b) $\sin\frac{21\pi}{4}$ (c) $\tan\left(\frac{3\pi}{4}\right)$

(d) $\sin\frac{17}{4}\pi$ (e) $\cos\frac{19}{3}\pi$

7. Draw the graph of cos x from $x = -\frac{\pi}{2}$ to $x = \frac{3\pi}{2}$.

8. Define a periodic function of x and show graphically that the period of tan x is π, i.e. the position of the graph from x = π to 2π is repetition of the portion from x = 0 to π.

ANSWERS

CHECK YOUR PROGRESS 16.1

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{12}$ (iii) $\frac{5\pi}{12}$ (iv) $\frac{7\pi}{12}$ (v) $\frac{3\pi}{2}$

2. (i) 45° (ii) 15° (iii) 9° (iv) 3° (v) 120°

3. $\frac{\pi}{4}, \frac{13\pi}{36}, \frac{14\pi}{36}$ 4. $\frac{5\pi}{6}$ 5. $\frac{\pi}{3}$

CHECK YOUR PROGRESS 16.2

1. (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{5\pi}{6}$

2. (a) 36° (b) 30° (c) 20°

3. $\frac{1}{6}$ radian; 9.55° 4. $\frac{1}{5}$ radian 5. 95.54 m

6. (a) 0.53 m (b) 38.22 cm (c) 0.002 radian
(d) 12.56 m (e) 31.4 cm (f) 3.75 radian
(g) 6.28 m (h) 2 radian (i) 19.11 m.

CHECK YOUR PROGRESS 16.3

1. (i) – ive (ii) – ive (iii) – ive (iv) + ive
(v) + ive (vi) – ive (vii) + ive (viii) – ive

2. (i) zero (ii) zero (iii) $-\frac{1}{2}$ (iv) – 1
(v) 1 (vi) Not defined (vii) Not defined (viii) 1

CHECK YOUR PROGRESS 16.4

2. $\sin\theta = \frac{1}{\sqrt{5}}$, $\cos\theta = \frac{2}{\sqrt{5}}$, $\cot\theta = 2$, $\operatorname{cosec}\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$

3. $\sin\theta = \frac{a}{b}$, $\cos\theta = \frac{\sqrt{b^2 - a^2}}{b}$, $\sec\theta = \frac{b}{\sqrt{b^2 - a^2}}$,

$\tan\theta = \frac{a}{\sqrt{b^2 - a^2}}$, $\cot\theta = \frac{\sqrt{b^2 - a^2}}{a}$ 6. $\frac{2m}{1 + m^2}$

CHECK YOUR PROGRESS 16.5

1. (i) $4\frac{1}{4}$ (ii) $6\frac{1}{2}$ (iii) -1 (iv) $\frac{22}{3}$ (v) Zero

CHECK YOUR PROGRESS 16.6

1. 1, −1 3. Graph of $y = 2\sin\theta$, $[0, \pi]$

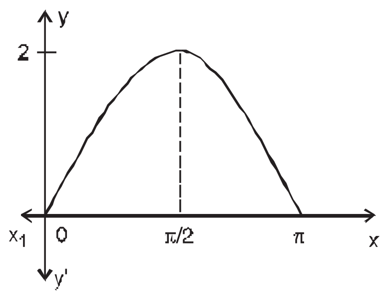


Fig. 16.36

4. (a) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{4\pi}{3}, \frac{5\pi}{3}$ 5. $y = \sin x$ from $-\pi$ to π

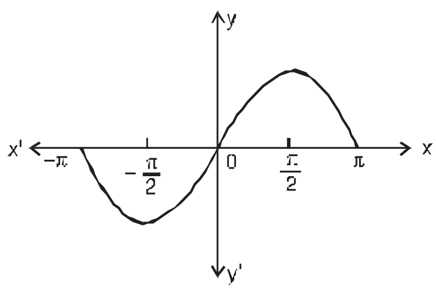


Fig. 16.37

CHECK YOUR PROGRESS 16.7

1. (a) $y = \cos\theta$ $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

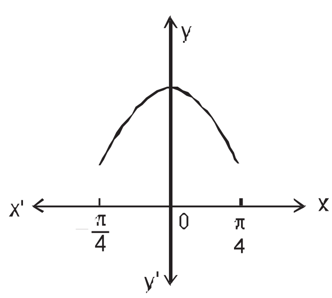


Fig. 16.38

(b) $y = 3\cos\theta$, 0 to 2π

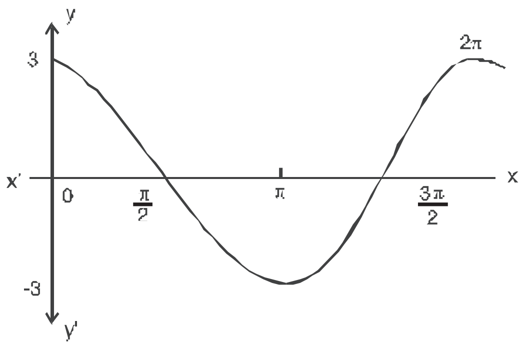


Fig. 16.39

(c) $y = \cos 3\theta$, $-\pi$ to π

$$\cos\theta = 0.87$$

$$\theta = \frac{\pi}{6}, -\frac{\pi}{6}$$

$$\cos\theta = -0.87$$

$$\theta = \frac{5\pi}{6}, -\frac{5\pi}{6}$$

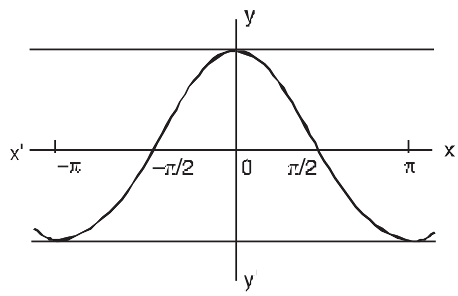


Fig. 16.40

(d) Graph of $y = \cos\theta$ in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ lies below the x-axis.

(e) $y = \cos\theta$

θ lies in 2π to 4π

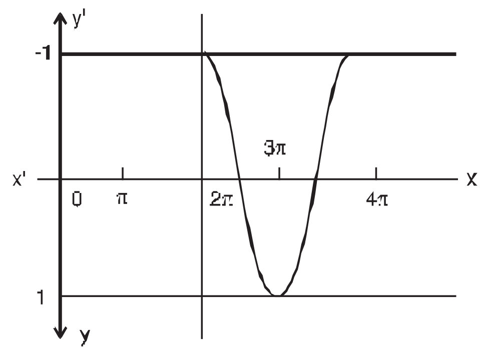


Fig. 16.41

CHECK YOUR PROGRESS 16.8

1. (a) Infinite (b) At $\frac{\pi}{2}, \frac{3\pi}{2}$ there are breaks in graphs.

(c) $y = \tan 2\theta$, $-\pi$ to π

At $\theta = \dfrac{\pi}{3}, \tan\theta = 1.7$

2. (a) Infinite (b) $\cot\theta = -1$ at $\theta = \dfrac{3\pi}{4}$

CHECK YOUR PROGRESS 16.9

1. (a) $y = \sec\theta$

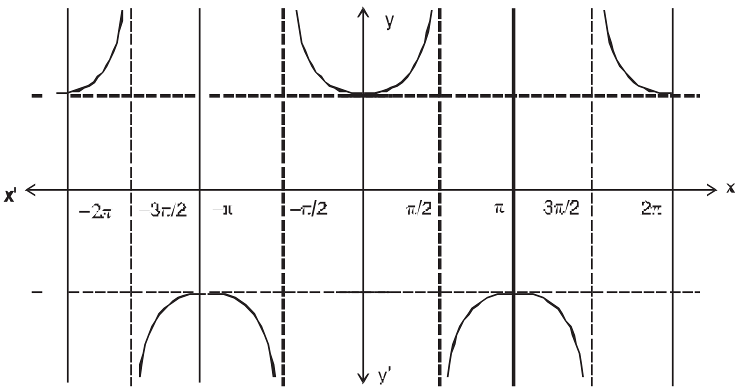


Fig. 16.42

Points of discontinuity of $\sec 2\theta$ are at $\dfrac{\pi}{4}, \dfrac{3\pi}{4}$ in the interval $[0, 2\pi]$.

(b) In tracing the graph from 0 to -2π, use $\text{cosec}(-\theta) = -\text{cosec}\ \theta$.

CHECK YOUR PROGRESS 16.10

1. (a) Period is $\dfrac{2\pi}{3}$ (b) Period is $\dfrac{2\pi}{2} = \pi$ (c) Period of y is $\dfrac{\pi}{3}$

(d) $y = \sin^2 2x = \dfrac{1-\cos 4x}{2} = \dfrac{1}{2} - \dfrac{1}{2}\cos 4x$; Period of y is $\dfrac{2\pi}{4}$ i.e $\dfrac{\pi}{2}$

(e) $y = 3\cot\left(\dfrac{x+1}{3}\right)$, Period of y is $\dfrac{\pi}{\frac{1}{3}} = 3\pi$

TERMINAL EXERCISE

1. $\dfrac{1}{2}$ radian 2. $20.45°$ 3. 15π cm

6. (a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{\sqrt{2}}$ (c) -1 (d) $\dfrac{1}{\sqrt{2}}$ (e) $\dfrac{1}{2}$

7.

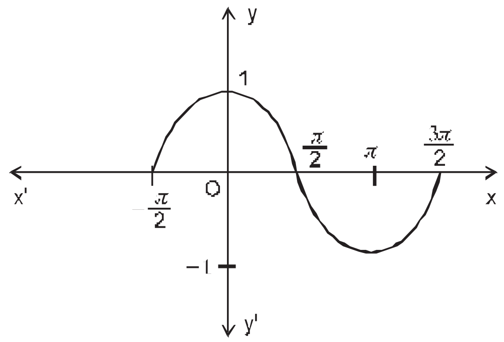
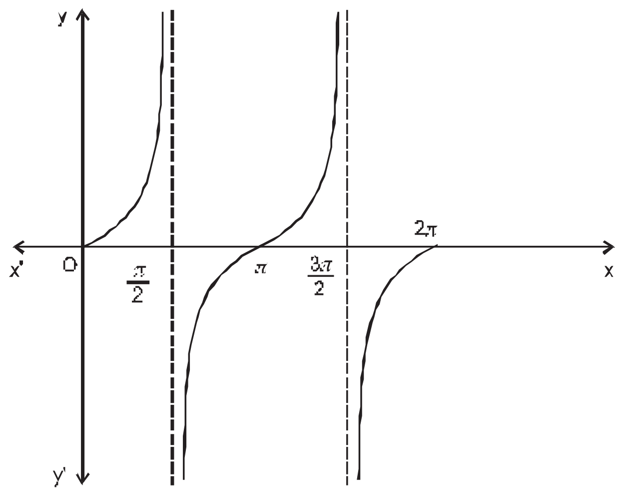


Fig. 16.43

8.



$y = \sec\theta$

Fig. 16.44

17

TRIGONOMETRIC FUNCTIONS-II

In the previous lesson, you have learnt trigonometric functions of real numbers, draw and interpret the graphs of trigonometric functions. In this lesson we will establish addition and subtraction formulae for $\cos(A \pm B)$, $\sin(A \pm B)$ and $\tan(A \pm B)$. We will also state the formulae for the multiple and sub multiples of angles and solve examples thereof. The general solutions of simple trigonometric functions also discussed in the lesson.

OBJECTIVES

After studying this lesson, you will be able to :

- write trigonometric functions of $-x, \frac{x}{2}, x \pm y, \frac{\pi}{2} \pm x, \pi \pm x$ where x, y are real nunbers;
- establish the addition and subtraction formulae for :

  $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$,

  $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ and $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- solve problems using the addition and subtraction formulae;
- state the formulae for the multiples and sub-multiples of angles such as cos2A, sin 2A, tan 2A, cos 3A, sin 3A, tan 3A, $\sin\frac{A}{2}, \cos\frac{A}{2}$ and $\tan\frac{A}{2}$; and
- solve simple trigonometric equations of the type :

  $\sin x = 0, \cos x = 0$, $\tan x = 0$,

  $\sin x = \sin\alpha$, $\cos x = \cos\alpha$, $\tan x = \tan\alpha$

EXPECTED BACKGROUND KNOWLEDGE

- Definition of trigonometric functions.
- Trigonometric functions of complementary and supplementary angles.
- Trigonometric identities.

17.1 ADDITION AND MULTIPLICATION OF TRIGONOMETRIC FUNCTIONS

In earlier sections we have learnt about circular measure of angles, trigonometric functions, values of trigonometric functions of specific numbers and of allied numbers.

You may now be interested to know whether with the given values of trigonometric functions of any two numbers *A* and *B*, it is possible to find trigonometric functions of sums or differences.

You will see how trigonometric functions of sum or difference of numbers are connected with those of individual numbers. This will help you, for instance, to find the value of trigonometric functions of $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ etc.

$\frac{\pi}{12}$ can be expressed as $\frac{\pi}{4}-\frac{\pi}{6}$

$\frac{5\pi}{12}$ can be expressed as $\frac{\pi}{4}+\frac{\pi}{6}$

How can we express $\frac{7\pi}{12}$ in the form of addition or subtraction?

In this section we propose to study such type of trigonometric functions.

17.1.1 Addition Formulae

For any two numbers A and B,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

In given figure trace out

$\angle SOP = A$

$\angle POQ = B$

$\angle SOR = -B$

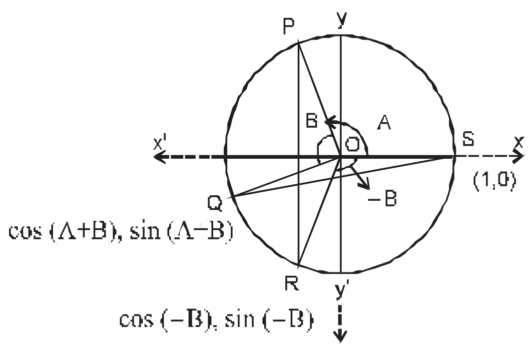


Fig. 17.1

where points P, Q, R, S lie on the unit circle.

Coordinates of P, Q, R, S will be (cos A, sin A), [cos (A + B), sin (A + B)],

$\left[\cos(-B), \sin(-B)\right]$, and (1, 0).

From the given figure, we have

side OP = side OQ

$\angle$POR = $\angle$QOS (each angle = $\angle$B + $\angle$ QOR)

side OR = side OS

ΔPOR $\cong$ ΔQOS (by SAS)

$\therefore$ PR = QS

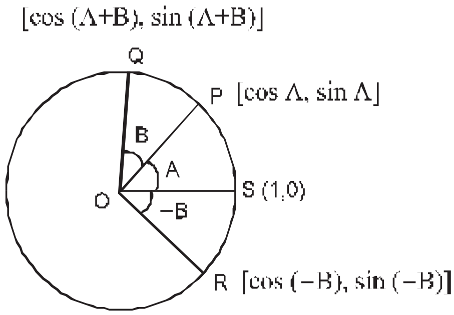


Fig. 17.2

$$PR = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin(-B))^2}$$

$$QS = \sqrt{(\cos(A+B) - 1)^2 + (\sin(A+B) - 0)^2}$$

Since $PR^2 = QS^2$

$$\therefore \cos^2 A + \cos^2 B - 2\cos A\cos B + \sin^2 A + \sin^2 B + 2\sin A\sin B$$

$$= \cos^2(A+B) + 1 - 2\cos(A+B) + \sin^2(A+B)$$

$$\Rightarrow \quad 1 + 1 - 2(\cos A\cos B - \sin A\sin B) = 1 + 1 - 2\cos(A+B)$$

$$\Rightarrow \quad \cos A\cos B - \sin A\sin B = \cos(A+B) \qquad \text{(I)}$$

Corollary 1

For any two numbers A and B, cos (A − B) = cos A cos B + sin A sin B

Proof : Replace B by $-B$ in (I)

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$[\because \cos(-B) = \cos B$ and $\sin(-B) = -\sin B]$

Corollary 2

For any two numbers A and B

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

Proof : We know that $\cos\left(\frac{\pi}{2} - A\right) = \sin A$

and $\qquad \sin\left(\frac{\pi}{2} - A\right) = \cos A$

$$\therefore \quad \sin(A+B) = \cos\left[\left(\frac{\pi}{2} - (A+B)\right)\right]$$

$$= \cos\left[\left(\frac{\pi}{2} - A\right) - B\right]$$

$$= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

or $\quad \sin(A+B) = \sin A \cos B + \cos A \sin B$ .....(II)

Corollary 3

For any two numbers A and B

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Proof : Replacing B by $-B$ in (2), we have

$$\sin(A+(-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

or $\quad \sin(A-B) = \sin A \cos B - \cos A \sin B$

Example 17.1

(a) Find the value of each of the following :

(i) $\sin\dfrac{5\pi}{12}$ (ii) $\cos\dfrac{\pi}{12}$ (iii) $\cos\dfrac{7\pi}{12}$

(b) If $\sin A = \dfrac{1}{\sqrt{10}}$, $\sin B = \dfrac{1}{\sqrt{5}}$ show that $A + B = \dfrac{\pi}{4}$

Solution :

(a) (i) $\sin\dfrac{5\pi}{12} = \sin\left(\dfrac{\pi}{4} + \dfrac{\pi}{6}\right) = \sin\dfrac{\pi}{4}\cdot\cos\dfrac{\pi}{6} + \cos\dfrac{\pi}{4}\cdot\sin\dfrac{\pi}{6}$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\therefore \quad \sin\dfrac{5\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

(ii) $\cos\dfrac{\pi}{12} = \cos\left(\dfrac{\pi}{4} - \dfrac{\pi}{6}\right)$

$$= \cos\frac{\pi}{4}\cdot\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\therefore \quad \cos\dfrac{\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

Observe that $\sin\dfrac{5\pi}{12} = \cos\dfrac{\pi}{12}$

(iii) $\cos\dfrac{7\pi}{12} = \cos\left(\dfrac{\pi}{3}+\dfrac{\pi}{4}\right)$

$$= \cos\frac{\pi}{3}\cdot\cos\frac{\pi}{4} \quad \sin\frac{\pi}{3}\cdot\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$\therefore \qquad \cos\dfrac{7\pi}{12} = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$

(b) $\quad \sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\cos A = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}} \text{ and } \cos B = \sqrt{1-\frac{1}{5}} = \frac{2}{\sqrt{5}}$$

Substituting all these values in (II), we get

$$\sin(A+B) = \frac{1}{\sqrt{10}}\frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}}\frac{1}{\sqrt{5}}$$

$$= \frac{5}{\sqrt{10}\sqrt{5}} + \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

or $\qquad A+B = \dfrac{\pi}{4}$

CHECK YOUR PROGRESS 17.1

1. (a) Find the values of each of the following :

   (i) $\sin\dfrac{\pi}{12}$ $\qquad$ (ii) $\sin\dfrac{\pi}{9}\cdot\cos\dfrac{2\pi}{9} + \cos\dfrac{\pi}{9}\cdot\sin\dfrac{2\pi}{9}$

   (b) Prove the following :

   (i) $\sin\left(\dfrac{\pi}{6}+A\right) = \dfrac{1}{2}\left(\cos A + \sqrt{3}\sin A\right)$ $\quad$ (ii) $\sin\left(\dfrac{\pi}{4}-A\right) = \dfrac{1}{\sqrt{2}}(\cos A - \sin A)$

   (c) If $\sin A = \dfrac{8}{17}$ and $\sin B = \dfrac{5}{13}$, find $\sin(A-B)$

2. (a) Find the value of $\cos\dfrac{5\pi}{12}$.

   (b) Prove the following :

   (i) $\cos\theta + \sin\theta = \sqrt{2}\cos\left(\theta - \dfrac{\pi}{4}\right)$ $\qquad$ (ii) $\sqrt{3}\sin\theta - \cos\theta = 2\sin\left(\theta - \dfrac{\pi}{6}\right)$

(iii) $\cos(n+1)A\cos(n-1)A+\sin(n+1)A\sin(n-1)A=\cos 2A$

(iv) $\cos\left(\frac{\pi}{4}+A\right)\cos\left(\frac{\pi}{4}-B\right)+\sin\left(\frac{\pi}{4}+A\right)\sin\left(\frac{\pi}{4}-B\right)=\cos(A+B)$

Corollary 4 : $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

Proof : $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

Dividing by cos A, cos B, we have

$$\tan(A+B)=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

or $\qquad \tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ ......(III)

Corollary 5 : $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

Proof : Replacing B by $-B$ in (III), we get the required result.

Corollary 6 : $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$

Proof : $\cot(A+B)=\dfrac{\cos(A+B)}{\sin(A+B)}=\dfrac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$

Dividing by sin A sin B, we have ......(IV)

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot B+\cot A}$$

Corollary 7 : $\tan\left(\frac{\pi}{4}+A\right)=\dfrac{1+\tan A}{1-\tan A}$

Proof : $\tan\left(\frac{\pi}{4}+A\right)=\dfrac{\tan\frac{\pi}{4}+\tan A}{1-\tan\frac{\pi}{4}\cdot\tan A}$

$$=\frac{1+\tan A}{1-\tan A}\quad \text{as } \tan\frac{\pi}{4}=1$$

Similarly, it can be proved that

$$\tan\left(\frac{\pi}{4}-A\right)=\frac{1-\tan A}{1+\tan A}$$

Example 17.2 Find $\tan\frac{\pi}{12}$

Solution : $\tan\frac{\pi}{12}=\tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\dfrac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}}$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+1.\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}=\frac{4-2\sqrt{3}}{2}$$

$$=2-\sqrt{3}\qquad\therefore\qquad\tan\frac{\pi}{12}=2-\sqrt{3}$$

Example 17.3 Prove the following :

(a) $\dfrac{\cos\frac{7\pi}{36}+\sin\frac{7\pi}{36}}{\cos\frac{7\pi}{36}-\sin\frac{7\pi}{36}}=\tan\frac{4\pi}{9}$

(b) $\tan 7A-\tan 4A-\tan 3A=\tan 7A\tan 4A\cdot\tan 3A$

(c) $\tan\frac{7\pi}{18}=\tan\frac{\pi}{9}+2\tan\frac{5\pi}{18}$

Solution : (a) Dividing numerator and denominator by $\cos\frac{7\pi}{36}$, we get

$$\text{L.H.S.}=\frac{\cos\frac{7\pi}{36}+\sin\frac{7\pi}{36}}{\cos\frac{7\pi}{36}-\sin\frac{7\pi}{36}}=\frac{1+\tan\frac{7\pi}{36}}{1-\tan\frac{7\pi}{36}}$$

$$=\frac{\tan\frac{\pi}{4}+\tan\frac{7\pi}{36}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{7\pi}{36}}$$

$$=\tan\left(\frac{\pi}{4}+\frac{7\pi}{36}\right)$$

$$= \tan\frac{16\pi}{36} = \tan\frac{4\pi}{9} = \text{R.H.S.}$$

(b) $\tan 7A = \tan(4A + 3A) = \dfrac{\tan 4A + \tan 3A}{1 - \tan 4A \tan 3A}$

or $\quad \tan 7A - \tan 7A \tan 4A \tan 3A = \tan 4A + \tan 3A$

or $\quad \tan 7A - \tan 4A - \tan 3A = \tan 7A \tan 4A \tan 3A$

(c) $$\tan\frac{7\pi}{18} = \tan\left(\frac{5\pi}{18} + \frac{2\pi}{18}\right) = \frac{\tan\frac{5\pi}{18} + \tan\frac{2\pi}{18}}{1 - \tan\frac{5\pi}{18}\tan\frac{2\pi}{18}}$$

$$\tan\frac{7\pi}{18} - \tan\frac{7\pi}{18}\tan\frac{5\pi}{18}\tan\frac{2\pi}{18} = \tan\frac{5\pi}{18} + \tan\frac{2\pi}{18} \qquad \text{.....(1)}$$

$$\tan\frac{7\pi}{18} = \tan\left(\frac{\pi}{2} - \frac{\pi}{9}\right) = \cot\frac{\pi}{9} = \cot\frac{2\pi}{18}$$

$\therefore$ (1) can be written as

$$\tan\frac{7\pi}{18} - \cot\frac{2\pi}{18}\tan\frac{5\pi}{18}\tan\frac{2\pi}{18} = \tan\frac{5\pi}{18} + \tan\frac{\pi}{9}$$

$$\therefore \quad \tan\frac{7\pi}{18} = \tan\frac{\pi}{9} + 2\tan\frac{5\pi}{18}$$

CHECK YOUR PROGRESS 17.2

1. Fill in the blanks :

(i) $\sin\left(\frac{\pi}{4} + A\right)\sin\left(\frac{\pi}{4} - A\right) = \text{.........}$ (ii) $\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \text{.........}$

2. (a) Prove the following :

(i) $\tan\left(\frac{\pi}{4} + \theta\right)\tan\left(\frac{\pi}{4} - \theta\right) = 1.$ (ii) $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(iii) $\tan\frac{\pi}{12} + \tan\frac{\pi}{6} + \tan\frac{\pi}{12}\cdot\tan\frac{\pi}{6} = 1$

(b) If $\tan A = \frac{a}{b}$; $\tan B = \frac{c}{d}$, Prove that

$$\tan(A + B) = \frac{ad + bc}{bd - ac}.$$

(c) Find the value of $\cos\frac{11\pi}{12}$.

3. (a) Prove the following :

(i) $\tan\left(\frac{\pi}{4}+A\right)\tan\left(\frac{3\pi}{4}+A\right)=-1$ (ii) $\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}=\tan\left(\frac{\pi}{4}+\theta\right)$

(iii) $\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\tan\left(\frac{\pi}{4}-\theta\right)$

17.2 TRANSFORMATION OF PRODUCTS INTO SUMS AND VICE VERSA

17.2.1 Transformation of Products into Sums or Differences

We know that

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

By adding and subtracting the first two formulae, we get respectively

$$2\sin A\cos B=\sin(A+B)+\sin(A-B) \qquad .....(1)$$

and $$2\cos A\sin B=\sin(A+B)-\sin(A-B) \qquad .....(2)$$

Similarly, by adding and subtracting the other two formulae, we get

$$2\cos A\cos B=\cos(A+B)+\cos(A-B) \qquad ....(3)$$

and $$2\sin A\sin B=\cos(A-B)-\cos(A+B) \qquad ....(4)$$

We can also quote these as

$$2\sin A\cos B=\sin(\text{sum})+\sin(\text{difference})$$

$$2\cos A\sin B=\sin(\text{sum})-\sin(\text{difference})$$

$$2\cos A\cos B=\cos(\text{sum})+\cos(\text{difference})$$

$$2\sin A\sin B=\cos(\text{difference})-\cos(\text{sum})$$

17.2.2 Transformation of Sums or Differences into Products

In the above results put

$$A + B = C$$

$$\text{and} \quad A - B = D$$

Then $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$ and (1), (2), (3) and (4) become

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$$

17.2.3 Further Applications of Addition and Subtraction Formulae

We shall prove that

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$ or $\cos^2 B - \sin^2 A$

Proof : (i) $\sin(A+B)\sin(A-B)$

$$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$$

$$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$$

$$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$$

$$= \sin^2 A - \sin^2 B$$

(ii) $\cos(A+B)\cos(A-B)$

$$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$$

$$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$$

$$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$$

$$= \cos^2 A - \sin^2 B$$

$$= (1-\sin^2 A) - (1-\cos^2 B)$$

$$= \cos^2 B - \sin^2 A$$

Example 17.4 Express the following products as a sum or difference

(i) $2\sin 3\theta\cos 2\theta$ (ii) $\cos 6\theta\cos\theta$ (iii) $\sin\dfrac{5\pi}{12}\sin\dfrac{\pi}{12}$

Solution :

(i) $2\sin 3\theta \cos 2\theta = \sin(3\theta + 2\theta) + \sin(3\theta - 2\theta)$

$= \sin 5\theta + \sin\theta$

(ii) $\cos 6\theta \cos\theta = \dfrac{1}{2}(2\cos 6\theta \cos\theta)$

$= \dfrac{1}{2}[\cos(6\theta + \theta) + \cos(6\theta - \theta)]$

$= \dfrac{1}{2}(\cos 7\theta + \cos 5\theta)$

(iii) $\sin\dfrac{5\pi}{12}\sin\dfrac{\pi}{12} = \dfrac{1}{2}\left[2\sin\dfrac{5\pi}{12}\sin\dfrac{\pi}{12}\right]$

$= \dfrac{1}{2}\left[\cos\left(\dfrac{5\pi - \pi}{12}\right) - \cos\left(\dfrac{5\pi + \pi}{12}\right)\right]$

$= \dfrac{1}{2}\left[\cos\dfrac{\pi}{3} - \cos\dfrac{\pi}{2}\right]$

Example 17.5 Express the following sums as products.

(i) $\cos\dfrac{5\pi}{9} + \cos\dfrac{7\pi}{9}$ (ii) $\sin\dfrac{5\pi}{36} + \cos\dfrac{7\pi}{36}$

Solution :

(i) $\cos\dfrac{5\pi}{9} + \cos\dfrac{7\pi}{9} = 2\cos\dfrac{5\pi + 7\pi}{9 \times 2}\cos\dfrac{5\pi - 7\pi}{9 \times 2}$

$= 2\cos\dfrac{2\pi}{3}\cos\dfrac{\pi}{9}$ $\left[\because \cos\left(-\dfrac{\pi}{9}\right) = \cos\dfrac{\pi}{9}\right]$

$= 2\cos\left(\pi - \dfrac{\pi}{3}\right)\cos\dfrac{\pi}{9}$

$= -2\cos\dfrac{\pi}{3}\cos\dfrac{\pi}{9}$

$= -\cos\dfrac{\pi}{9}$ $\left[\because \cos\dfrac{\pi}{3} = \dfrac{1}{2}\right]$

(ii) $\sin\dfrac{5\pi}{36} + \cos\dfrac{7\pi}{36} = \sin\left(\dfrac{\pi}{2} - \dfrac{13\pi}{36}\right) + \cos\dfrac{7\pi}{36}$

$= \cos\dfrac{13\pi}{36} + \cos\dfrac{7\pi}{36}$

$$= 2\cos\frac{13\pi + 7\pi}{36 \times 2}\cos\frac{13\pi - 7\pi}{36 \times 2}$$

$$= 2\cos\frac{5\pi}{18}\cos\frac{\pi}{12}$$

Example 17.6 Prove that $\dfrac{\cos 7A - \cos 9A}{\sin 9A - \sin 7A} = \tan 8A$

Solution :

$$\text{L.H.S.} = \frac{2\sin\dfrac{7A + 9A}{2}\sin\dfrac{9A - 7A}{2}}{2\cos\dfrac{9A + 7A}{2}\sin\dfrac{9A - 7A}{2}}$$

$$= \frac{\sin 8A \sin A}{\cos 8A \sin A} = \frac{\sin 8A}{\cos 8A} = \tan 8A = \text{R.H.S.}$$

Example 17.7 Prove the following :

(i) $\cos^2\left(\dfrac{\pi}{4} - A\right) - \sin^2\left(\dfrac{\pi}{4} - B\right) = \sin(A + B)\cos(A - B)$

(ii) $\sin^2\left(\dfrac{\pi}{8} + \dfrac{A}{2}\right) - \sin^2\left(\dfrac{\pi}{8} - \dfrac{A}{2}\right) = \dfrac{1}{\sqrt{2}}\sin A$

Solution :

(i) Applying the formula

$\cos^2 A - \sin^2 B = \cos(A + B)\cos(A - B)$, we have

$$\text{L.H.S.} = \cos\left[\frac{\pi}{4} - A + \frac{\pi}{4} - B\right]\cos\left[\frac{\pi}{4} - A - \frac{\pi}{4} + B\right]$$

$$= \cos\left[\frac{\pi}{2} - (A + B)\right]\cos\left[-(A - B)\right]$$

$$= \sin(A + B)\cos(A - B) = \text{R.H.S.}$$

(ii) Applying the formula

$\sin^2 A - \sin^2 B = \sin(A + B)\sin(A - B)$, we have

$$\text{L.H.S.} = \sin\left(\frac{\pi}{8} + \frac{A}{2} + \frac{\pi}{8} - \frac{A}{2}\right)\sin\left(\frac{\pi}{8} + \frac{A}{2} - \frac{\pi}{8} + \frac{A}{2}\right)$$

$$= \sin\frac{\pi}{4}\sin A$$

$$= \frac{1}{\sqrt{2}}\sin A = \text{R.H.S.}$$

Example 17.8 Prove that

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\pi}{3}\cos\frac{4\pi}{9}=\frac{1}{16}$$

Solution : L.H.S. $\cos\frac{\pi}{3}\left[\cos\frac{2\pi}{9}\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}$

$$=\frac{1}{2}\cdot\frac{1}{2}\left[2\cos\frac{2\pi}{9}\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9} \qquad \left[\because \cos\frac{\pi}{3}=\frac{1}{2}\right]$$

$$=\frac{1}{4}\left[\cos\frac{\pi}{3}+\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}$$

$$=\frac{1}{8}\cos\frac{4\pi}{9}+\frac{1}{8}\left[2\cos\frac{4\pi}{9}\cos\frac{\pi}{9}\right]$$

$$=\frac{1}{8}\cos\frac{4\pi}{9}+\frac{1}{8}\left[\cos\frac{5\pi}{9}+\cos\frac{\pi}{3}\right]$$

$$=\frac{1}{8}\cos\frac{4\pi}{9}+\frac{1}{8}\cos\frac{5\pi}{9}+\frac{1}{16} \qquad .....(1)$$

Now $\cos\frac{5\pi}{9}=\cos\left[\pi-\frac{4\pi}{9}\right]=-\cos\frac{4\pi}{9}$ .....(2)

From (1) and (2), we get

$$\text{L.H.S.}=\frac{1}{16}=\text{R.H.S.}$$

CHECK YOUR PROGRESS 17.3

1. Express each of the following as sums or differences :

(a) $2\cos 3\theta\sin 2\theta$ (b) $2\sin 4\theta\sin 2\theta$

(c) $2\cos\frac{\pi}{4}\cos\frac{\pi}{12}$ (d) $2\sin\frac{\pi}{3}\cos\frac{\pi}{6}$

2. Express each of the following as a product :

(a) $\sin 6\theta+\sin 4\theta$ (b) $\sin 7\theta-\sin 3\theta$

(c) $\cos 2\theta-\cos 4\theta$ (d) $\cos 7\theta+\cos 5\theta$

3. Prove the following :

(a) $\sin\frac{5\pi}{18}+\cos\frac{4\pi}{9}=\cos\frac{\pi}{9}$ (b) $\dfrac{\cos\frac{\pi}{9}-\cos\frac{7\pi}{18}}{\sin\frac{7\pi}{18}-\sin\frac{\pi}{9}}=1$

(c) $\sin\frac{5\pi}{18}-\sin\frac{7\pi}{18}+\sin\frac{\pi}{18}=0$ (d) $\cos\frac{\pi}{9}+\cos\frac{5\pi}{9}+\cos\frac{7\pi}{9}=0$

4. Prove the following :

(a) $\sin^2(n+1)\theta-\sin^2 n\theta=\sin(2n+1)\theta\cdot\sin\theta$

(b) $\cos\beta\cos(2\alpha-\beta)=\cos^2\alpha-\sin^2(\alpha-\beta)$

(c) $\cos^2\frac{\pi}{4}-\sin^2\frac{\pi}{12}=\frac{\sqrt{3}}{4}$

5. Show that $\cos^2\left(\frac{\pi}{4}+\theta\right)-\sin^2\left(\frac{\pi}{4}-\theta\right)$ is independent of θ.

6. Prove the following :

(a) $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}=\tan 4\theta$

(b) $\sin\frac{\pi}{18}\sin\frac{5\pi}{6}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}=\frac{1}{16}$

(c) $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=4\cos^2\frac{\alpha-\beta}{2}$

17.3 TRIGONOMETRIC FUNCTIONS OF MULTIPLES OF ANGLES

(a) To express sin 2A in terms of sin A, cos A and tan A.

We know that

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

By putting B = A, we get

$$\sin 2A=\sin A\cos A+\cos A\sin A$$
$$=2\sin A\cos A$$

$\therefore$ sin 2A can also be written as

$$\sin 2A=\frac{2\sin A\cos A}{\cos^2 A+\sin^2 A}\qquad(\because\ 1=\cos^2 A+\sin^2 A)$$

Dividing numerator and denomunator by $\cos^2 A$, we get

$$\sin 2A = \frac{2\left(\dfrac{\sin A \cos A}{\cos^2 A}\right)}{\dfrac{\cos^2 A}{\cos^2 A} + \dfrac{\sin^2 A}{\cos^2 A}} = \frac{2 \tan A}{1 + \tan^2 A}$$

(b) To express cos 2A in terms of sin A, cos A and tan A.

We know that

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Putting B = A, we have

$$\cos 2A = \cos A \cos A - \sin A \sin A$$

or $$\cos 2A = \cos^2 A - \sin^2 A$$

Also $$\cos 2A = \cos^2 A - \left(1 - \cos^2 A\right)$$

$$= \cos^2 A - 1 + \cos^2 A$$

i.e, $$\cos 2A = 2\cos^2 A - 1 \quad \Rightarrow \quad \cos^2 A = \frac{1 + \cos 2A}{2}$$

Also $$\cos 2A = \cos^2 A - \sin^2 A$$

$$= 1 - \sin^2 A - \sin^2 A$$

i.e., $$\cos 2A = 1 - 2\sin^2 A \quad \Rightarrow \quad \sin^2 A = \frac{1 - \cos 2A}{2}$$

$\therefore$ $$\cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

Dividing the numerator and denominator of R.H.S. by $\cos^2 A$, we have

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(c) To express tan 2A in terms of tan A.

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2 \tan A}{1 - \tan^2 A}$$

Thus we have derived the following formulae :

$$\sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\cos^2 A = \frac{1+\cos 2A}{2}, \quad \sin^2 A = \frac{1-\cos 2A}{2}$$

Example 17.9 If $A = \frac{\pi}{6}$, verify the following :

(i) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Solution :

(i) $\qquad \sin 2A = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$2\sin A\cos A = 2\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\times\frac{1}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\frac{2\tan A}{1+\tan^2 A} = \frac{2\tan\frac{\pi}{6}}{1+\tan^2\frac{\pi}{6}} = \frac{\left(2\times\frac{1}{\sqrt{3}}\right)}{1+\frac{1}{3}} = \frac{2}{\sqrt{3}}\times\frac{3}{4} = \frac{\sqrt{3}}{2}$$

Thus, it is verified that

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

(ii) $\qquad \cos 2A = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\cos^2 A - \sin^2 A = \cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

$$2\cos^2 A - 1 = 2\cos^2\frac{\pi}{6} - 1 = 2\times\left(\frac{\sqrt{3}}{2}\right)^2 - 1$$

$$= 2\times\frac{3}{4} - 1 = \frac{1}{2}$$

$$1 - 2\sin^2 A = 1 - 2\sin\frac{\pi}{6} = 1 - 2\times\left(\frac{1}{2}\right)^2 = 1 - 2\times\frac{1}{4} = \frac{1}{2}$$

$$\frac{1-\tan^2 A}{1+\tan^2 A} = \frac{1-\tan^2\frac{\pi}{6}}{1+\tan^2\frac{\pi}{6}} = \frac{1-\left(\frac{1}{\sqrt{3}}\right)^2}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{2}{3}\times\frac{3}{4} = \frac{1}{2}$$

Thus, it is verified that

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

(iii) $\tan 2A = \tan\frac{\pi}{3} = \sqrt{3}$

$$\frac{2\tan A}{1-\tan^2 A} = \frac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}} = \frac{2\times\frac{1}{\sqrt{3}}}{1-\frac{1}{3}} = \frac{2}{\sqrt{3}}\times\frac{3}{2} = \sqrt{3}.$$

Thus, it is verified that $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Example 17.10 Prove that $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$

Solution : $\dfrac{\sin 2A}{1+\cos 2A} = \dfrac{2\sin A\cos A}{2\cos^2 A}$

$$= \frac{\sin A}{\cos A} = \tan A$$

Example 17.11 Prove that $\cot A - \tan A = 2\cot 2A$.

Solution : $\cot A - \tan A = \dfrac{1}{\tan A} - \tan A$

$$= \frac{1-\tan^2 A}{\tan A}$$

$$= \frac{2\left(1-\tan^2 A\right)}{2\tan A}$$

$$= \frac{2}{\left(\frac{2\tan A}{1-\tan^2 A}\right)}$$

$$= \frac{2}{\tan 2A} = 2\cot 2A.$$

Example 17.12 Evaluate $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8}$.

Solution : $\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}=\dfrac{1+\cos\frac{\pi}{4}}{2}+\dfrac{1+\cos\frac{3\pi}{4}}{2}$

$$=\frac{1+\frac{1}{\sqrt{2}}}{2}+\frac{1-\frac{1}{\sqrt{2}}}{2}$$

$$=\frac{\left(\sqrt{2}+1\right)+\left(\sqrt{2}-1\right)}{2\sqrt{2}}=1$$

Example 17.13 Prove that $\dfrac{\cos A}{1-\sin A}=\tan\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)$.

Solution : R.H.S. $=\tan\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)=\dfrac{\tan\frac{\pi}{4}+\tan\frac{A}{2}}{1-\tan\frac{\pi}{4}\tan\frac{A}{2}}$

$$=\frac{1+\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}{1-\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}=\frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}$$

$$=\frac{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)}{\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)^2}$$

[Multiplying Numerator and Denominator by $\left(\dfrac{\cos A}{2}-\dfrac{\sin A}{2}\right)$]

$$=\frac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}-2\cos\frac{A}{2}\sin\frac{A}{2}}$$

$$=\frac{\cos A}{1-\sin A}=\text{L.H.S}.$$

Example 17.14 Prove that

$$(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=4\sin^2\frac{\alpha-\beta}{2}$$

Solution : $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$$= \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta$$

$$= 2 - 2(\cos\alpha\cos\beta + \text{in } \alpha\sin\beta)$$

$$= 2\{1 - \cos(\alpha - \beta)\}$$

$$= 2 \times 2\sin^2\frac{\alpha-\beta}{2} = 4\sin^2\frac{\alpha-\beta}{2}$$

CHECK YOUR PROGRESS 17.4

1. If $A = \frac{\pi}{3}$, verify that

   (a) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$

   (b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

2. Find the value of sin 2A when (assuming $0 < A < \frac{\pi}{2}$)

   (a) $\cos A = \frac{3}{5}$ (b) $\sin A = \frac{12}{13}$ (c) $\tan A = \frac{16}{63}$.

3. Find the value of cos 2A when

   (a) $\cos A = \frac{15}{17}$ (b) $\sin A = \frac{4}{5}$ (c) $\tan A = \frac{5}{12}$

4. Find the value of tan 2A when

   (a) $\tan A = \frac{3}{4}$ (b) $\tan A = \frac{a}{b}$

5. Evaluate $\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8}$.

6. Prove the following :

   (a) $\dfrac{1+\sin 2A}{1-\sin 2A} = \tan^2\left(\dfrac{\pi}{4} + A\right)$ (b) $\dfrac{\cot^2 A + 1}{\cos^2 A - 1} = \sec 2A$

7. (a) Prove that $\dfrac{\sin 2A}{1-\cos 2A} = \cos A$ (b) Prove that $\tan A + \cot A = 2\operatorname{cosec} 2A$.

8. (a) Prove that $\dfrac{\cos A}{1+\sin A} = \tan\left(\dfrac{\pi}{4} - \dfrac{A}{2}\right)$

   (b) Prove that $(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\cos^2\dfrac{\alpha-\beta}{2}$

17.3.1 Trigonometric Functions of 3A in Terms of those of A

(a) sin 3A in terms of sin A

Substituting 2A for B in the formula

$\sin(A+B) = \sin A\cos B + \cos A\sin B$, we get

$$\sin(A+2A) = \sin A\cos 2A + \cos A\sin 2A$$

$$= \sin A\left(1-2\sin^2 A\right) + \left(\cos A \times 2\sin A\cos A\right)$$

$$= \sin A - 2\sin^3 A + 2\sin A\left(1-\sin^2 A\right)$$

$$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$$

$$\therefore \quad \sin 3A = 3\sin A - 4\sin^3 A \qquad \text{....(1)}$$

(b) cos 3A in terms of cos A

Substituting 2A for B in the formula

$\cos(A+B) = \cos A\cos B - \sin A\sin B$, we get

$$\cos(A+2A) = \cos A\cos 2A - \sin A\sin 2A$$

$$= \cos A\left(2\cos^2 A - 1\right) - (\sin A)\times 2\sin A\cos A$$

$$= 2\cos^3 A - \cos A - 2\cos A\left(1-\cos^2 A\right)$$

$$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$$

$$\therefore \quad \cos 3A = 4\cos^3 A - 3\cos A \qquad \text{....(2)}$$

(c) tan 3A in terms of tan A

Putting B = 2A in the formula

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$, we get

$$\tan(A+2A) = \frac{\tan A + \tan 2A}{1 - \tan A\tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A \times \dfrac{2\tan A}{1-\tan^2 A}}$$

$$= \frac{\dfrac{\tan A - \tan^3 A + 2\tan A}{1-\tan^2 A}}{\dfrac{1-\tan^2 A - 2\tan^2 A}{1-\tan^2 A}}$$

$\therefore \quad = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$ ....(3)

(d) **Formulae for** $\sin^3 A$ **and** $\cos^3 A$

$\because \quad \sin 3A = 3\sin A - 4\sin^3 A$

$\therefore \quad 4\sin^3 A = 3\sin A - \sin 3A$

or $\quad \sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$

Similarly, $\cos 3A = 4\cos^3 A - 3\cos A$

$\therefore \quad 3\cos A + \cos 3A = 4\cos^3 A$

or $\quad \cos^3 A = \dfrac{3\cos A + \cos 3A}{4}$

Thus, we have derived the following formulae :

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

$$\cos^3 A = \frac{3\cos A + \cos 3A}{4}$$

Example 17.15 If $A = \dfrac{\pi}{4}$, verify that

(i) $\sin 3A = 3\sin A - 4\sin^3 A$ (ii) $\cos 3A = 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Solution :

(i) $\sin 3A = \sin\dfrac{3\pi}{4} = \dfrac{1}{\sqrt{2}}$

$$3\sin A - 4\sin^3 A = 3\sin\frac{\pi}{4} - 4\sin^3\frac{\pi}{4}$$

$$= 3\times\frac{1}{\sqrt{2}} - 4\times\left(\frac{1}{\sqrt{2}}\right)^3$$

$$= \frac{3}{\sqrt{2}} - \frac{4}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Thus, it is verified that $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $$\cos 3A = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$4\cos^3 A - 3\cos A = 4\times\left(\frac{1}{\sqrt{2}}\right)^3 - 3\times\frac{1}{\sqrt{2}}$$

$$= \frac{4}{2\sqrt{2}} - \frac{3}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

Thus, it is verified that $\cos 3A = 4\cos^3 A - 3\cos A$

(iii) $$\tan 3A = \tan\frac{3\pi}{4} = -1,$$

$$\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = \frac{3\times 1 - 1^3}{1 - 3\times 1^2} = \frac{2}{-2} = -1$$

Thus, it is verified that $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Example 17.16 If $\cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$, prove that $\cos 3\theta = \frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$

Solution : $$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$= 4\left\{\frac{1}{2}\left(a + \frac{1}{a}\right)\right\}^3 - 3\times\frac{1}{2}\left(a + \frac{1}{a}\right)$$

$$= 4\times\frac{1}{8}\left(a^3 + 3a^2\cdot\frac{1}{a} + 3a\cdot\frac{1}{a^2} + \frac{1}{a^3}\right) - \frac{3a}{2} - \frac{3}{2a}$$

$$= \frac{a^3}{2} + \frac{1}{2a^3} = \frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$$

Example 17.17 Prove that

$$\sin\alpha\sin\left(\frac{\pi}{3} + \alpha\right)\sin\left(\frac{\pi}{3} - \alpha\right) = \frac{1}{4}\sin 3\alpha$$

Solution : $$\sin\alpha\sin\left(\frac{\pi}{3} + \alpha\right)\sin\left(\frac{\pi}{3} - \alpha\right)$$

$$= \frac{1}{2}\sin\alpha\left[\cos 2\alpha - \cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\sin\alpha\left[1 - 2\sin^2\alpha - \left(1 - 2\sin^2\frac{\pi}{3}\right)\right]$$

$$= 2\frac{1}{2}\sin\alpha\left[\sin^2\frac{\pi}{3} - \sin^2\alpha\right]$$

$$= \sin\alpha\left[\frac{3}{4} - \sin^2\alpha\right] = \frac{3\sin\alpha - 4\sin^3\alpha}{4} = \frac{1}{4}\sin 3\alpha$$

Example 17.18 Prove that

$$\cos^3 A\sin 3A + \sin^3 A\cos 3A = \frac{3}{4}\sin 4A$$

Solution : $\cos^3 A\sin 3A + \sin^3 A\cos 3A$

$$= \cos^3 A\left(3\sin A - 4\sin^3 A\right) + \sin^3 A\left(4\cos^3 A - 3\cos A\right)$$

$$= 3\sin A\cos^3 A - 4\sin^3 A\cos^3 A + 4\sin^3 A\cos^3 A - 3\sin^3 A\cos A$$

$$= 3\sin A\cos^3 A - 3\sin^3 A\cos A$$

$$= 3\sin A\cos A\left(\cos^2 A - \sin^2 A\right) = \left(3\sin A\cos A\right)\cos 2A$$

$$= \frac{3\sin 2A}{2}\times\cos 2A$$

$$= \frac{3}{2}\frac{\sin 4A}{2} = \frac{3}{4}\sin 4A.$$

Example 17.19 Prove that $\cos^3\frac{\pi}{9} + \sin^3\frac{\pi}{18} = \frac{3}{4}\left(\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right)$

Solution : L.H.S. $= \frac{1}{4}\left[3\cos\frac{\pi}{9} + \cos\frac{\pi}{3}\right] + \frac{1}{4}\left(3\sin\frac{\pi}{18} - \sin\frac{\pi}{6}\right)$

$$= \frac{3}{4}\left[\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right] + \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\right)$$

$$= \frac{3}{4}\left[\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right] = \text{R.H.S.}$$

CHECK YOUR PROGRESS 17.5

1. If $A = \frac{\pi}{3}$, verify that

(a) $\sin 3A = 3\sin A - 4\sin^3 A$ (b) $\cos 3A = 4\cos^3 A - 3\cos A$

(c) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

2. Find the value of sin 3A when (a) $\sin A = \dfrac{2}{3}$ (b) $\sin A = \dfrac{p}{q}$.

3. Find the value of cos 3A when (a) $\cos A = -\dfrac{1}{3}$ (b) $\cos A = \dfrac{c}{d}$.

4. Prove that $\cos\alpha \cos\left(\dfrac{\pi}{3} - \alpha\right)\cos\left(\dfrac{\pi}{3} + \alpha\right) = \dfrac{1}{4}\cos 3\alpha$.

5. (a) Prove that $\sin^3 \dfrac{2\pi}{9} - \sin^3 \dfrac{\pi}{9} = \dfrac{3}{4}\left(\sin\dfrac{2\pi}{9} - \sin\dfrac{\pi}{9}\right)$

(b) Prove that $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A}$ is constant.

6. (a) Prove that $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

(b) Prove that

$$\cos 10A + \cos 8A + 3\cos 4A + 3\cos 2A = 8\cos A \cos^3 3A$$

17.4 TRIGONOMETRIC FUNCTIONS OF SUBMULTIPLES OF ANGLES

$\dfrac{A}{2}, \dfrac{A}{3}, \dfrac{A}{4}$ are called submultiples of A.

It has been proved that

$$\sin^2 A = \frac{1 - \cos 2A}{2},\ \cos^2 A = \frac{1 + \cos 2A}{2},\ \tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$$

Replacing A by $\dfrac{A}{2}$, we easily get the following formulae for the sub-multiple $\dfrac{A}{2}$:

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}},\ \cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}} \text{ and } \tan\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

We will choose either the positive or the negative sign depending on whether corresponding value of the function is positive or negative for the value of $\dfrac{A}{2}$. This will be clear from the following examples

Example 17.20 Find the values of $\cos\frac{\pi}{12}$ and $\cos\frac{\pi}{24}$.

Solution : We use the formulae $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$ and take the positive sign, because $\cos\frac{\pi}{12}$ and $\cos\frac{\pi}{24}$ are both positive.

$$\cos\frac{\pi}{12} = \pm\sqrt{\frac{1+\cos\frac{\pi}{6}}{2}}$$

$$= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2+\sqrt{3}}{2\times 2}}$$

$$= \sqrt{\frac{4+2\sqrt{3}}{8}}$$

$$= \sqrt{\frac{\left(\sqrt{3}+1\right)^2}{8}} \qquad \left(\because 4+2\sqrt{3} = 1+3+2\sqrt{3} = \left(1+\sqrt{3}\right)^2\right)$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos\frac{\pi}{24} = \sqrt{\frac{1+\cos\frac{\pi}{12}}{2}}$$

$$= \sqrt{\frac{\left(1+\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{2}}$$

$$= \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}}$$

$$= \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}}$$

Example 17.21 Find the values of $\sin\left(-\frac{\pi}{8}\right)$ and $\cos\left(-\frac{\pi}{8}\right)$.

Solution : We use the formula $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

and take the lower sign, i.e., negative sign, because $\sin\left(-\frac{\pi}{8}\right)$ is negative.

$$\sin\left(-\frac{\pi}{8}\right) = -\sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}}$$

$$= -\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$$

$$= -\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = -\frac{\sqrt{2-\sqrt{2}}}{2}$$

Similarly, $$\cos\left(-\frac{\pi}{8}\right) = \pm\sqrt{\frac{1+\cos\left(-\frac{\pi}{4}\right)}{2}}$$

$$= \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}$$

$$= \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{2}$$

Example 17.22 If $\cos A = \frac{7}{25}$ and $\frac{3\pi}{2} < A < 2\pi$, find the values of

(i) $\sin\frac{A}{2}$ (ii) $\cos\frac{A}{2}$ (iii) $\tan\frac{A}{2}$

Solution : $\because$ A lies in the 4th-quadrant, $\frac{3\pi}{2} < A < 2\pi$

$$\Rightarrow \qquad \frac{3\pi}{4} < \frac{A}{2} < \pi$$

$$\therefore \qquad \sin\frac{A}{2} > 0,\ \cos\frac{A}{2} < 0,\ \tan\frac{A}{2} < 0.$$

$$\therefore \qquad \sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-\frac{7}{25}}{2}} = \sqrt{\frac{18}{50}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}} = -\sqrt{\frac{1+\frac{7}{25}}{2}} = -\sqrt{\frac{32}{50}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

and $$\tan\frac{A}{2} = -\sqrt{\frac{1-\cos A}{1+\cos A}} = -\sqrt{\frac{1-\frac{7}{25}}{1+\frac{7}{25}}} = -\sqrt{\frac{18}{32}} = -\sqrt{\frac{9}{16}} = -\frac{3}{4}$$

CHECK YOUR PROGRESS 17.6

1. If $A = \frac{\pi}{3}$, verify that

   (a) $\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$ (b) $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$

   (c) $\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$

2. Find the values of $\sin\frac{\pi}{12}$ and $\sin\frac{\pi}{24}$.

3. Determine the values of

   (a) $\sin\frac{\pi}{8}$ (b) $\cos\frac{\pi}{8}$ (c) $\tan\frac{\pi}{8}$.

Example 17.23 Prove that following :

(a) $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$ and $\cos\frac{\pi}{10} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

(b) $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$ and $\sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$

Solution : (a) Let $A = \frac{\pi}{10} \Rightarrow 5A = \frac{\pi}{2}$

$\therefore$ $$2A = \frac{\pi}{2} - 3A$$

$\therefore$ $$\sin 2A = \sin\left(\frac{\pi}{2} - 3A\right) = \cos 3A$$

$\therefore$ $2\sin A\cos A = 4\cos^3 A - 3\cos A$

or $$\cos A\left[2\sin A - 4\cos^2 A + 3\right] = 0 \qquad .....(1)$$

As $\quad \cos A \neq 0$ and $1 - \sin^2 A = \cos^2 A$

$\therefore$ (1) becomes $2\sin A - 4\left(1 - \sin^2 A\right) + 3 = 0$

$$4\sin^2 A + 2\sin A - 1 = 0$$

$\Rightarrow \qquad \sin A = \dfrac{-2 \pm \sqrt{4+16}}{8} = \dfrac{-1 \pm \sqrt{5}}{4}$

$\therefore \qquad \sin A = \dfrac{\sqrt{5}-1}{4}$

$\Rightarrow \qquad \sin\dfrac{\pi}{10} = \dfrac{\sqrt{5}-1}{4}$

Now $\quad \cos\dfrac{\pi}{10} = \sqrt{1 - \sin^2\dfrac{\pi}{10}} = \sqrt{1 - \left(\dfrac{\sqrt{5}-1}{4}\right)^2} = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

(b) Let $\quad A = \dfrac{\pi}{10},\ 2A = \dfrac{\pi}{5}$

$$\cos 2A = 1 - 2\sin^2 A$$

$\therefore \qquad \cos\dfrac{\pi}{5} = 1 - 2\left(\dfrac{\sqrt{5}-1}{4}\right)^2 = 1 - 2\left(\dfrac{6-2\sqrt{5}}{16}\right) = \dfrac{2+2\sqrt{5}}{8}$

$$= \frac{\sqrt{5}+1}{4}$$

Now $\quad \sin\dfrac{\pi}{5} = \sqrt{1 - \cos^2\dfrac{\pi}{5}} = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

Example 17.24 Prove the following :

$$\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$$

Solution : We have to prove that

$$\tan\alpha - \cot\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$$

or $\qquad \left(\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\cos\alpha}{\sin\alpha}\right) + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$

or $\qquad -\dfrac{2\cos 2\alpha}{2\sin\alpha\cos\alpha} + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$

or $\qquad -2\dfrac{\cos 2\alpha}{\sin 2\alpha} + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$

or $\quad -2\cot 2\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$ ......(1)

Combining $\quad -2\cot 2\alpha + 2\tan 2\alpha$

(1) becomes $\quad -4\cot 4\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$ ......(2)

Combining $\quad -4\cot 4\alpha + 4\tan 4\alpha$; L.H.S. of (2) becomes

$$-8\cot 8\alpha + 8\cot 8\alpha = 0 = \text{R.H.S. of (2)}$$

17.5 TRIGONOMETRIC EQUATIONS

You are familiar with the equations like simple linear equations, quadratic equations in algebra. You have also learnt how to solve the same.

Thus, (i) $x - 3 = 0$ gives one value of x as a solution.

(ii) $x^2 - 9 = 0$ gives two values of x.

You must have noticed, the number of values depends upon the degree of the equation.

Now we need to consider as to what will happen in case x's and y's are replaced by trigonometric functions.

Thus solution of the equation $\sin\theta - 1 = 0$, will give

$$\sin\theta = 1 \text{ and } \theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots.$$

Clearly, the solution of simple equations with only finite number of values does not necessarily hold good in case of trigonometric equations.

So, we will try to find the ways of finding solutions of such equations.

17.5.1 To find the general solution of the equation sin θ = 0

It is given that $\sin\theta = 0$

But we know that $\sin 0, \sin\pi, \sin 2\pi, \ldots, \sin n\pi$ are equal to 0

$\therefore \quad \theta = n\pi,\ n \in N$

But we know $\sin(-\theta) = -\sin\theta = 0$

$\therefore \quad \sin(-\pi), \sin(-2\pi), \sin(-3\pi), \ldots, \sin(-n\pi) = 0$

$\therefore \quad \theta = n\pi, \quad n \in I.$

Thus, the general solution of equations of the type $\sin\theta = 0$ is given by $\theta = n\pi$ where n is an integer.

17.5.2 To find the general solution of the equation cos θ = 0

It is given that $\cos\theta = 0$

But in practice we know that $\cos\frac{\pi}{2} = 0$. Therefore, the first value of θ is

$$\theta = \frac{\pi}{2} \qquad .....(1)$$

We know that $\cos(\pi+\theta) = -\cos\theta$ or $\cos\left(\pi+\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0.$

or $\cos\frac{3\pi}{2} = 0$

In the same way, it can be found that

$$\cos\frac{5\pi}{2}, \frac{\cos 7\pi}{2}, \cos\frac{9\pi}{2}, ..., \cos(2n+1)\frac{\pi}{2} \text{ are all zero.}$$

$\therefore \quad \theta = (2n+1)\frac{\pi}{2}, \; n \in N$

But we know that $\cos(-\theta) = \cos\theta$

$$\therefore \cos\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{3\pi}{2}\right) = \cos\left(-\frac{5\pi}{2}\right) = ... = \cos\left\{-(2n+1)\frac{\pi}{2}\right\} = 0$$

$\therefore \quad \theta = (2n+1)\frac{\pi}{2}, \; n \in I$

Therefore, $\theta = (2n+1)\frac{\pi}{2}$ is the solution of equations $\cos\theta = 0$ for all numbers whose cosine is 0.

17.5.3 To find a general solution of the equation tan θ = 0

It is given that $\tan\theta = 0$

or $\frac{\sin\theta}{\cos\theta} = 0$ or $\sin\theta = 0$

i.e. $\theta = n\pi, \; n \in I.$

We have consider above the general solution of trigonometric equations, where the right hand is zero. In the following, we take up cases where right hand side is non-zero.

17.5.4 To find the general solution of the equation sin θ = sin α

It is given that $\sin\theta = \sin\alpha$

$\Rightarrow \quad \sin\theta - \sin\alpha = 0$

or $2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\therefore$ Either $\cos\left(\frac{\theta+\alpha}{2}\right) = 0$ or $\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$$\Rightarrow \quad \frac{\theta+\alpha}{2} = (2p+1)\frac{\pi}{2} \quad \text{or} \quad \frac{\theta-\alpha}{2} = q\pi,\ p,\ q \in I$$

$$\Rightarrow \quad \theta = (2p+1)\pi - \alpha \quad \text{or} \quad \theta = 2q\pi + \alpha \qquad ....(1)$$

From (1), we get

$\theta = n\pi + (-1)^n \alpha,\ n \in I$ as the geeneral solution of the equation $\sin\theta = \sin\alpha$

17.5.5 To find the general solution of the equation cos θ = cos α

It is given that, $\cos\theta = \cos\alpha$

$$\Rightarrow \quad \cos\theta - \cos\alpha = 0$$

$$\Rightarrow \quad -2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$$

$$\therefore \text{Either,} \quad \sin\frac{\theta+\alpha}{2} = 0 \quad \text{or} \quad \sin\frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \quad \frac{\theta+\alpha}{2} = p\pi \quad \text{or} \quad \frac{\theta-\alpha}{2} = q\pi,\ p,q \in I$$

$$\Rightarrow \quad \theta = 2p\pi - \alpha \quad \text{or} \quad \theta = 2p\pi + \alpha \qquad ....(1)$$

From (1), we have

$\theta = 2n\pi \pm \alpha, n \in I$ as the general solution of the equation cos θ = cos α

17.5.6 To find the general solution of the equation tan θ = tan α

It is given that, $\tan\theta = \tan\alpha$

$$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha} = 0$$

$$\Rightarrow \quad \sin\theta\cos\alpha - \sin\alpha\cos\theta = 0$$

$$\Rightarrow \quad \sin(\theta-\alpha) = 0$$

$$\Rightarrow \quad \theta - \alpha = n\pi,\ n \in I$$

$$\Rightarrow \quad \theta = n\pi + \alpha \quad n \in I$$

Similarly, for $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$, the general solution is

$$\theta = n\pi + (-1)^n \alpha$$

and, for $\sec\theta = \sec\alpha$, the general solution is

$$\theta = 2n\pi \pm \alpha$$

and for $\cot\theta = \cot\alpha$

$\theta = n\pi + \alpha$ is its general solution

If $\sin^2\theta = \sin^2\alpha$, then

$$\frac{1-\cos 2\theta}{2} = \frac{1-\cos 2\alpha}{2}$$

$\Rightarrow$ $\cos 2\theta = \cos 2\alpha$

$\Rightarrow$ $2\theta = 2n\pi \pm 2\alpha, \quad n \in I$

$\Rightarrow$ $\theta = n\pi \pm \alpha$

Similarly, if $\cos^2\theta = \cos^2\alpha$, then

$$\alpha = n\pi \pm \alpha, \quad n \in I$$

Again, if $\tan^2\theta = \tan^2\alpha$, then

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

$\Rightarrow$ $\cos 2\theta = \cos 2\alpha$

$\Rightarrow$ $2\theta = 2n\pi \pm 2\alpha$

$\Rightarrow$ $\theta = n\pi \pm \alpha, \quad n \in I$ is the general solution.

Example 17.25 Find the general solution of the following equations :

(a) (i) $\sin\theta = \frac{1}{2}$ (ii) $\sin\theta = -\frac{\sqrt{3}}{2}$

(b) (i) $\cos\theta = \frac{\sqrt{3}}{2}$ (ii) $\cos\theta = -\frac{1}{2}$

(c) $\cot\theta = -\sqrt{3}$ (d) $4\sin^2\theta = 1$

Solution :

(a) (i) $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

$\therefore$ $\theta = n\pi + (-1)^n \frac{\pi}{6}, \quad n \in I$

(ii) $\sin\theta = \frac{-\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin\frac{4\pi}{3}$

$\therefore$ $\theta = n\pi + (-1)^n \frac{4\pi}{3}, \quad n \in I$

(b) (i) $\cos\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$

$$\therefore \quad \theta = 2n\pi \pm \frac{\pi}{6}, \quad n \in I$$

(ii) $$\cos\theta = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\therefore \quad \theta = 2n\pi \pm \frac{2\pi}{3}, \quad n \in I$$

(c) $$\cot\theta = -\sqrt{3}$$

$$\tan\theta = -\frac{1}{\sqrt{3}} = -\tan\frac{\pi}{6} = \tan\left(\pi - \frac{\pi}{6}\right) = \tan\frac{5\pi}{6}$$

$$\therefore \quad \theta = n\pi + \frac{5\pi}{6}, \quad n \in I$$

(d) $$4\sin^2\theta = 1 \quad \Rightarrow \quad \sin^2\theta = \frac{1}{4} = \left(\frac{1}{2}\right)^2 = \sin^2\frac{\pi}{6}$$

$$\Rightarrow \quad \sin\theta = \sin\left(\pm\frac{\pi}{6}\right)$$

$$\theta = n\pi \pm \frac{\pi}{6}, \quad n \in I$$

Example 17.26 Solve the following :

(a) $2\cos^2\theta + 3\sin\theta = 0$ (b) $\cos 4x = \cos 2x$

(c) $\cos 3x = \sin 2x$ (d) $\sin 2x + \sin 4x + \sin 6x = 0$

Solution :

(a) $$2\cos^2\theta + 3\sin\theta = 0$$

$$\Rightarrow \quad 2\left(1 - \sin^2\theta\right) + 3\sin\theta = 0$$

$$\Rightarrow \quad 2\sin^2\theta - 3\sin\theta - 2 = 0$$

$$\Rightarrow \quad (2\sin\theta + 1)(\sin\theta - 2) = 0$$

$$\Rightarrow \quad \sin\theta = -\frac{1}{2} \quad \text{or} \quad \sin\theta = 2$$

Since $\sin\theta = 2$ is not possible.

$$\therefore \quad \sin\theta = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$$

$$\therefore \quad \theta = n\pi + (-1)^n \cdot \frac{7\pi}{6}, \ n \in I$$

(b) $\quad \cos 4x = \cos 2x$

i.e., $\quad \cos 4x - \cos 2x = 0$

$\Rightarrow \quad -2\sin 3x \sin x = 0$

$\Rightarrow \quad \sin 3x = 0 \quad$ or $\quad \sin x = 0$

$\Rightarrow \quad 3x = n\pi \quad$ or $\quad x = n\pi$

$\Rightarrow \quad x = \dfrac{n\pi}{3} \quad$ or $\quad x = n\pi \qquad n \in I$

(c) $\quad \cos 3x = \sin 2x$

$$\Rightarrow \quad \cos 3x = \cos\left(\frac{\pi}{2} - 2x\right)$$

$$\Rightarrow \quad 3x = 2n\pi \pm \left(\frac{\pi}{2} - 2x\right) \qquad n \in I$$

Taking positive sign only, we have

$$3x = 2n\pi + \frac{\pi}{2} - 2x$$

$$\Rightarrow \quad 5x = 2n\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad x = \frac{2n\pi}{5} + \frac{\pi}{10}$$

Now taking negative sign, we have

$$3x = 2n\pi - \frac{\pi}{2} + 2x \quad \Rightarrow \quad x = 2n\pi - \frac{\pi}{2} \quad n \in I$$

(d) $\quad \sin 2x + \sin 4x + \sin 6x = 0$

or $\quad (\sin 6x + \sin 2x) + \sin 4x = 0$

or $\quad 2\sin 4x \cos 2x + \sin 4x = 0$

or $\quad \sin 4x\,[2\cos 2x + 1] = 0$

$$\therefore \quad \sin 4x = 0 \quad \text{or} \quad \cos 2x = -\frac{1}{2} = \cos\frac{2\pi}{3}$$

$$\Rightarrow \quad 4x = n\pi \quad \text{or} \quad 2x = 2n\pi \pm \frac{2\pi}{3},\ n \in I$$

$$x = \frac{n\pi}{4} \quad \text{or} \quad x = n\pi \pm \frac{\pi}{3} \quad n \in I$$

CHECK YOUR PROGRESS 17.7

1. Find the most general value of θ satisfying :

(i) $\sin\theta = \frac{\sqrt{3}}{2}$ (ii) $\operatorname{cosec}\theta = \sqrt{2}$

(ii) $\sin\theta = -\frac{\sqrt{3}}{2}$ (ii) $\sin\theta = -\frac{1}{\sqrt{2}}$

2. Find the most general value of θ satisfying :

(i) $\cos\theta = -\frac{1}{2}$ (ii) $\sec\theta = -\frac{2}{\sqrt{3}}$

(iii) $\cos\theta = \frac{\sqrt{3}}{2}$ (iv) $\sec\theta = -\sqrt{2}$

3. Find the most general value of θ satisfying :

(i) $\tan\theta = -1$ (ii) $\tan\theta = \sqrt{3}$ (iii) $\cot\theta = -1$

4. Find the most general value of θ satisfying :

(i) $\sin 2\theta = \frac{1}{2}$ (ii) $\cos 2\theta = \frac{1}{2}$ (iii) $\tan 3\theta = \frac{1}{\sqrt{3}}$

(iv) $\cos 3\theta = -\frac{\sqrt{3}}{2}$ (v) $\sin^2\theta = \frac{3}{4}$ (vi) $\sin^2 2\theta = \frac{1}{4}$

(vii) $4\cos^2\theta = 1$ (viii) $\cos^2 2\theta = \frac{3}{4}$

5. Find the general solution of the following :

(i) $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$ (ii) $4\cos^2\theta - 4\sin\theta = 1$

(iii) $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$

LET US SUM UP

- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B,$

  $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

  $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}, \quad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot B+\cot A},\quad \cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}$$

- $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

  $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

  $2\cos A\cos B=\cos(A+B)-\cos(A-B)$

  $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

- $\sin C+\sin D=2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

  $\sin C-\sin D=2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

  $\cos C+\cos D=2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

  $\cos C-\cos D=2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$

- $\sin(A+B)\cdot\sin(A-B)=\sin^2 A-\sin^2 B$

  $\cos(A+B)\cdot\cos(A-B)=\cos^2 A-\sin^2 B$

- $\sin 2A=2\sin A\cos A=\dfrac{2\tan A}{1+\tan^2 A}$

- $\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$

- $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$

- $\sin^2 A=\dfrac{1-\cos 2A}{2},\quad \cos^2 A=\dfrac{1+\cos 2A}{2},\quad \tan^2 A=\dfrac{1-\cos 2A}{1+\cos 2A}$

- $\sin 3A=3\sin A-4\sin^3 A,\quad \cos 3A=4\cos^3 A-3\cos A$

  $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

- $\sin^3 A=\dfrac{3\sin A-\sin 3A}{4},\quad \cos^3 A=\dfrac{3\cos A+\cos 3A}{4}$

- $\sin\dfrac{A}{2}=\pm\sqrt{\dfrac{1-\cos A}{2}},\quad \cos\dfrac{A}{2}=\pm\sqrt{\dfrac{1+\cos A}{2}}$

  $\tan\dfrac{A}{2}=\pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$

- $\sin\dfrac{\pi}{10} = \dfrac{\sqrt{5}-1}{4}$, $\cos\dfrac{\pi}{5} = \dfrac{\sqrt{5}+1}{4}$
- $\sin\theta = 0 \quad \Rightarrow \quad \theta = n\pi, \quad n \in I$

  $\cos\theta = 0 \quad \Rightarrow \quad \theta = (2n+1)\dfrac{\pi}{2}, \quad n \in I$

  $\tan\theta = 0 \quad \Rightarrow \quad \theta = n\pi, \quad n \in I$
- $\sin\theta = \sin\alpha \quad \Rightarrow \quad \theta = n\pi + (-1)^n \alpha, \quad n \in I$
- $\cos\theta = \cos\alpha \quad \Rightarrow \quad \theta = 2n\pi \pm \alpha, \quad n \in I$
- $\tan\theta = \tan\alpha \quad \Rightarrow \quad \theta = n\pi + \alpha, \quad n \in I$
- $\sin^2\theta = \sin^2\alpha \quad \Rightarrow \quad \theta = n\pi \pm \alpha, \quad n \in I$
- $\cos^2\theta = \cos^2\alpha \quad \Rightarrow \quad \theta = n\pi \pm \alpha, \quad n \in I$
- $\tan^2\theta = \tan^2\alpha \quad \Rightarrow \quad \theta = n\pi \pm \alpha, \quad n \in I$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. Prove that $\tan(A+B)\times\tan(A-B) = \dfrac{\cos^2 B - \cos^2 A}{\cos^2 B - \sin^2 A}$

2. Prove that $\cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \dfrac{\pi}{3}\right)$

3. If $A + B = \dfrac{\pi}{4}$

   Prove that $(1+\tan A)(1+\tan B) = 2$ and $(\cot A - 1)(\cos B - 1) = 2$

4. Prove each of the following :

   (i) $\dfrac{\sin(A-B)}{\cos A\cos B} + \dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A} = 0$

(ii) $\cos\left(\frac{\pi}{10}-A\right)\cdot\cos\left(\frac{\pi}{10}+A\right)+\cos\left(\frac{2\pi}{5}-A\right)\cdot\cos\left(\frac{2\pi}{5}+A\right)=\cos 2A$

(iii) $\cos\frac{2\pi}{9}\cdot\cos\frac{4\pi}{9}\cdot\cos\frac{9\pi}{9}=-\frac{1}{8}$ (iv) $\cos\frac{13\pi}{45}+\cos\frac{17\pi}{45}+\cos\frac{43\pi}{45}=0$

(v) $\tan\left(A+\frac{\pi}{6}\right)+\cot\left(A-\frac{\pi}{6}\right)=\frac{1}{\sin 2A-\sin\frac{\pi}{3}}$

(vi) $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta$ (vii) $\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}=\tan 2\theta+\sec 2\theta$

(viii) $\left(\frac{1-\sin\theta}{1+\sin\theta}\right)^2=\tan^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

(ix) $\cos^2 A+\cos^2\left(A+\frac{\pi}{3}\right)+\cos^2\left(A-\frac{\pi}{3}\right)=\frac{3}{2}$

(x) $\frac{\sec 8A-1}{\sec 4A-1}=\frac{\tan 8A}{\tan 2A}$ (xi) $\cos\frac{\pi}{30}\cos\frac{7\pi}{30}\cos\frac{11\pi}{30}\cos\frac{13\pi}{30}=\frac{11}{16}$

(xii) $\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}=-\frac{1}{2}$

5. Find the general value of 'θ' satisfying

(a) $\sin\theta=\frac{1}{\sqrt{2}}$ (b) $\sin\theta=\frac{\sqrt{3}}{2}$

(c) $\sin\theta=-\frac{1}{\sqrt{2}}$ (d) $\operatorname{cosec}\theta=\sqrt{2}$

6. Find the general value of 'θ' satisfying

(a) $\cos\theta=\frac{1}{2}$ (b) $\sec\theta=\frac{2}{\sqrt{3}}$

(c) $\cos\theta=\frac{-\sqrt{3}}{2}$ (d) $\sec\theta=-2$

7. Find the most general value of 'θ' satisfying

(a) $\tan\theta=1$ (b) $\tan\theta=-1$ (c) $\cot\theta=-\frac{1}{\sqrt{3}}$

8. Find the general value of 'θ' satisfying

(a) $\sin^2\theta=\frac{1}{2}$ (b) $4\cos^2\theta=1$ (c) $2\cot^2\theta=\operatorname{cosec}^2\theta$

9. Solve the following for θ :

(a) $\cos p\theta = \cos q\theta$ (b) $\sin 9\theta = \sin \theta$

(c) $\tan 5\theta = \cot \theta$

10. Solve the following for θ :

(a) $\sin m\theta + \sin n\theta = 0$ (b) $\tan m\theta + \cot n\theta = 0$

(c) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ (d) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

ANSWERS

CHECK YOUR PROGRESS 17.1

1. (a) (i) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\frac{\sqrt{3}}{2}$ (c) $\frac{21}{221}$

2. (a) $\frac{\sqrt{3}-1}{2\sqrt{2}}$

CHECK YOUR PROGRESS 17.2

1. (i) $\frac{\cos^2 A - \sin^2 A}{2}$ (ii) $-\frac{1}{4}$ 2. (c) $-\frac{(\sqrt{3}+1)}{2\sqrt{2}}$

CHECK YOUR PROGRESS 17.3

1. (a) $\sin 5\theta - \sin\theta$; (b) $\cos 2\theta - \cos 6\theta$

   (c) $\cos\frac{\pi}{3} + \cos\frac{\pi}{6}$ (d) $\sin\frac{\pi}{2} + \sin\frac{\pi}{6}$

2. (a) $2\sin 5\theta \cos\theta$ (b) $2\cos 5\theta \cdot \sin 2\theta$

   (c) $2\sin 3\theta \cdot \sin\theta$ (d) $2\cos 6\theta \cdot \cos\theta$

CHECK YOUR PROGRESS 17.4

2. (a) $\frac{24}{25}$ (b) $\frac{120}{169}$ (c) $\frac{2016}{4225}$

3. (a) $\frac{161}{289}$ (b) $\frac{-7}{25}$ (c) $\frac{119}{169}$

4. (a) $\frac{24}{7}$ (b) $\frac{2ab}{b^2 - a^2}$ 5. 1

CHECK YOUR PROGRESS 17.5

2. (a) $\frac{22}{27}$ (b) $\frac{(3pq^2 - 4p^3)}{q^3}$ 3. (a) $\frac{23}{27}$ (b) $\frac{4c^3 - 3cd^2}{d^3}$

CHECK YOUR PROGRESS 17.6

2. (a) $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\frac{\sqrt{(4-\sqrt{2}-\sqrt{6})}}{2\sqrt{2}}$

3. (a) $\dfrac{\sqrt{2-\sqrt{2}}}{2}$ (b) $\dfrac{\sqrt{2+\sqrt{2}}}{2}$ (c) $\sqrt{2}-1$

CHECK YOUR PROGRESS 17.7

1. (i) $\theta = n\pi + (-1)^n \dfrac{\pi}{3}, n \in I$ (ii) $\theta = n\pi + (-1)^n \dfrac{\pi}{4}, n \in I$

   (iii) $\theta = n\pi + (-1)^n \dfrac{4\pi}{3}, n \in I$ (iv) $\theta = n\pi + (-1)^n \dfrac{5\pi}{4}, n \in I$

2. (i) $\theta = 2n\pi \pm \dfrac{2\pi}{3}, n \in I$ (ii) $\theta = 2n\pi \pm \dfrac{5\pi}{6}, n \in I$

   (iii) $\theta = 2n\pi \pm \dfrac{\pi}{6}, n \in I$ (iv) $\theta = 2n\pi \pm \dfrac{3\pi}{4}, n \in I$

3. (i) $\theta = n\pi + \dfrac{3\pi}{4}, n \in I$ (ii) $\theta = n\pi + \dfrac{\pi}{3}, n \in I$

   (iii) $\theta = n\pi - \dfrac{\pi}{4}, n \in I$

4. (i) $\theta = \dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{12}, n \in I$ (ii) $\theta = n\pi \pm \dfrac{\pi}{6}, n \in I$

   (iii) $\theta = \dfrac{n\pi}{3} + \dfrac{\pi}{18}, n \in I$ (iv) $\theta = \dfrac{2n\pi}{3} \pm \dfrac{5\pi}{18}, n \in I$

   (v) $\theta = n\pi \pm \dfrac{\pi}{3}, n \in I$ (vi) $\theta = \dfrac{n\pi}{2} \pm \dfrac{\pi}{12}, n \in I$

   (vii) $\theta = n\pi \pm \dfrac{\pi}{3}, n \in I$ (viii) $\theta = \dfrac{n\pi}{2} \pm \dfrac{\pi}{12}, n \in I$

5. (i) $\theta = 2n\pi \pm \dfrac{5\pi}{6}, n \in I$ (ii) $\theta = n\pi + (-1)^n \dfrac{\pi}{6}, n \in I$

   (iii) $\theta = 2n\pi \pm \dfrac{\pi}{3}, n \in I$

TERMINAL EXERCISE

5. (a) $\theta = n\pi + (-1)^n \dfrac{\pi}{4}, n \in I$ (b) $\theta = n\pi + (-1)^n \dfrac{\pi}{3}, n \in I$

   (c) $\theta = n\pi + (-1)^n \dfrac{5\pi}{4}, n \in I$ (d) $\theta = n\pi + (-1)^n \dfrac{\pi}{4}, n \in I$

6. (a) $\theta = 2n\pi \pm \dfrac{\pi}{3}, n \in I$ (b) $\theta = 2n\pi \pm \dfrac{\pi}{6}, n \in I$

(c) $\theta = 2n\pi \pm \frac{5\pi}{6}, n \in I$ (d) $\theta = 2n\pi \pm \frac{2\pi}{3}, n \in I$

7. (a) $\theta = n\pi + \frac{\pi}{4}, n \in I$ (b) $\theta = n\pi + \frac{3\pi}{4}, n \in I$

(c) $\theta = n\pi + \frac{2\pi}{3}, n \in I$

8. (a) $\theta = n\pi \pm \frac{\pi}{4}, n \in I$ (b) $\theta = n\pi \pm \frac{\pi}{3}, n \in I$

(c) $\theta = n\pi \pm \frac{\pi}{4}, n \in I$

9. (a) $\theta = \frac{2n\pi}{p \mp q}, n \in I$ (b) $\theta = \frac{n\pi}{4}$ or $(2n+1)\frac{\pi}{10}, n \in I$

(c) $\theta = (2n+1)\frac{\pi}{12}, n \in I$

10. (a) $\theta = \frac{(2k+1)\pi}{m-n}$ or $\frac{2k\pi}{m+n}, k \in I$ (b) $\theta = \frac{(2k+1)\pi}{2(m-n)}, k \in I$

(c) $\theta = (2n+1)\frac{\pi}{4}$ or $2n\pi \pm \frac{2\pi}{3}, n \in I$

(d) $\theta = \frac{2n\pi}{5}$ or $\theta = n\pi \pm \frac{\pi}{2}, n \in I$ or $\theta = (2n-1)\pi, n \in I$

18

INVERSE TRIGONOMETRIC FUNCTIONS

In the previous lesson, you have studied the definition of a function and different kinds of functions. We have defined inverse function.

Let us briefly recall :

Let f be a one-one onto function from A to B.

Let y be an arbitary element of B. Then, f being onto, $\exists$ an element $x \in A$ such that $f(x) = y$. Also, f being one-one, then x must be unique. Thus for each $y \in B$, $\exists$ a unique element $x \in A$ such that $f(x) = y$. So we may define a function, denoted by f^{-1} as $f^{-1} : B \to A$

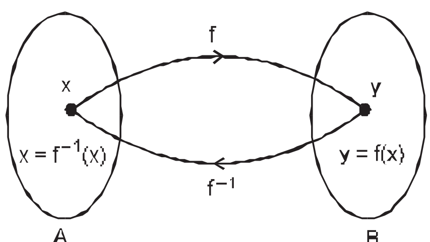


Fig. 18.1

$$\therefore \qquad f^{-1}(y) = x \Leftrightarrow f(x) = y$$

The above function f^{-1} is called the inverse of f. A function is invertiable if and only if f is one-one onto.

In this case the domain of f^{-1} is the range of f and the range of f^{-1} is the domain f.

Let us take another example.

We define a function : f : Car $\to$ Registration No.

If we write, g : Registration No. $\to$ Car, we see that the domain of f is range of g and the range of f is domain of g.

So, we say g is an **inverse function** of f, i.e., $g = f^{-1}$.

In this lesson, we will learn more about inverse trigonometric function, its domain and range, and simplify expressions involving inverse trigonometric functions.

OBJECTIVES

After studying this lesson, you will be able to :

- define inverse trigonometric functions;

- state the condition for the inverse of trigonometric functions to exist;
- define the principal value of inverse trigonometric functions;
- find domain and range of inverse trigonometric functions;
- state the properties of inverse trigonometric functions; and
- simplify expressions involving inverse trigonometric functions.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of function and their types, domain and range of a function
- Formulae for trigonometric functions of sum, difference, multiple and sub-multiples of angles.

18.1 IS INVERSE OF EVERY FUNCTION POSSIBLE ?

Take two ordered pairs of a function (x_1, y) and (x_2, y)

If we invert them, we will get (y, x_1) and (y, x_2)

This is not a function because the first member of the two ordered pairs is the same.

Now let us take another function :

$$\left(\sin\frac{\pi}{2}, 1\right), \left(\sin\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\sin\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$$

Writing the inverse, we have

$$\left(1, \sin\frac{\pi}{2}\right), \left(\frac{1}{\sqrt{2}}, \sin\frac{\pi}{4}\right) \text{ and } \left(\frac{\sqrt{3}}{2}, \sin\frac{\pi}{3}\right)$$

which is a function.

Let us consider some examples from daily life.

f : Student $\rightarrow$ Score in Mathematics

Do you think f^{-1} will exist ?

It may or may not be because the moment two students have the same score, f^{-1} will cease to be a function. Because the first element in two or more ordered pairs will be the same. So we conclude that

every function is not invertible.

Example 18.1 If $f : R \rightarrow R$ defined by $f(x) = x^3 + 4$. What will be f^{-1} ?

Solution : In this case f is one-to-one and onto both.

$\Rightarrow f$ is invertible.

Let $\quad y = x^3 + 4$

$\therefore \quad y - 4 = x^3 \quad \Rightarrow x \quad \sqrt[3]{y \quad 4}$

So f^{-1}, inverse function of f i.e., $f^{-1}(y) = \sqrt[3]{y - 4}$

The functions that are one-to-one and onto will be invertible.

Let us extend this to trigonometry :

Take $y = \sin x$. Here domain is the set of all real numbers. Range is the set of all real numbers lying between -1 and 1, including -1 and 1 i.e. $-1 \le y \le 1$.

We know that there is a unique value of y for each given number x.

In inverse process we wish to know a number corresponding to a particular value of the sine.

Suppose $\qquad y = \sin x = \dfrac{1}{2}$

$$\Rightarrow \qquad \sin x = \sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{13\pi}{6} = ...$$

x may have the values as $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6} = ...$

Thus there are infinite number of values of x.

$y = \sin x$ can be represented as

$$\left(\frac{\pi}{6}, \frac{1}{2}\right), \left(\frac{5\pi}{6}, \frac{1}{2}\right), ...$$

The inverse relation will be

$$\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), ...$$

It is evident that it is not a function as first element of all the ordered pairs is $\dfrac{1}{2}$, which contradicts the definition of a function.

Consider $y = \sin x$, where x $\in$ R (domain) and y $\in [-1, 1]$ or $-1 \le y \le 1$ which is called range. This is many-to-one and onto function, therefore it is not invertible.

Can $y = \sin x$ be made invertible and how? Yes, if we restrict its domain in such a way that it becomes one-to-one and onto taking x as

(i) $\quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}, \qquad y \in [-1, 1] \qquad$ or

(ii) $\quad \dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2} \qquad y \in [-1, 1] \qquad$ or

(iii) $\quad -\dfrac{5\pi}{2} \le x \le -\dfrac{3\pi}{2} \qquad y \in [-1, 1] \qquad$ etc.

Now consider the inverse function $y = \sin^{-1} x$.

We know the domain and range of the function. We interchange domain and range for the inverse of the function. Therefore,

(i) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ $\quad x \in [-1, 1]$ $\quad$ or

(ii) $\frac{3\pi}{2} \le y \le \frac{5\pi}{2}$ $\quad x \in [-1, 1]$ $\quad$ or

(iii) $-\frac{5\pi}{2} \le y \le -\frac{3\pi}{2}$ $\quad x \in [-1, 1]$ $\quad$ etc.

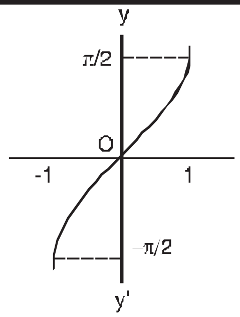
Here we take the least numerical value among all the values of the real number whose sine is x which is called the principle value of $\sin^{-1} x$.

For this the only case is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Therefore, for principal value of $y = \sin^{-1} x$, the domain is $[-1, 1]$ i.e. $x \in [-1, 1]$ and range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

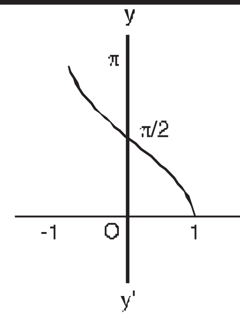
Similarly, we can discuss the other inverse trigonometric functions.

	Function	**Domain**	**Range (Principal value)**
1.	$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3.	$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1} x$	R	$[0, \pi]$
5.	$y = \sec^{-1} x$	$x \ge 1$ or $x \le -1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
6.	$y = \operatorname{cosec}^{-1} x$	$x \ge 1$ or $x \le -1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

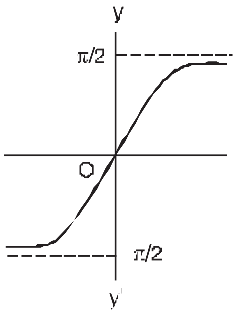
18.2 GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS



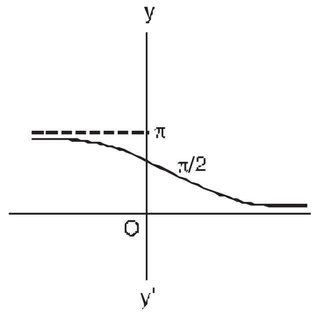
$y = \sin^{-1} x$



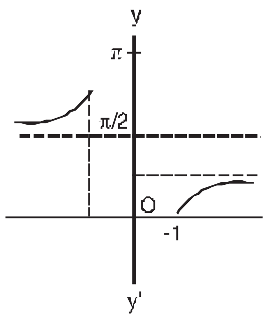
$y = \cos^{-1} x$



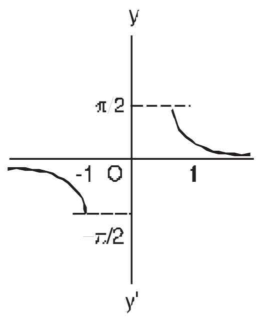
$y = \tan^{-1} x$



$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \text{cosec}^{-1} x$

Fig. 18.2

Example 18.2 Find the principal value of each of the following :

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution : (i) Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$

or $\sin\theta = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right)$ or $\theta = \frac{\pi}{4}$

(ii) Let $\cos^{-1}\left(-\frac{1}{2}\right) = \theta$

$\Rightarrow$ $\cos\theta = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$ or $\theta = \frac{2\pi}{3}$

(iii) Let $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \theta$ or $-\frac{1}{\sqrt{3}} = \tan\theta$ or $\tan\theta = \tan\left(-\frac{\pi}{6}\right)$

$\Rightarrow$ $\theta = -\frac{\pi}{6}$

Example 18.3 Find the principal value of each of the following :

(a) (i) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\tan^{-1}(-1)$

(b) Find the value of the following using the principal value :

$$\sec\left[\cos^{-1}\frac{\sqrt{3}}{2}\right]$$

Solution : (a) (i) Let $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\theta$, then

$$\frac{1}{\sqrt{2}}=\cos\theta \qquad \text{or} \qquad \cos\theta=\cos\frac{\pi}{4}$$

$$\Rightarrow \qquad \theta=\frac{\pi}{4}$$

(ii) Let $\tan^{-1}(-1)=\theta$, then

$$-1=\tan\theta \qquad \text{or} \qquad \tan\theta=\tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \qquad \theta=-\frac{\pi}{4}$$

(b) Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\theta$, then

$$\frac{\sqrt{3}}{2}=\cos\theta \qquad \text{or} \qquad \cos\theta=\cos\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \qquad \theta=\frac{\pi}{6}$$

$$\therefore \qquad \sec\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)=\sec\theta=\sec\left(\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}$$

Example 18.4 Simplify the following :

(i) $\cos\left[\sin^{-1}x\right]$ (ii) $\cot\left[\mathrm{cosec}^{-1}x\right]$

Solution : (i) Let $\sin^{-1}x=\theta$

$$\Rightarrow \qquad x=\sin\theta$$

$$\therefore \qquad \cos\left[\sin^{-1}x\right]=\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$$

(ii) Let $\mathrm{cosec}^{-1}x=\theta$

$$\Rightarrow \qquad x=\mathrm{cosec}\,\theta$$

Also $\qquad \cot\theta = \sqrt{\operatorname{cosec}^2\theta - 1}$

$$= \sqrt{x^2 - 1}$$

CHECK YOUR PROGRESS 18.1

1. Find the principal value of each of the following :

(a) $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ (b) $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ (c) $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

(d) $\tan^{-1}\left(-\sqrt{3}\right)$ (e) $\cot^{-1}(1)$

2. Evaluate each of the following :

(a) $\cos\left(\cos^{-1}\dfrac{1}{3}\right)$ (b) $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\dfrac{\pi}{4}\right)$ (c) $\cos\left(\operatorname{cosec}^{-1}\dfrac{2}{\sqrt{3}}\right)$

(d) $\tan\left(\sec^{-1}\sqrt{2}\right)$ (e) $\operatorname{cosec}\left[\cot^{-1}\left(-\sqrt{3}\right)\right]$

3. Simplify each of the following expressions :

(a) $\sec\left(\tan^{-1}x\right)$ (b) $\tan\left(\operatorname{cosec}^{-1}\dfrac{x}{2}\right)$ (c) $\cot\left(\operatorname{cosec}^{-1}x^2\right)$

(d) $\cos\left(\cot^{-1}x^2\right)$ (e) $\tan\left(\sin^{-1}\left(\sqrt{1-x}\right)\right)$

18.3 PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property 1 $\sin^{-1}(\sin\theta) = \theta,\ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

Solution : Let $\sin\theta = x$

$\Rightarrow \qquad \theta = \sin^{-1}x$

$$= \sin^{-1}(\sin\theta) = \theta$$

Also $\qquad \sin\left(\sin^{-1}x\right) = x$

Similarly, we can prove that

(i) $\qquad \cos^{-1}(\cos\theta) = \theta,\ 0 \le \theta \le \pi$

(ii) $\qquad \tan^{-1}(\tan\theta) = \theta,\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

Property 2 (i) $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\dfrac{1}{x}\right)$ (ii) $\cot^{-1}x = \tan^{-1}\left(\dfrac{1}{x}\right)$

(iii) $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Solution : (i) Let $\operatorname{cosec}^{-1} x = \theta$

$\Rightarrow \quad x = \operatorname{cosec}\theta$

$\Rightarrow \quad \frac{1}{x} = \sin\theta$

$\therefore \quad \theta = \sin^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \quad \operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

(ii) Let $\cot^{-1} x = \theta$

$\Rightarrow \quad x = \cot\theta$

$\Rightarrow \quad \frac{1}{x} = \tan\theta$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$

(iii) $\sec^{-1} x = \theta$

$\Rightarrow \quad x = \sec\theta$

$\therefore \quad \frac{1}{x} = \cos\theta \quad$ or $\quad \theta = \cos^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Property 3 (i) $\sin^{-1}(-x) = -\sin^{-1} x$ (ii) $\tan^{-1}(-x) = -\tan^{-1} x$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Solution : (i) Let $\sin^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \sin\theta \quad$ or $\quad x = -\sin\theta = \sin(-\theta)$

$\therefore \quad -\theta = \sin^{-1} x \quad$ or $\quad \theta = -\sin^{-1} x$

or $\quad \sin^{-1}(-x) = -\sin^{-1} x$

(ii) Let $\tan^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \tan\theta \quad$ or $\quad x = -\tan\theta = \tan(-\theta)$

$\therefore \quad \theta = -\tan^{-1} x \quad$ or $\quad \tan^{-1}(-x) = -\tan^{-1} x$

(iii) Let $\cos^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \cos\theta \quad$ or $\quad x = -\cos\theta = \cos(\pi - \theta)$

$\therefore \quad \cos^{-1} x = \pi - \theta$

$\therefore \quad \cos^{-1}(-x) = \pi - \cos^{-1} x$

Property 4 (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

Soluton : (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Let $\sin^{-1} x = \theta \;\Rightarrow\; x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

or $\qquad \cos^{-1} x = \left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow \qquad \theta + \cos^{-1} x = \frac{\pi}{2} \qquad$ or $\qquad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(ii) Let $\cot^{-1} x = \theta \;\Rightarrow x = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

$\therefore \qquad \tan^{-1} x = \frac{\pi}{2} - \theta \qquad$ or $\qquad \theta + \tan^{-1} x = \frac{\pi}{2}$

or $\qquad \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}$

(iii) Let $\operatorname{cosec}^{-1} x = \theta$

$\Rightarrow \qquad \_\_\_ \; x = \operatorname{cosec}\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

$\therefore \qquad \sec^{-1} x = \frac{\pi}{2} - \theta \qquad$ or $\qquad \theta + \sec^{-1} x = \frac{\pi}{2}$

$\Rightarrow \operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

Property 5 (i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Solution : (i) Let $\tan^{-1} x = \theta$, $\tan^{-1} y = \phi \;\Rightarrow x = \tan\theta, y = \tan\phi$

We have to prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

By substituting that above values on L.H.S. and R.H.S., we have

$$\text{L.H.S.} = \theta + \phi \text{ and } \text{R.H.S.} = \tan^{-1}\left[\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\right]$$

$$= \tan^{-1}\left[\tan(\theta+\phi)\right] = \theta + \phi = \text{L.H.S.}$$

$\therefore$ The result holds.
Simiarly (ii) can be proved.

Property 6 $2\tan^{-1} x = \sin^{-1}\left[\frac{2x}{1+x^2}\right] = \cos^{-1}\left[\frac{1-x^2}{1+x^2}\right] = \tan^{-1}\left[\frac{2x}{1-x^2}\right]$

(i) (ii) (iii) (iv)

Let $x = \tan\theta$

Substituting in (i), (ii), (iii), and (iv) we get

$$2\tan^{-1} x = 2\tan^{-1}(\tan\theta) = 2\theta \qquad \text{......(i)}$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta)$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta \qquad \text{.....(ii)}$$

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}\left(\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}\right)$$

$$= \cos^{-1}(\cos^2\theta - \sin^2\theta)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta \qquad \text{......(iii)}$$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) = 2\theta \qquad \text{.....(iv)}$$

From (i), (ii), (iii) and (iv), we get

$$2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Property 7

(i) $\sin^{-1} x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$

$$= \sec^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right]$$

$$= \cot^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$$

$$= \operatorname{cosec}^{-1}\left[\frac{1}{x}\right]$$

(ii) $$\cos^{-1} x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$$

$$= \operatorname{cosec}^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right]$$

$$= \cot^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

$$= \sec^{-1}\left[\frac{1}{x}\right]$$

Proof : Let $\sin^{-1} x = \theta \Rightarrow \sin\theta = x$

(i) $\cos\theta = \sqrt{1-x^2}$, $\tan\theta = \frac{x}{\sqrt{1-x^2}}$, $\sec\theta = \frac{1}{\sqrt{1-x^2}}$, $\cot\theta = \frac{\sqrt{1-x^2}}{x}$ and $\operatorname{cosec}\theta = \frac{1}{x}$

$\therefore$ $$\sin^{-1} x = \theta = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) Let $\cos^{-1} x = \theta \Rightarrow$ $x = \cos\theta$

$\therefore$ $\sin\theta = \sqrt{1-x^2}$, $\tan\theta = \frac{\sqrt{1-x^2}}{x}$, $\sec\theta = \frac{1}{x}$, $\cot\theta = \frac{x}{\sqrt{1-x^2}}$

and $\operatorname{cosec}\theta = \frac{1}{\sqrt{1-x^2}}$

$$\cos^{-1} x = \sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \sec^{-1}\left(\frac{1}{x}\right)$$

Property 8

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$

(iii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right]$

Proof (i) : Let $x = \sin\theta$, $y = \sin\phi$, then

$$\text{L.H.S.} = \theta + \phi$$

$$\text{R.H.S.} = \sin^{-1}\left(\sin\theta\cos\phi + \cos\theta\sin\phi\right)$$

$$= \sin^{-1}\left[\sin\left(\theta+\phi\right)\right] = \theta + \phi$$

$\therefore$ L.H.S. = R.H.S.

(ii) Let $x = \cos\theta$ and $y = \cos\phi$

$$\text{L.H.S.} = \theta + \phi$$

$$\text{R.H.S.} = \cos^{-1}\left(\cos\theta\cos\phi - \sin\theta\sin\phi\right)$$

$$= \cos^{-1}\left[\cos\left(\theta+\phi\right)\right] = \theta + \phi$$

$\therefore$ L.H.S. = R.H.S.

(iii) Let $x = \sin\theta$, $y = \sin\phi$

$$\text{L.H.S.} = \theta - \phi$$

$$\text{R.H.S.} = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$$

$$= \sin^{-1}\left[\sin\theta\sqrt{1-\sin^2\phi} - \sin\phi\sqrt{1-\sin^2\theta}\right]$$

$$= \sin^{-1}\left[\sin\theta\cos\phi - \cos\theta\sin\phi\right]$$

$$= \sin^{-1}\left[\sin\left(\theta-\phi\right)\right] = \theta - \phi$$

$\therefore$ L.H.S. = R.H.S.

(iv) Let $x = \cos\theta$, $y = \cos\phi$

$\therefore$ L.H.S $= \theta - \phi$

$$\text{R.H.S.} = \cos^{-1}\left[\cos\theta\cos\phi + \sin\theta\sin\phi\right]$$

$$= \cos^{-1}\left[\cos\left(\theta-\phi\right)\right] = \theta - \phi$$

$\therefore$ L.H.S. = R.H.S.

Example 18.5 Evaluate :

$$\cos\left[\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right]$$

Soluton : Let $\sin^{-1}\left(\frac{3}{5}\right)=\theta$ and $\sin^{-1}\left(\frac{5}{13}\right)=\phi$, then

$$\sin\theta=\frac{3}{5} \text{ and } \sin\phi=\frac{5}{13}$$

$$\Rightarrow \qquad \cos\theta=\frac{4}{5} \text{ and } \cos\phi=\frac{12}{13}$$

$\therefore$ The given expression becomes $\cos[\theta+\phi]$

$$=\cos\theta\cos\phi-\sin\theta\sin\phi$$

$$=\frac{4}{5}\cdot\frac{12}{13}-\frac{3}{5}\cdot\frac{5}{13}=\frac{33}{65}$$

Example 18.6 Prove that

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)$$

Solution : Applying the formula :

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ we have}$$

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right)=\tan^{-1}\left(\frac{20}{90}\right)=\tan^{-1}\left(\frac{2}{9}\right)$$

Example 18.7 Prove that

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$$

Applying the property

$$\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right), \text{ we have}$$

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\frac{16}{25}}\sqrt{1-\frac{144}{169}}\right)$$

$$=\cos^{-1}\left(\frac{33}{65}\right)$$

Example 18.8 Prove that

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

Solution : Let $\sqrt{x} = \tan\theta$ then

$$\text{L.H.S.} = \theta \text{ and R.H.S.} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta)$$

$$= \frac{1}{2}\times 2\theta = \theta$$

$\therefore$ L.H.S. = R.H.S.

Example 18.9 Solve the equation

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x, x > 0$$

Solution : Let $x = \tan\theta$, then

$$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$$

$$\Rightarrow \quad \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right] = \frac{1}{2}\theta$$

$$\Rightarrow \quad \frac{\pi}{4}-\theta = \frac{1}{2}\theta$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}\times\frac{2}{3} = \frac{\pi}{6}$$

$$\therefore \quad x = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Example 18.10 Show that

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

Solution : Let $x^2 = \cos 2\theta$, then

$$2\theta = \cos^{-1}(x^2)$$

$$\Rightarrow \quad \theta = \frac{1}{2}\cos^{-1}x^2$$

Substituting $x^2 = \cos 2\theta$ in L.H.S. of the given equation, we have

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1 + \tan\theta}{1 - \tan\theta}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right]$$

$$= \frac{\pi}{4} + \theta$$

$$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1}\left(x^2\right)$$

CHECK YOUR PROGRESS 18.2

1. Evaluate each of the following :

(a) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ (b) $\cot\left(\tan^{-1}\alpha + \cot^{-1}\alpha\right)$

(c) $\tan\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right)$

(d) $\tan\left(2\tan^{-1}\frac{1}{5}\right)$ (e) $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$

2. If $\cos^{-1}x + \cos^{-1}y = \beta$, prove that

$$x^2 - 2xy\cos\beta + y^2 = \sin^2\beta$$

3. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, prove that

$$x^2 + y^2 + z^2 + 2xyz = 1$$

4. Prove each of the following :

(a) $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}} = \frac{\pi}{2}$ (b) $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$

(c) $\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}$ (d) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

LET US SUM UP

- Inverse of a trigonometric function exists if we restrict the domain of it.

(i) $\sin^{-1} x = y$ if $\sin y = x$ where $-1 \le x \le 1, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii) $\cos^{-1} x = y$ if $\cos y = x$ where $-1 \le x \le 1, 0 \le y \le \pi$

(iii) $\tan^{-1} x = y$ if $\tan y = x$ where $x \in R, -\frac{\pi}{2} < y < \frac{\pi}{2}$

(iv) $\cot^{-1} x = y$ if $\cot y = x$ where $x \in R, 0 < y < \pi$

(v) $\sec^{-1} x = y$ if $\sec y = x$ where $x \ge 1, \quad 0 \le y < \frac{\pi}{2}$ or $x \le -1, \frac{\pi}{2} < y \le \pi$

(vi) $\operatorname{cosec}^{-1} x = y$ if $\operatorname{cosec} y = x$ where $x \ge 1, 0 < y \le \frac{\pi}{2}$

or $\quad x \le -1, -\frac{\pi}{2} \le y < 0$

- Graphs of inverse trigonometric functions can be represented in the given intervals by interchanging the axes as in case of $y = \sin x$, etc.
- **Properties :**

(i) $\sin^{-1}(\sin\theta) = \theta$, $\tan^{-1}(\tan\theta) = \theta$, $\tan(\tan^{-1}\theta) = \theta$ and $\sin(\sin^{-1}\theta) = \theta$

(ii) $\operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$, $\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$, $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

(iii) $\sin^{-1}(-x) = -\sin^{-1} x$, $\tan^{-1}(-x) = -\tan^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(iv) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

(v) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

(vi) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(vii) $\sin^{-1} x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(viii) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(ix) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp y\sqrt{1-x^2}\sqrt{1-y^2}\right]$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. Prove each of the following :

   (a) $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{8}{17}\right)=\sin^{-1}\left(\frac{77}{85}\right)$ (b) $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{1}{9}\right)=\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

   (c) $\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{27}{11}\right)$

2. Prove each of the following :

   (a) $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{23}{11}\right)$ (b) $\tan^{-1}\left(\frac{1}{2}\right)+2\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}2$

   (c) $\tan^{-1}\left(\frac{1}{8}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{1}{3}\right)$

3. (a) Prove that $2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

   (b) Prove that $2\cos^{-1}x=\cos^{-1}\left(2x^2-1\right)$

   (c) Prove that $\cos^{-1}x=2\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)=2\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

4. Prove the following :

   (a) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}$ (b) $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)=\frac{\pi}{4}-x$

   (c) $\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)=0$

5. Solve each of the following :

   (a) $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$ (b) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec}x)$

   (c) $\cos^{-1}x+\sin^{-1}\left(\frac{1}{2}x\right)=\frac{\pi}{6}$ (d) $\cot^{-1}x-\cot^{-1}(x+2)=\frac{\pi}{12}, x>0$

ANSWERS

CHECK YOUR PROGRESS 18.1

1. (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$ (d) $-\frac{\pi}{3}$ (e) $\frac{\pi}{4}$

2. (a) $\frac{1}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{1}{2}$ (d) 1 (e) -2

3. (a) $\sqrt{1+x^2}$ (b) $\frac{2}{\sqrt{x^2-4}}$ (c) $\sqrt{x^4-1}$ (d) $\frac{x^2}{\sqrt{x^4+1}}$ (e) $\sqrt{\frac{1-x}{x}}$

CHECK YOUR PROGRESS 18.2

1. (a) 1 (b) 0 (c) $\frac{x+y}{1-xy}$ (d) $\frac{5}{12}$ (e) $-\frac{7}{17}$

TERMINAL EXERCISE

5. (a) $\frac{1}{6}$ (b) $\frac{\pi}{4}$ (c) ± 1 (d) $\sqrt{3}$

Hyperbolic Functions

Objectives

If we take $x = a\cos\theta, y = a\sin\theta\ (\theta \in R)$ Then $x^2 + y^2 = a^2$, In other words, for any real value of θ the point $(a\cos\theta, a\sin\theta)$ lies on the circle $x^2 + y^2 = a^2$. For this reason, the trigonometric function we are also known as circular functions.

$x = a\left(\dfrac{e^{\theta}+e^{-\theta}}{2}\right),\ y = b\left(\dfrac{e^{\theta}+e^{-\theta}}{2}\right)$ $(\theta \in R)$ then we get that $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1.$ This is the equation of "hyperbpla" (Hyperbolas will be discussed in the second year intermediate course). This mean that points on the hyperbola are in the $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ form

$$\left[a\left(\frac{e^{\theta}+e^{-\theta}}{2}\right),\ b\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)\right]\ (\theta \in R)$$

Keeping this fact in view, the "Hyperbolic functions" are introduced. The number e is defined as

$$e = 1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+.....\infty = \sum_{n=0}^{\infty}\frac{1}{n!}\ \forall\ x \in R$$

For any it is known that

$$e^x = 1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+.....\infty = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

e is also give by $e = \underset{n\to\infty}{\text{lt}}\left(1+\dfrac{1}{x}\right)^x$

the approximate value of e is given by $e \sim 2.7118287$.

18.4 Definition

18.4.1 Definition : 1) The function $f : R \to R$ defined by $f(x) = \dfrac{e^x - e^{-x}}{2}$ for all $x \in R$ is called the "hyperbolic sine" function. It is denoted by $\sin hx$.

Thus $\quad \sinh x = \dfrac{e^x - e^{-x}}{2}$ for all $x \in R$

ii) $\cosh x = \dfrac{e^x + e^{-x}}{2}$ for all $x \in R$

iii) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ for all $x \in R$

iv) $\cot h\, x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ for all $x \in R$

v) $\sec h\, x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$ for all $x \in R$

vi) $\operatorname{cosec} h\, x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$ for all $x \in R$

18.4.2 Identities

i) $\cosh^2 x - \sinh^2 x = 1$ for all $x \in R$

ii) $1 - \tanh^2 x = \operatorname{sech}^2 x$ for all $x \in R$

iii) $\coth^2 x - 1 = \operatorname{cosech}^2 x$ for all $x \in R - \{0\}$

18.5 Inverse Hyperbolic function

18.5.1 Definition

The function $f: \mathrm{R} \to \mathrm{R}$ defined by $f(x) = \sinh x$ nfor all $x \in \mathrm{R}$, is a bijection. Thus the inverse of this function exists and it is denoted by $\sinh^{-1}$. Thus, if x, y are real numbers then

$$\sinh^{-1} x = y \Leftrightarrow \sinh y = x$$

$$\Rightarrow \quad \cosh^{-1} x = y \Leftrightarrow \cosh y = x$$

$$\tanh^{-1} x = y \Leftrightarrow \tanh y = x$$

$$\coth^{-1} x = y \Leftrightarrow \coth y = x$$

$$\mathrm{sech}^{-1} x = y \Leftrightarrow \mathrm{sech}\, y = x$$

$$\mathrm{cosech}^{-1} x = y \Leftrightarrow \mathrm{cosech}\, y = x \text{ we define.}$$

18.6 Properties of hyperbolic function

18.6.1 for $x, y \in \mathrm{R}$

i) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sin hy$

ii) $\sinh(x-y) = \sinh x \cosh y - \cosh x \cosh y$

iii) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

iv) $\cos(x-y) = \mathrm{cohsh} x \cosh y - \sinh x \sinh y$

18.6.2 for any $x \in \mathrm{R}$

i) $\sinh 2x = 2\sinh x \cosh x = \dfrac{2\tanh x}{1-\tanh^2 x}$

ii) $\cosh 2x = \cosh^2 x + \sinh^2 x$

$$= 2\cosh^2 x - 1$$

$$= 1 + 2\sinh^2 x$$

$$= \frac{1+\tanh^2 x}{1-\tanh^2 x}$$

18.6.3 for any $x, y \in$ R

i) $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

ii) $\tanh(x-y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

iii) $\coth(x+y) = \dfrac{\coth x \coth y + 1}{\coth y + \coth x}$

iv) $\coth(x-y) = \dfrac{\coth x \coth y - 1}{\coth y - \coth x}$

18.6.4 for any $x \in$ R

i) $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

ii) $\coth 2x = \dfrac{\coth^2 x + 1}{2\coth x}$ if $x \neq 0$

Example 1 : Prove that, for any $x \in$ R $\sinh(3x) = 3\sinh x + 4\sinh^3 x$

Sol: $\sinh 3x \quad = \sinh(2x + x)$

$= \sinh 2x \,.\, \cosh x + \cosh 2x \sinh x$

$= 2\sinh x(1 + \sinh^2 x) + (1 + 2\sinh^2 x)\sinh x$

$= 3\sinh x + 4\sinh^3 x$

Example 2: If $\cosh x = \dfrac{5}{2}$ find the value of i) $\cosh(2x)$ ii) $\sinh(2x)$

Sol : i) $\cosh(2x) = 2\cosh^2 x - 1$

$= 2.\dfrac{25}{4} - 1 = \dfrac{23}{2}$

ii) we know that $\cosh^2(2x) - \sinh^2(2x) = 1$

Therefore $\sinh^2(2x) = \cosh^2(2x) - 1 = \left(\frac{23}{2}\right)^2 - 1$

$$= \frac{25.24}{4}$$

$$\sinh(2x) = \pm\frac{5\sqrt{21}}{2}$$

Check your progress 1

1. If $\sinh x = \frac{3}{4}$ find the value of $\cosh(2x), \sinh(2x)$.

2. Prove that $\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$.

3. Prove that $\cosh^4(x) - \sinh^4 x = \cosh(2x)$.

Letus sum up

1. i) $\sinh x = \frac{e^x - e^{-x}}{2}$,

   ii) $\cosh x = \frac{e^x + e^{-x}}{2}$

2. i) $\tanh x = \frac{\sinh x}{\cosh x}$;

   ii) $\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$

   iii) $\sec hx = \frac{1}{\cosh x}$; $\quad$ $\text{cosec } hx = \frac{1}{\sinh x}$

3. i) $\cosh^2 x - \sinh^2 x = 1$

   ii) $1 - \tanh^2 x = \text{sech}^2 x$

   iii) $\coth^2 x - 1 = \text{cosech}^2 x$.

4. i) $\sinh 2x = 2\sinh x \cosh x$,

ii) $\cosh 2x = \cosh^2 x + \sinh^2 x$

5. i) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

ii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

iii) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + \tanh^3 x}$

Supportive Websites

- http://www.wikipedia.org
- http://mathworld.wolfram.com

Answers

Check your progress 1

1. $\dfrac{17}{8}, \dfrac{15}{8}$

19

RELATIONS BETWEEN SIDES AND ANGLES OF A TRIANGLE

In an earlier lesson, we have learnt about trigonometric functions of real numbers, relations between them, drawn the graphs of trigonometric functions, studied the characteristics from their graphs, studied about trigonometric functions of sum and difference of real numbers, and deduced trigonometric functions of multiple and sub-multiples of real numbers. We also studied about inverse trigonometric functions, some of their properties and solved problems based on their concepts.

In this lesson, we shall try to establish some results which will give the relationship between sides and angles of a triangle and will help in finding unknown parts of a triangle.

OBJECTIVES

After studying this lesson, you will be able to :

- derive sine formula, cosine formula and projection formula
- apply these formulae to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Trigonometric and inverse trigonometric functions.
- Formulae for sum and difference of trigonometric functions of real numbers.
- Trigonometric functions of multiples and sub-multiples of real numbers.

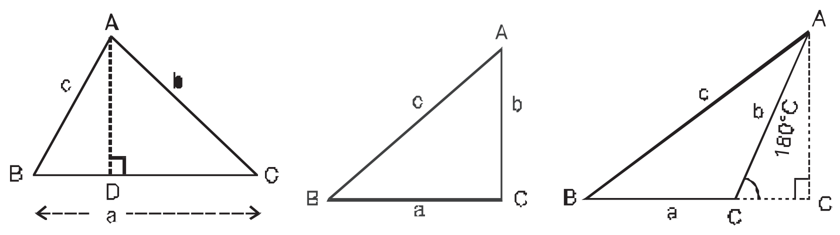
19.1 SINE FORMULA

In a ΔABC, the angles corresponding to the vertices A, B, and C are denoted by A, B, and C and the sides opposite to these vertices are denoted by a, b and c respectively. These angles and sides are called six elements of the triangle.

Result 1 : Prove that in any triangle, the lengths of the sides are proportional to the sines of the angles opposite to the sides,

i.e. $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof : In ΔABC, in Fig. 19.1 [(i), (ii) and (iii)], BC = a, CA = b and AB = c and $\angle C$ is acute angle in (i), right angle in (ii) and obtuse angle in (iii).



(i) (ii) (iii)

Fig. 19.1

Draw AD prependicular to BC (or BC produced, if need be)

In ΔABC, $\frac{AD}{AB} = \sin B$

or $\frac{AD}{c} = \sin B \quad \Rightarrow \quad AD = c \sin B$ .....(i)

In ΔADC, $\frac{AD}{AC} = \sin C$ in Fig 19.1 (i)

or, $\frac{AD}{b} = \sin C \quad \Rightarrow \quad AD = b \sin C$ ....(ii)

If Fig. 19.1 (ii), or $\frac{AD}{AC} = 1 = \sin\frac{\pi}{2} = \sin C$

$AD = b \sin C$

and in Fig. 19.1 (iii), $\frac{AD}{AC} = \sin(\pi - C) = \sin C$

or $\frac{AD}{b} = \sin C$ or $AD = b \sin C$

Thus, in all three figures, AD = b sin C

From (i) and (ii), we get

$$c \sin B = b \sin C$$

$$\Rightarrow \quad \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{....(iii)}$$

Similarly, by drawing prependicular from C on AB, we can prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad .....(iv)$$

From (iii) and (iv), we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad .....(A)$$

(A) is called the sine-formula

Note : (A) is sometimes written as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad ....(A')$$

The relations (A) and (A') help us in finding unknown angles and sides, when some others are given.

Let us take some examples :

Example 19.1 Prove that $a\cos\frac{B-C}{2} = (b+c)\sin\frac{A}{2}$, using sine-formula.

Solution : R.H.S. $= (b+c)\sin\frac{A}{2}$

We know that, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$\Rightarrow \qquad a = k\sin A,\ b = k\sin B,\ c = k\sin C$

$\therefore \qquad$ R.H.S. $= k(\sin B + \sin C)\sin\frac{A}{2}$

$$= k\ 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}$$

Now $\frac{B+C}{2} = 90^\circ - \frac{A}{2} \qquad (\because\ A + B + C = \pi)$

$\therefore \quad \sin\frac{B+C}{2} = \cos\frac{A}{2}$

$\therefore \qquad$ R.H.S. $= 2k\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}$

$$= k\cdot\sin A\cdot\cos\frac{B-C}{2}$$

$$= a\cdot\cos\frac{B-C}{2} = \text{L.H.S}$$

Example 19.2 Using sine formula, prove that

$$a\left(\cos C-\cos B\right)=2\left(b-c\right)\cos^2\frac{A}{2}$$

Solution : We have

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k \quad \text{(say)}$$

$\Rightarrow$ $\quad a = k\sin A,\ b = k\sin B,\ c = k\sin C$

$\therefore$ $$\text{R.H.S} = 2k\left(\sin B-\sin C\right)\cdot\cos^2\frac{A}{2}$$

$$= 2k\cdot 2\cos\frac{B+C}{2}\cdot\sin\frac{B-C}{2}\cdot\cos^2\frac{A}{2}$$

$$= 4k\sin\frac{A}{2}\cdot\sin\frac{B-C}{2}\cdot\cos^2\frac{A}{2} = 2a\sin\frac{B-C}{2}\cdot\cos\frac{A}{2}$$

$$= 2a\sin\frac{B+C}{2}\cdot\sin\frac{B-C}{2}$$

$$= a\left(\cos C-\cos B\right)$$

$$= \text{L.H.S.}$$

Example 19.3 In any triangle ABC, show that

$$a\sin A-b\sin B=c\sin\left(A-B\right)$$

Solution : We have

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k \quad \text{(say)}$$

L.H.S. $$= k\sin A\cdot\sin A - k\sin B\cdot\sin B$$

$$= k\left[\sin^2 A-\sin^2 B\right]$$

$$= k\sin\left(A+B\right)\cdot\sin\left(A-B\right)$$

$$A+B=\pi-C \Rightarrow \sin\left(A+B\right)=\sin C$$

$\therefore$ $$\text{L.H.S.} = k\sin C\cdot\sin\left(A-B\right)$$

$$= c\sin\left(A-B\right)=\text{R.H.S.}$$

Example 19.4 In any triangle, show that

$$a\left(b\cos C-c\cos B\right)=b^2-c^2$$

Solution : We have, $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k$ (say)

L.H.S. $= k\sin A\left(k\sin B\cos C - k\sin C\cos B\right)$

$= k^2\cdot\sin A\left[\sin\left(B-C\right)\right]$

$= k^2\ \sin\left(B+C\right)\cdot\sin\left(B-C\right)$ $\qquad\left[\because\ \sin A = \sin\left(B+C\right)\right]$

$= k^2\left(\sin^2 B - \sin^2 C\right)$

$= k^2\sin^2 B - k^2\sin^2 C$

$= b^2 - c^2 = \text{R.H.S}$

CHECK YOUR PROGRESS 19.1

1. Using sine-formula, show that each of the following hold :

(i) $\dfrac{\tan\dfrac{A-B}{2}}{\tan\dfrac{A+B}{2}} = \dfrac{a-b}{a+b}$ (ii) $b\cos B + c\cos C = a\cos\left(B-C\right)$

(iii) $a\sin\dfrac{B-C}{2} = \left(b-c\right)\cos\dfrac{A}{2}$ (iv) $\dfrac{b+c}{b-c} = \tan\dfrac{B+C}{2}\ \cot\dfrac{B-C}{2}$

(v) $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

2. In any triangle if $\dfrac{a}{\cos A} = \dfrac{b}{\cos B}$, prove that the tiangle is isosceles.

19.2 COSINE FORMULA

Result 2 : In any triangle, prove that

(i) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ (ii) $\cos B = \dfrac{c^2+a^2-b^2}{2ac}$ (iii) $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Proof :

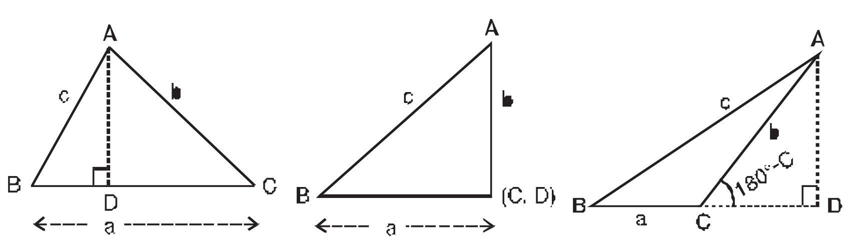


Fig. 19.2

Three cases arise :

(i) When $\angle C$ is acute (ii) When $\angle C$ is a right angle

(iii) When $\angle C$ is obtuse

Let us consider these one by one :

Case (i) When $\angle C$ is acute

$$\frac{AD}{AC} = \sin C \quad \Rightarrow \quad AD = b \sin C$$

Also $BD = BC - DC = a - b\cos C$ $\left[\because \frac{DC}{b} = \cos C\right]$

From Fig. 19.2 (i)

$$c^2 = (b\sin C)^2 + (a - b\cos C)^2$$

$$= b^2 \sin^2 C + a^2 + b^2 \cos^2 C - 2ab\cos C$$

$$= a^2 + b^2 - 2ab\cos C$$

$$\Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Case (ii) When $\angle C = 90°$

$$c^2 = AD^2 + BD^2 = b^2 + a^2$$

As $C = 90° \Rightarrow \cos C = 0$

$$\therefore \quad c^2 = b^2 + a^2 - 2ab \cdot \cos C$$

$$\Rightarrow \quad \cos C = \frac{b^2 + a^2 - c^2}{2ab}$$

Case (iii) When $\angle C$ is obtuse

$$\frac{AD}{AC} = \sin(180° - C) = \sin C$$

$$\therefore \quad AD = b\sin C$$

Also, $BD = BC + CD = a + b\cos(180° - C)$

$$= a - b\cos C$$

$$\therefore \quad c^2 = (b\sin C)^2 + (a - b\cos C)^2$$

$$= a^2 + b^2 - 2ab\cos C$$

$$\Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$\therefore$ In all the three cases, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Similarly, it can be proved that

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac} \quad \text{and} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Let us take some examples to show its application.

Example 19.5 In any triangle ABC, show that

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

Solution : We know that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},\ \cos B = \frac{c^2 + a^2 - b^2}{2ac},\ \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \quad \text{L.H.S.} = \frac{b^2 + c^2 - a^2}{2abc} + \frac{c^2 + a^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}$$

$$= \frac{1}{2abc}\left[b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2\right]$$

$$= \frac{a^2 + b^2 + c^2}{2abc} = \text{R.H.S.}$$

Example 19.6 If $\angle A = 60^\circ$, show that in ΔABC

$$(a + b + c)(b + c - a) = 3bc$$

Solution : $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ .....(i)

$$A = 60^\circ \Rightarrow \cos A = \cos 60^\circ = \frac{1}{2}$$

$\therefore$ (i) becomes $\dfrac{1}{2} = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow$ $\qquad b^2 + c^2 - a^2 = bc$

or $\qquad b^2 + c^2 + 2bc - a^2 = 3bc$

or $\qquad (b + c)^2 - a^2 = 3bc$

or $\qquad (b + c + a)(b + c - a) = 3bc$.

Example 19.7 If the sides of a triangle are 3 cm, 5 cm and 7 cm find the greatest angle of a triangle.

Solution : Here a = 3 cm, b = 5 cm, c = 7 cm

We know that in a triangle, the angle opposite to the largest side is greatest

$\therefore \angle C$ is the greatest angle.

$$\therefore \qquad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$= \frac{9+25-49}{30} = \frac{-15}{30} = \frac{-1}{2}$$

$$\therefore \qquad \cos C = \frac{-1}{2} \quad \Rightarrow \quad C = \frac{2\pi}{3}$$

$\therefore$ The greatest angle of the triangle is $\frac{2\pi}{3}$ or $120°$.

Example 19.8 In ΔABC, if $\angle A = 60°$, prove that $\frac{b}{c+a} + \frac{c}{a+b} = 1$.

Solution : $\cos A = \frac{b^2+c^2-a^2}{2bc}$

or $$\cos 60° = \frac{1}{2} = \frac{b^2+c^2-a^2}{2bc}$$

$$\therefore \qquad b^2+c^2-a^2 = bc$$

or $$b^2+c^2 = a^2+bc \qquad .....(i)$$

$$\text{R.H.S.} = \frac{b}{c+a} + \frac{c}{a+b} = \frac{ab+b^2+c^2+ac}{(c+a)(a+b)}$$

$$= \frac{ab+ac+a^2+bc}{(c+a)(a+b)} \qquad \text{[Using (i)]}$$

$$= \frac{a(a+b)+c(a+b)}{(a+c)(a+b)}$$

$$= \frac{(a+c)(a+b)}{(a+c)(a+b)} = 1$$

CHECK YOUR PROGRESS 19.2

1. In any triangle ABC, show that

(i) $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$

(ii) $\left(a^2-b^2+c^2\right)\tan B=\left(b^2-c^2+a^2\right)\tan C=\left(c^2-a^2+b^2\right)\tan A$

(iii) $\frac{k}{2}(\sin 2A+\sin 2B+\sin 2C)=\frac{a^2+b^2+c^2}{2abc}$

where $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

(iv) $\left(b^2-c^2\right)\cot A+\left(c^2-a^2\right)\cot B+\left(a^2-b^2\right)\cot C=0$

2. The sides of a triangle are a = 9 cm, b = 8 cm, c = 4 cm. Show that

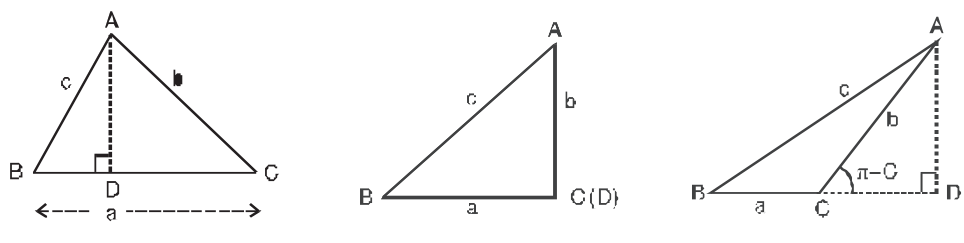
$$6\cos C = 4 + 3\cos B.$$

19.3 PROJECTION FORMULA

Result 3 : In ΔABC, if BC = a, CA = b and AB = c, then prove that

(i) $a=b\cos C+c\cos B$ (ii) $b=c\cos A+a\cos C$ (iii) $c=a\cos B+b\cos A$

Proof :



(i) (ii) (iii)

Fig. 19.3

As in previous result, three cases arise. We will discuss them one by one.

(i) When $\angle C$ is acute :

In Δ ADB, $\frac{BD}{c}=\cos B \quad\Rightarrow\quad BD=c\cos B$

In Δ ADC, $\frac{DC}{b}=\cos C \quad\Rightarrow\quad DC=b\cos C$

$a = BD + DC = c\cos B + b\cos C$

$\therefore \quad a = c\cos B + b\cos c$

(ii) When $\angle C = 90^\circ$

$$a = BC = \frac{BC}{AB}\cdot AB = \cos B \cdot c$$
$$= c\cos B + 0$$
$$= c\cos B + b\cos 90^\circ \qquad (\because \cos 90^\circ = 0)$$
$$= c\cos B + b\cos C$$

(iii) When $\angle C$ is obtuse

In $\Delta\, ADB$, $\quad \frac{BD}{c} = \cos B \Rightarrow BD = c\cos B$

In $\Delta\, ADC$, $\quad \frac{CD}{b} = \cos(\pi - C) = -\cos C$

$\Rightarrow \quad CD = -b\cos C$

In Fig.19.3 (iii),

$$BC = BD - CD$$
$$a = c\cos B - (-b\cos C)$$
$$= c\cos B + b\cos C$$

Thus in all cases, $a = b\cos C + c\cos B$

Similarly, we can prove that

$$b = c\cos A + a\cos C \quad \text{and} \quad c = a\cos B + b\cos A$$

Let us take some examples, to show the application of these results.

Example 19.9 In any triangle ABC, show that

$$(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a + b + c$$

Solution : L.H.S. $= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C$

$$= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (c\cos B + b\cos C)$$
$$= c + b + a$$
$$= a + b + c = \text{R.H.S.}$$

Example 19.10 In any $\Delta\, ABC$, prove that

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Solution : L.H.S. $= \frac{1 - 2\sin^2 A}{a^2} - \frac{1 - 2\sin^2 B}{b^2}$

$$= \frac{1}{a^2} - \frac{2\sin^2 A}{a^2} - \frac{1}{b^2} + \frac{2\sin^2 B}{b^2}$$
$$= \frac{1}{a^2} - \frac{1}{b^2} - 2k^2 + 2k^2 = \frac{1}{a^2} - \frac{1}{b^2} \qquad \left(\because \frac{\sin A}{a} = \frac{\sin B}{b} = k\right)$$
$$= \text{R. H. S.}$$

Example 19.11 In $\Delta\,ABC$, if $a\cos A = b\cos B$, where $a \neq b$ prove that $\Delta\,ABC$ is a right angled triangle.

Solution : $a\cos A = b\cos B$

$\therefore$ $$a\left[\frac{b^2+c^2-a^2}{2bc}\right] = b\left[\frac{a^2+c^2-b^2}{2ac}\right]$$

or $$a^2\left(b^2+c^2-a^2\right) = b^2\left(a^2+c^2-b^2\right)$$

or $$a^2b^2+a^2c^2-a^4 = a^2b^2+b^2c^2-b^4$$

or $$c^2\left(a^2-b^2\right) = \left(a^2-b^2\right)\left(a^2+b^2\right)$$

$\Rightarrow$ $$c^2 = a^2+b^2$$

$\therefore\ \Delta\,ABC$ is a right triangle.

Example 19.12 If a = 2, b = 3, c = 4, find cos A, cos B and cos C.

Solution : $$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{9+16-4}{2\times3\times4} = \frac{21}{24} = \frac{7}{8}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ac} = \frac{16+4-9}{2\times4\times2} = \frac{11}{16}$$

and $$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{4+9-16}{2\times2\times3} = \frac{-3}{12} = \frac{-1}{4}$$

CHECK YOUR PROGRESS 19.3

1. If a = 3, b = 4 and c = 5, find cos A, cos B and cos C.
2. The sides of a triangle are 7 cm, $4\sqrt{3}$ cm and $\sqrt{13}$ cm. Find the smallest angle of the triangle.
3. If a : b : c = 7 : 8 : 9, prove that
   cos A : cos B : cos C = 14 : 11 : 6.
4. If the sides of a triangle are x^2+x+1, $2x+1$ and x^2-1. Show that the greatest angle of the triangle is 120°.
5. In a triangle, b cos A = a cos B, prove that the triangle is isosceles.
6. Deduce sine formula from the projection formula.

LET US SUM UP

It is possible to find out the unknown elements of a triangle, if the relevent elements are given by using

Sine-formula :

(i) $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine foumulae :

(ii) $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Projection formulae :

$$a = b\cos C + c\cos B$$

$$b = c\cos A + a\cos C$$

$$c = a\cos B + b\cos A$$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

In a triangle ABC, prove the following (1-10) :

1. $a\sin(B - C) + b\sin(C - A) + c\sin(A - B) = 0$

2. $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

3. $\frac{b^2 - c^2}{a^2} \cdot \sin 2A + \frac{c^2 - a^2}{b^2} \cdot \sin 2B + \frac{a^2 - b^2}{c^2} \cdot \sin 2C = 0$

4. $$\frac{c^2+a^2}{b^2+c^2}=\frac{1+\cos B\cos(C-A)}{1+\cos A\cos(B-C)}$$

5. $$\frac{c-b\cos A}{b-c\cos A}=\frac{\cos B}{\cos C}$$

6. $$\frac{a-b\cos C}{c-b\cos A}=\frac{\sin C}{\sin A}$$

7. $$(a+b+c)\left[\tan\frac{A}{2}+\tan\frac{B}{2}\right]=2c\cot\frac{C}{2}$$

8. $$\sin\frac{A-B}{2}=\frac{a-b}{c}\cos\frac{C}{2}$$

9. (i) $b\cos B+c\cos C=a\cos(B-C)$

   (ii) $a\cos A+b\cos B=c\cos(A-B)$

10. $$b^2=(c-a)^2\cos^2\frac{B}{2}+(c+a)^2\sin^2\frac{B}{2}$$

11. In a triangle, if $b = 5$, $c = 6$, $\tan\frac{A}{2}=\frac{1}{\sqrt{2}}$, then show that $a=\sqrt{41}$.

12. In any $\Delta\, ABC$, show that

$$\frac{\cos A}{\cos B}=\frac{b-a\cos C}{a-b\cos C}$$

ANSWERS

CHECK YOUR PROGRESS 19.3

1. $\cos A = \frac{4}{5}$

   $\cos B = \frac{3}{5}$

   $\cos C = \text{zero}$

2. The smallest angle of the triangle is 30°.

Heights and Distances

Objectives

We have studied about triangles in general and right angled triangles in particular in this chapter we will combine the principles of both these topics to study some applications of trigonometry to problems of daily life and some problems of scientific interest.

Suppose a boy is sitting on the balcony of this building located on one bank of a canal. He is looking at a small pup standing on a step of the temple on the other bank of the canal. By Imagining a right angled triangle to fit with the geometry of the situation and knowing the height at which the boy is sitting. Can you determine the width of the canal.

To solve such problems we introduce two fundamental concepts - i) angle of elevation and ii) angle of depression.

19.4 Angle of elevation and angle of depression

In this discussion that follows in this section, we discuss some elementary ideas like obesver's line of right, the angle of elevation and the angle of depression and illustrate then diagrams. Later we shall discuss some physical problem involving single plane which are based on these concepts and the trigonometric ratios and obtain their solutions.

19.4.1 Definition (Angle of elevation)

The line of right is the line drawn from the eye of an observer to the object viewed by the obeserver.

The anlge of elevation is defined as the anlge $\angle AOB$ formed by the horizontal OA with the line of right OB, when the object being viewed is above the horizontal level.

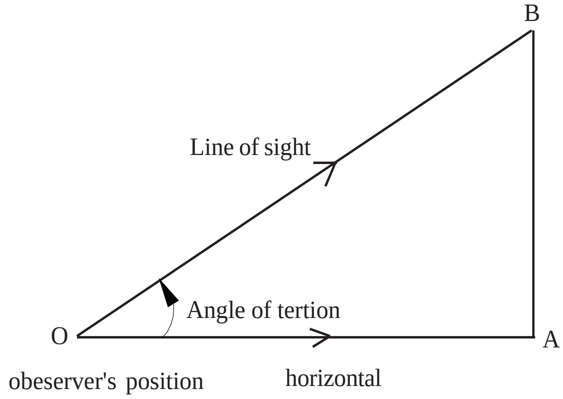


Fig. 19.4

19.4.2 Definition (Angle of depresition)

The angle of depression of an object being viewed is defined as the angle $\angle CBO$ formed by the horizontal CB with the line of sight BO when the object viewed is below the horizontal level.

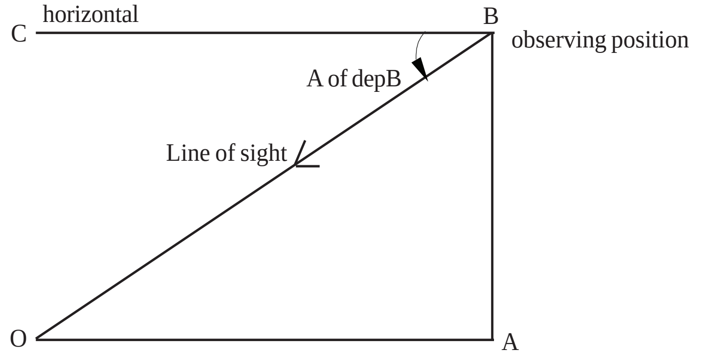


Fig. 19.5

Example 1

From a point on the level ground, the angle of elevation of the top of a hill is α. After moving through a distance of 'd' meters towards the foot of the hill. The angle of elevation of its top is found to be β. Find the height of the hill.

Sol: As shown in fig 19.5 Let 'B' be the foot of the hill AB. Let C be the initial point of observation and D be the final point of observation on the horizontal. Let 'h' be the height of the hill and DB = x.

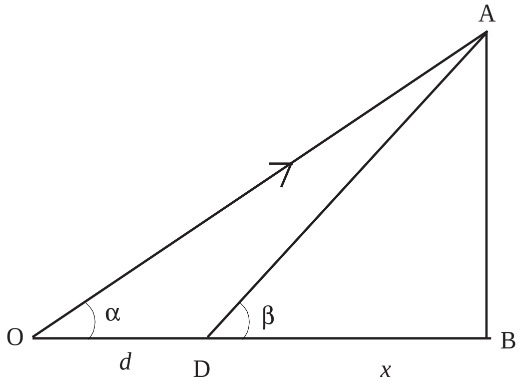


Fig. 19.5

By hypothesis CD $= d$ $\underline{|ACB} = \alpha$, $\underline{|ADB} = \beta$

From Δ ACB, $\cot\alpha = \dfrac{CB}{AB} = \dfrac{x+d}{h}$

$$\therefore x + d = h\cot\alpha \qquad \text{... (1)}$$

From Δ ADB $\cot\beta = \dfrac{DB}{AB} = \dfrac{x}{h}$ $\quad \therefore x = h\cot\beta$ ... (2)

From equation (1) and (2) we have $d = h(\cot\alpha - \cot\beta)$ ... (3)

$$h = \frac{d}{\cot\alpha - \cot\beta}$$

Height of the hill $= \dfrac{d}{\cot\alpha - \cot\beta}$

Example 2

From the cliff of a mountain, 120 meters above the sea level, the angles of depression of two boats of the same side of the mountain are 45^0 and 35^0 respectively. Find the distance between the two boats. (Assume that the boats lie along a straight line from the foot of the hill)

Sol : Take MN as the mountain and consider a horizontal MX through the foot of the mountian M. The horizontal MX can be taken as the sea level. Draw NY parallel to the horizontal line such that it passes through the observers eye. Denote the two boats by A and B.

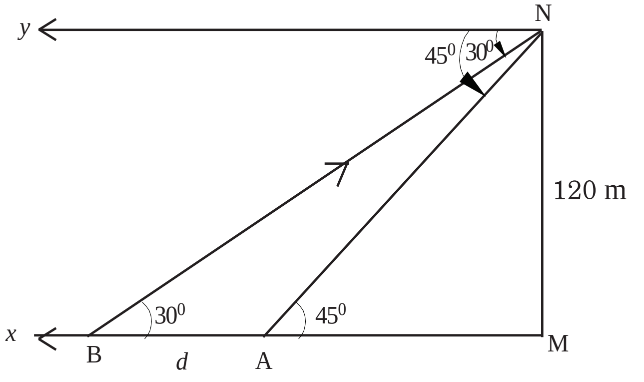


Fig. 19.6

By hypothesis $\angle YNA = 45^0$; $\angle YNB = 30^0$

By from fig 11.4, $\angle YNA = \angle NAN$

From Δ MBN, $\cot 30^0 = \dfrac{BM}{MN} = \dfrac{BM}{120} \Rightarrow BM = 120 \cot 30^0$

From Δ MAN, $\cot 45^0 = \dfrac{AM}{MN} = \dfrac{AM}{120} \Rightarrow AM = 120^0 \cot 45^0$

$AB = BM - AM = 120 (\cot 30^0 - \cot 45^0)$

$= 120 (\sqrt{3} - 1)$ mts.

Example 3

One end of the ladder is in contact with a wall and the other and is in count with the level ground making an angle α. When the foot of the ladder is moved through a distance 'a' cms away from the wall, the end in contact with the walls slides through 'b' cms along the wall and the angle made by the ladder with the level grounds is now β. Show that $a = b \text{ Tan}\left(\dfrac{\alpha+\beta}{2}\right)$

Sol: Let l be the length of the ladder, OA = x, OB = y. Let the intiati position of the ladder be AB and the final position after its foot moved through a distance 'a' cms be $A'B'$. Then AB = A'B' = l.

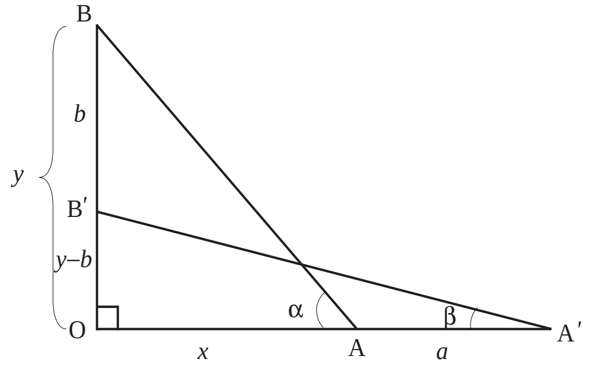


Fig. 19.7

From fig 19.7 $OB = OB - BB' = y - b$

Also from Δ OAB, $\cos\alpha = \dfrac{x}{l} \Rightarrow x = l\cos\alpha$ ...(1)

similarly $OB = y = l\sin\alpha$ ...(2)

In Δ OA'B', $OA' = x + a = l\cos\beta$ ...(3)

and $OB' = y - b = l\sin\beta$ ...(4)

we now make out a relation between a, b, c, d using relation (1) - (4)

From (1) and (3); $a = l(\cos\beta - \cos\alpha)$

From (2) and (4) ; $b = l(\sin\alpha - \sin\beta)$

$$\frac{a}{b} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \frac{2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)}{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)} = \text{Tan}\left(\frac{\alpha+\beta}{2}\right)$$

$$\frac{a}{b} = \text{Tan}\left(\frac{\alpha+\beta}{2}\right) \Rightarrow a = b\ \text{Tan}\left(\frac{\alpha+\beta}{2}\right)$$

Check your progress 1

1. From the top of a light house 200 meters high on the sea - shore, the angle of depression of a ship in the sea is 36^0. How far is the ship from the sea - shore ?

2. The angles of elevation of the top of a tower from two points at a distance of 4m and 9 m. From the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6m.

3. The angle of elevation of the top of a building from the foot of the tower is 30^0 and the angle of elevation of the top of the tower from the foot of the building is 60^0. If the tower is 50 meters high, find the hight of the buildings.

Letus sum up

1. The line of right is the line drawn from the eye of the observe to the object that is viewed by the observer.

2. The angle of elevation of an object viewed, is the angle formed by the line of sight with the horizontal, when the object that viewed is above the horizontal level. In this case, the observe raises his/her head to look at the object.

3. The angle of depression of an object viewed is the angle formed by the line of sight with the horizontal, when the object that viewed is below the horizontal level. In this case, the observer lowers his/her hand to look at the object.

Supportive Websites

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

Answers

Check your progress 1

(1) 275.26 meters

(3) $\frac{50}{3}$ meters

De Moivre's Theorem

Objectives

We learn that $\text{cis}\theta_1 \cdot \text{cis}\theta_2 = \text{cis}(\theta_1 + \theta_2)$ and hence $(\text{cis}\theta)^2 = \text{cis}\,2\theta$. In this chapter we extend this result for any integer. The extension is called De Moivre's theorem for integral indices. We use it in studing the n^{th} roots of unity. We also obtain a verson of De Moivre's theorem for rational indices.

19.5 De Moivre's theorem for Integral Index and for rational Index

By using this theorem all the n^{th} roots of a complex number $Z \neq 0$ are determined. As a particular case, all the n^{th} roots of unity are determined and their geometrical representations are obtained.

19.5.1 De Moivre's Theorem for Integral Index

for any real number θ and any Integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\sin n\theta$$

19.5.2

i) It is customary to write $\text{cis}\,\theta$ for $\cos\theta + i\sin\theta$. Thus we may state the DeMoivre's theorm as $(\text{cis}\,\theta)^n = \text{cis}\,(n\theta)$ if $n \in Z$.

ii) $(\cos\theta + i\sin\theta)^{-n} = \cos(-n)\,\theta + i\sin(-n)\,\theta = \cos n\,\theta - i\sin n\theta$ provided 'n' is an integer.

iii) $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta - i^2\sin^2\theta$

$$= \cos^2\theta + \sin^2\theta = 1 \; (i^2 = -1)$$

iv) $(\cos\theta - i\sin\theta)^n = \left(\dfrac{1}{\cos\theta + i\sin\theta}\right)^n = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin^2 n\theta$

v) $\text{cis}\,\theta \,.\, \text{cis}\,\varphi = \text{cis}\,(\theta + \varphi)$ for any $\theta, \varphi \in$ R.

Example 1 : If m, n are integers and $x = \cos\alpha + i\sin\alpha$

Then prove that $y = \cos\beta + i\sin\beta$ then $x^m y^n + \dfrac{1}{x^m y^n} = \cos(m\alpha + n\beta)$

$$x^m y^n - \frac{1}{x^m y^n} = 2i\sin(m\alpha + n\beta)$$

Sol : $x^m = (\cos\alpha + i\sin\alpha)^m = \cos m\alpha + i\sin m\alpha$

$y^n = (\cos\beta + i\sin\beta)^n = \cos n\beta + i\sin n\beta$

$$\therefore\ x^m y^n = (\cos m\alpha + i\sin m\alpha)(\cos n\beta + i\sin n\beta)$$

$$= \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta) \qquad ...(1)$$

$$\frac{1}{x^m y^n} = \frac{1}{\cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)} = \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta) \qquad ...(2)$$

$$(1) + (2)\quad x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$$

$$(1) - (2)\quad x^m y^n - \frac{1}{x^m y^n} = 2i\sin(m\alpha + n\beta)$$

Example 2 :

If n is a positive integer, show that $(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cos\left(\dfrac{n\pi}{4}\right)$.

Sol : $1+i = \sqrt{2}\left(\dfrac{1}{\sqrt{2}} + i\,\dfrac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

$$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$(1+i)^n = (\sqrt{2})^n \left(\cos\frac{\pi}{n} + i\,\frac{\pi}{n}\right)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \qquad \ldots(1)$$

$$(1-i)^n = (\sqrt{2})^n \left(\cos\frac{\pi}{n} - i\,\frac{\pi}{n}\right)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} - i\,\sin\frac{n\pi}{4}\right) \qquad \ldots(2)$$

By adding (1) & (2) we get

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}}\left(2\cos\frac{n\pi}{4}\right) = 2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$$

Example 3:

If n is an integer then show that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

$$= 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$$

Sol : Take L.H.S $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = \left(2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$

$$+\left(2\cos^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$$

$$= 2^n\cos^n\frac{\theta}{2}\left[\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^n + \left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)^n\right]$$

$$= 2^n\cos^n\frac{\theta}{2}\left[\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2} + \cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right]$$

$$= 2^n\cos^n\frac{\theta}{2}\left(2\cos\frac{n\theta}{2}\right) = 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}.$$

Check Your Progress 1

1) i) $(1+i\sqrt{3})^3$ iv) $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$ find the values

2) If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$ show that

i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

ii) $\sin 3\alpha + \sin 3\beta + \sin\gamma = 3\sin(\alpha+\beta+\gamma)$

19.6 n^{th} roots of Unity

Let n be a positive integer greater than one and $w = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, then $1, w, w^2 \ldots w^{n-1}$ are all the distinct n^{th} roots of unity.

i) The sum of the n^{th} roots of unity is zero

$$1 + w + w^2 + ... + w^{n-1} = 0$$

ii) product of the n^{th} roots of unity is $(-1)^{n-1}$

19.7 Cube roots of Unity

In this section we find all the cube roots of unity and all the cube roots of a positive real numbers.

19.7.1

$1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$ are the cube roots of the unity

$$w = \text{cis } \frac{2\pi}{n} .$$

Example 4 : Find the values of $\left(\sqrt{3} + i\right)^{\frac{1}{4}}$.

Sol : The modulus ampllitude form of $\sqrt{3} + i$

$$= 2(\cos 30^0 + i \sin 30^0)$$

Hence $\left(\sqrt{3} + i\right)^{\frac{1}{4}} = 2^{\frac{1}{4}} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)^{\frac{1}{4}}$

$$= 2^{\frac{1}{4}} \text{cis} \left(\frac{2k\pi + \frac{\pi}{6}}{6}\right), \ k = 0,1, 2, 3$$

$$= 2^{\frac{1}{4}} \text{cis} \left(\frac{12k\pi + \pi}{24}\right), \ k = 0,1, 2, 3$$

$$= 2^{\frac{1}{4}} \text{cis } (12k+1) \frac{\pi}{24}, \ k = 0,1, 2, 3$$

$\left(\sqrt{3}+i\right)^{\frac{1}{4}}$ is all values $2^{\frac{1}{4}}\text{ cis }\frac{\pi}{24},\ 2^{\frac{1}{4}}\text{ cis }\frac{13\pi}{4},\ 2^{\frac{1}{4}}\text{ cis }\frac{25\pi}{24},\ 2^{\frac{1}{4}}\text{ cis }\frac{37\pi}{4}$

Example 5.

Find all the roots of the equation $x^{11}-x^7+x^4-1=0$

Sol : $x^{11}-x^7+x^4-1=x^7(x^4-1)+1(x^4-1)$

$$=(x^4-1)(x^7+1)$$

Therefore the roots of the given equations are precisely the 4th roots of unity and 7th roots of -1. They are

$$\text{cis }\frac{2k\pi}{4}=\text{cis }\frac{k\pi}{2}\ k\in\{0,1,2,3\}\text{ and}$$

$$\text{cis }\frac{2k\pi+\pi}{7}=\text{cis }(2k+1)\frac{\pi}{7},\ \ k\in\{0,1,2,3,4,5,6\}$$

The value of x are

$$\text{cis }0=1,\ \text{cis }\frac{\pi}{2}=i,\ \text{cis }\pi=-1\qquad[\because\ -1=\text{cis }\pi]$$

$$\text{cis }\frac{3\pi}{2}=-i$$

$$\text{cis }\frac{\pi}{7},\ \text{cis }\frac{3\pi}{7},\ \text{cis }\frac{5\pi}{7},\ \text{cis }\pi=-1,\ \text{cis }\frac{9\pi}{7},\ \text{cis }\frac{11\pi}{7},\ \text{cis }\frac{13\pi}{7}$$

i.e., $\pm$ i, ±1, $\text{cis }\frac{\pi}{7},\ \text{cis }\frac{3\pi}{7},\ \text{cis }\frac{5\pi}{7},\ \text{cis}\frac{9\pi}{7},\ \text{cis }\frac{11\pi}{7},\ \text{cis }\frac{13\pi}{7}$

Example 6

If $1,\ w,\ w^2$ are the cube roots of unity, prove that $x^2+4x+7=0$ where $x=w\ \ w^2-2$

Sol : $x=w-w^2-2$

$\Rightarrow\ (x+2)=w-w^2$

$\Rightarrow\ (x+2)^2=w^2+w^4-2w^3$

$\Rightarrow\ x^2+4x+4=w^2+w-2=-1-2=-3$

$\Rightarrow\ x^2+4x+7=0$

Check your Progress 2

1. Find all the value of $(-16)^{1/4}$.
2. If $1, w, w^2$ are the cube roots of unity the prove that $\frac{1}{2+w} + \frac{1}{1+2w} = \frac{1}{1+w}$.
3. Solve $x^4 - 1 = 0$ the equation.
4. Solve the equation $x^9 - x^5 + x^4 - 1 = 0$.

Letus sum up

1. If 'n' is an inteter, then $(\text{cis}\theta)^n = \text{cis}\,(n\theta)$ [De-Moiver's theorem for integral index]
2. If $Z_0 = r_0 \text{ cis } \theta_0 \neq 0$ then the n^{th} roots of Z_0 are

$$a_k = r_0^{\frac{1}{n}} \text{ cis} \left(\frac{2kn + \theta_0}{n} \right), \; k = 0, 1, 2, \ldots.(n-1)$$

3. The n^{th} roots of unity are $\text{cis } \frac{2k\pi}{n}$ $k = 0, 1, 2, 3, \ldots. (n-1)$
4. Cube roots of unity are $1, \; w = \frac{-1 + i\sqrt{3}}{2}$, $w^2 = \frac{-1-1\sqrt{3}}{2}$

Suportive Websites

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

Answers

Check your Progress 1

(1) 8 (3) i

Check your Progress 2

1) $2 \text{ cis } (2k+1) \frac{\pi}{4}$ $k = 0, 1, 2, 3$

3) $\pm 1, \pm i$

4) $\pm 1, \pm i, \text{ cis} \left(\pm \frac{\pi}{4} \right), \text{cis} \left(\pm \frac{3\pi}{5} \right)$

Trigonometric Expansions

Objectives

In this section we discuss expansions for $\sin(n\theta)\cos(n\theta)$ and $\tan(n\theta)$ in terms of powers of $\sin\theta$, $\cos\theta$, $\tan\theta$ when 'n' is a positive integer. These expressions are consequence of De-Moiver's theorem and have finitely many terms.

19.8 Expansion of trigonometric functions $\sin n\theta$, $\cos n\theta$, $\tan n\theta$, $\cot n\theta$

Here we are De Moiver's theorem and Binomial theorem for expressing $\sin n\theta, \cos n\theta,$ interms of powers of $\sin\theta, \cos\theta$ and then using these expressions we express $\tan n\theta, \cot n\theta$ in terms of powers of $\tan\theta, \cot\theta$ where n is a positive integer.

19.8.1

If n is a positive integer, θ is real number then

i) $\sin n\theta = n_{C_1}\cos_\theta^{n-1}\sin\theta - n_{C_3}\cos_\theta^{n-3}\sin^3\theta + n_{C_5}\cos_\theta^{n-5}\sin^5\theta + ...$

where n is an odd integer and

$\sin n\theta = n_{C_1}\cos_\theta^{n-1}\sin\theta - n_{C_3}\cos_\theta^{n-3}\sin^3\theta + n_{C_5}\cos_\theta^{n-5}\sin^5\theta \; ....$

when n is an even integer.

ii) $\cos n\theta = \cos^n\theta - n_{C_2}\cos^{n-2}\sin^2\theta + n_{C_4}\cos^{n-4}\sin^4\theta + \; .....$

Where 'n' is an odd integer and

$$\cos n\theta = \cos^n\theta - n_{C_2}\cos^{n-2}\sin^2\theta + n_{C_4}\cos^{n-4}\sin^4\theta + \; .....(-1)^{\frac{n}{2}}\sin^n\theta$$

Where n is an even integer.

19.8.2

If n is a positive integer, then

$$\text{Tan } n\theta = \frac{n_{C_1}\text{Tan}\,\theta - n_{C_3}\text{Tan}^3\theta + n_{C_5}\text{Tan}^5\theta - n_{C_7}\text{Tan}^7\theta + \ldots.}{1 - n_{C_2}\text{Tan}^2\theta + n_{C_4}\text{Tan}^4\theta - n_{C_6}\text{Tan}^6\theta \ \ldots.}$$

provided $\cos n\theta \neq 0$

19.8.3 Note: If n is positive integer and $\sin n\theta \neq 0$ then we have $\cot n\theta = \dfrac{\cos n\theta}{\sin n\theta}$

$$\cot n\theta = \frac{\cot^n\theta - n_{C_2}\cot^{n-2}\theta + n_{C_4}\cot^{n-4}\theta - n_{C_6}\cot^{n-6}\theta + \ldots.}{n_{C_1}\cot^{n-1}\theta - n_{C_3}\cot^{n-3}\theta + n_{C_5}\cot^{n-5}\theta - n_{C_7}\cot^{n-7}\theta + \ldots.}$$

Example 1 : Expand i) $\cos 6\theta$, ii) $\sin 6\theta$, iii) $\tan 6\theta$

Sol: i) $\cos n\theta = \cos^n\theta - n_{C_2}\cos^{n-2}\theta\sin^2\theta + n_{C_4}\cos^{n-4}_{\theta}\sin^4\theta + \ldots..$

$$\cos 6\theta = \cos^6\theta - 6_{C_2}\cos^4\theta\sin^2\theta + 6_{C_4}\cos^2\theta\sin^4\theta - 6_{C_6}\cos^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

ii) $\sin n\theta = n_{C_1}\cos^{n-1}\theta\sin\theta - n_{C_3}\cos^{n-3}\theta\sin^3\theta + \ldots\ldots...$

$$\sin 6\theta = 6_{C_1}\cos^5\theta\sin\theta - 6_{C_3}\cos^3\theta\sin^3\theta + 6_{C_5}\cos\theta\sin^5\theta$$

$$= 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

iii) $\text{Tan } n\theta = \dfrac{n_{C_1}\text{Tan}\theta - n_{C_3}\text{Tan}^3\theta + n_{C_5}\text{Tan}^5\theta - \ldots.}{1 - n_{C_2}\text{Tan}^2\theta + n_{C_4}\text{Tan}^4\theta - n_{C_6}\text{Tan}^6\theta + \ldots.}$ కనుక

$$\text{Tan } 6\theta = \frac{6_{C_1}\text{Tan}\theta - 6_{C_3}\text{Tan}^3\theta + 6_{C_5}\text{Tan}^5\theta}{1 - 6_{C_2}\text{Tan}^2\theta + 6_{C_4}\text{Tan}^4\theta - 6_{C_6}\text{Tan}^6\theta}$$

$$= \frac{6\text{Tan}\theta - 20\text{Tan}^3\theta + 6\text{Tan}^5\theta}{1 - 15\text{Tan}^2\theta + 15\text{Tan}^4\theta - \text{Tan}^6\theta}$$

Example 2 : Show that $\dfrac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$ 63 when $\sin\theta \neq 0$

Sol: $\sin 6\theta = 6_{C_1}\cos^5\sin\theta - 6_{C_3}\cos^3\theta\sin^3\theta + 6_{C_5}\cos\theta\sin^5\theta$

$$\frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta\sin^2\theta + 6\cos\theta\sin 4\theta$$

$$= 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$$

$$\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta .$$

19.9 Expressing $\sin^n\theta$ and $\cos^n\theta$ interms of sines and cosines of multiples of θ

$x = \cos\theta + i\sin\theta$ then $\dfrac{1}{x} = \cos\theta - i\sin\theta$

$$x + \frac{1}{x} = 2\cos\theta\ ;\ x - \frac{1}{x} = 2i\sin\theta \qquad \ldots(1)$$

Let n be a positive integer, then by De-Moiver's theorem

$$x^n + \frac{1}{x^n} = 2\cos n\theta,\ x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$\frac{1}{x^n} = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta,\ x^n - \frac{1}{x^n} = 2i\sin n\theta \qquad \ldots(2)$$

19.9.1

i) If 'n' is a positive integer then

$$2^{n-1}\cos^n\theta = \cos n\theta + n_{C_1}\cos(n-2)\theta + n_{C_2}\cos(n-4)\theta + \ldots$$

19.9.2

Let n be a positive integer

i) If n is an even number then

$$2^{n-1}(-1)^{n/2}\sin^n\theta = \cos n\theta - n_{C_1}\cos(n-2)\theta + n_{C_2}(n-4)\theta - \ldots$$

ii) If n is an odd number then

$$2^{n-1}(-1)^{n-1/2}\sin^n\theta = \sin n\theta - n_{C_1}\sin(n-2)\theta + n_{C_2}(n-4)\,\theta - \ldots$$

Example 3

Show that $2^7\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35$

Sol : We have every even 'n'

$$2^{n-1}\cos^n\theta = \cos n\theta + n_{C_1}\cos(n-2)\theta + n_{C_2}\cos(n-4)\,\theta + \ldots + \frac{(-1)^{\frac{n}{2}}}{2}\, n_{C_{\frac{n}{2}}}$$

putting $n = 8$ in the above expansion

$$2^7\cos^8\theta = \cos 8\theta + 8_{C_1}\cos 6\theta + 8_{C_2}\cos 4\theta + 8_{C_3}\cos 2\theta + \frac{1}{2}\,8_{C_4}$$

$$2^7\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35$$

Example 4

Show that $2^6\sin^4\theta\cos^3\theta = \cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta$.

Sol : $x = \cos\theta + i\sin\theta$ then $x + \frac{1}{x} = 2\cos\theta$; $x - \frac{1}{x} = 2i\sin\theta$

$$x^n + \frac{1}{x^n} = 2\cos n\theta,\ x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$(2i\sin\theta)^4\,(2\cos\theta)^3 = \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^3$$

$$= \left(x^2 - \frac{1}{x^2}\right)^3\left(x - \frac{1}{x}\right)$$

$$2^7\sin^4\theta\cos^3\theta = \left[x^6 - \frac{1}{x^6} + \frac{3}{x^2} - 3x^2\right]\left[x - \frac{1}{x}\right]$$

$$= \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$= 2\cos 7\theta - 2\sin 5\theta - 3(2\cos 3\theta) + 3(2\cos\theta)$$

$$2^6\sin^4\theta\cos^3\theta = \cos 7\theta - \cos 5\theta - 3\cos 3\,\theta + 3\cos\theta.$$

Check Your Progress 1

1. Show that $\dfrac{\sin 4\theta}{\sin\theta} = 8\cos^3\theta - 4\cos\theta$

2. Show that $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

3. Show that $64\sin^5\theta\cos^2\theta = \sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta$

Letus sum up

1. $\sin n\theta = n_{C_1}\cos^{n-1}\theta\,\sin\theta - n_{C_3}\cos^{n-3}\theta\,\sin^3\theta + n_{C_5}\cos^{n-5}\theta\,\sin^5\theta\ ......$

2. $\cos n\theta = \cos^n\theta - n_{C_2}\cos^{n-2}\theta\,\sin^2\theta + n_{C_4}\cos^{n-4}\theta\,\sin^4\theta - ......$

3. $\text{Tan } n\theta = \dfrac{n_{C_1}\text{Tan}\,\theta - n_{C_3}\text{Tan}^3\theta + n_{C_5}\text{Tan}^5\theta - n_{C_7}\text{Tan}^7\theta + ...}{1 - n_{C_2}\text{Tan}^2\theta + n_{C_4}\text{Tan}^4\theta - n_{C_6}\text{Tan}^6\theta + ...}$

4. If n is a positive integer then

   (i) If 'n' is even then

   $$2^{n-1}(-1)^{n/2}\sin^n\theta = \cos n\theta - n_{C_1}\cos(n-2)\theta + n_{C_2}\cos(n-4)\theta - ...$$

   (ii) If 'n' is odd then

   $$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - n_{C_1}\sin(n-2)\theta + n_{C_2}\sin(n-4)\theta - ...$$

Supportive Websites

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

MODULE-V

CALCULUS

20

LIMIT AND CONTINUITY

Consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$

You can see that the function f(x) is not defined at x = 1 as $x - 1$ is in the denominator. Take the value of x very nearly equal to but not equal to 1 as given in the tables below. In this case $x - 1 \neq 0$ as $x \neq 1$.

$\therefore$ We can write $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x+1)(x-1)}{(x-1)} = x + 1$, because $x - 1 \neq 0$ and so division by $(x-1)$ is possible.

Table -1

x	f(x)
0.5	1.5
0.6	1.6
0.7	1.7
0.8	1.8
0.9	1.9
0.91	1.91
:	:
:	:
0.99	1.99
:	:
:	:
0.9999	1.9999

Table - 2

x	f (x)
1.9	2.9
1.8	2.8
1.7	2.7
1.6	2.6
1.5	2.5
:	:
:	:
1.1	2.1
1.01	2.01
1.001	2.001
:	:
:	:
1.00001	2.00001

In the above tables, you can see that as x gets closer to 1, the corresponding value of f (x) also gets closer to 2.

However, in this case f(x) is not defined at x = 1. The idea can be expressed by saying that the limiting value of f(x) is 2 when x approaches to 1.

Let us consider another function f (x) =2x. Here, we are interested to see its behavior near the point 1 and at x = 1. We find that as x gets nearer to 1, the corresponding value of f (x) gets closer to 2 at x = 1 and the value of f (x) is also 2.

So from the above findings, what more can we say about the behaviour of the function near x = 2 and at x = 2 ?

In this lesson we propose to study the behaviour of a function near and at a particular point where the function may or may not be defined.

OBJECTIVES

After studying this lesson, you will be able to :

- illustrate the notion of limit of a function through graphs and examples;
- define and illustrate the left and right hand limits of a function y =f (x) at x = a;
- define limit of a function y = f (x) at x = a;
- state and use the basic theorems on limits;
- establish the following on limits and apply the same to solve problems :

(i) $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x - a} = na^{n-1} \; (x \neq a)$

(ii) $\lim\limits_{x\to 0} \sin x = 0$ and $\lim\limits_{x\to 0} \cos x = 1$

(iii) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ (iv) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(vi) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$ (vii) $\lim\limits_{x\to 0} \dfrac{\log(1+x)}{x} = 1$

- define and interprete geometrically the continuity of a function at a point;
- define the continuity of a function in an interval;
- determine the continuity or otherwise of a function at a point; and
- state and use the theorems on continuity of functions with the help of examples.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a function
- Drawing the graph of a function
- Concept of trigonometric function
- Concepts of exponential and logarithmic functions

20.1 LIMIT OF A FUNCTION

In the introduction, we considered the function $f(x)=\dfrac{x^2-1}{x-1}$. We have seen that as x approaches l, f (x) approaches 2. In general, if a function f (x) approaches L when x approaches 'a', we say that L is the limiting value of f (x)

Symbolically it is written as

$$\lim_{x\to a} f(x) = L$$

Now let us find the limiting value of the function $(5x-3)$ when x approaches 0.

i.e. $$\lim_{x\to 0} (5x-3)$$

For finding this limit, we assign values to x from left and also from right of 0.

x	–0.1	–0.01	–0.001	–0.0001..........
5x – 3	–3.5	–3.05	–3.005	–3.0005..........

x	0.1	0.01	0.001	0.0001..........
5x – 3	–2.5	–2.95	–2.995	–2.9995.........

It is clear from the above that the limit of $(5x-3)$ as $x \to 0$ is -3

i.e., $$\lim_{x\to 0} (5x-3) = -3$$

This is illustrated graphically in the Fig. 20.1

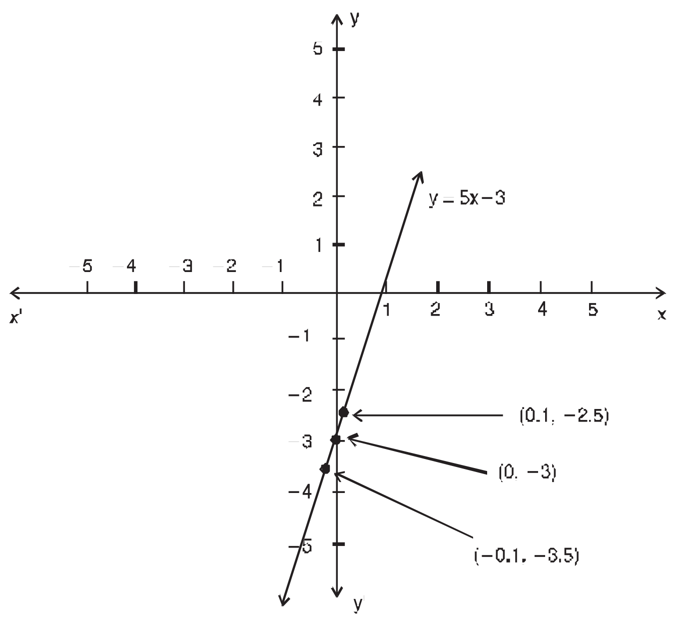


Fig. 20.1

The method of finding limiting values of a function at a given point by putting the values of the variable very close to that point may not always be convenient.

We, therefore, need other methods for calculating the limits of a function as x (independent variable) ends to a finite quantity, say a

Consider an example : Find $\lim_{x\to 3} f(x)$, where $f(x)=\dfrac{x^2-9}{x-3}$

We can solve it by the method of substitution. Steps of which are as follows :

Step 1: We consider a value of x close to a say $x = a + h$, where h is a very small positive number. Clearly, as $x \to a,\ h \to 0$	For $f(x)=\dfrac{x^2-9}{x-3}$ we write $x = 3 + h$, so that as $x \to 3, h \to 0$
Step 2 : Simplify $f(x) = f(a + h)$	Now $f(x) = f(3+h)$ $=\dfrac{(3+h)^2-9}{3+h-3}$ $=\dfrac{h^2+6h}{h}$ $= h + 6$
Step 3 : Put $h = 0$ and get the requried result	$\therefore \lim_{x\to 3} f(x) = \lim_{h\to 0}(6+h)$ As $x \to 0,\ h \to 0$ Thus, $\lim_{x\to 3} f(x) = 6 + 0 = 6$ by putting $h = 0$.

Remarks : It may be noted that f (3) is not defined, however, in this case the limit of the function $f(x)$ as $x \to 3$ is 6.

Now we shall discuss other methods of finding limits of different types of functions.

Consider the example :

Find $\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases}\dfrac{x^3-1}{x^2-1}, & x \neq 1\\ 1, & x=1\end{cases}$

Here, for $x \neq 1, f(x)=\dfrac{x^3-1}{x^2-1}$

$$=\frac{(x-1)\left(x^2+x+1\right)}{(x-1)(x+1)}$$

It shows that if f (x) is of the form $\frac{g(x)}{h(x)}$, then we may be able to solve it by the method of factors. In such case, we follow the following steps :

Step 1. Factorise g (x) and h (x)	**Sol.** $f(x)=\frac{x^3-1}{x^2-1}$ $=\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$ ($\because x\neq 1, \therefore x-1\neq 0$ and as such can be cancelled)
Step 2 : Simplify f (x)	$\therefore \quad f(x)=\frac{x^2+x+1}{x+1}$
Step 3 : Putting the value of x, we get the required limit.	$\therefore \lim\limits_{x\to 1}\frac{x^3-1}{x^2-1}=\frac{1+1+1}{1+1}=\frac{3}{2}$ Also $f(1)=1(\text{given})$ In this case, $\lim\limits_{x\to 1} f(x)\neq f(1)$

Thus, the limit of a function f (x) as $x\to a$ may be different from the value of the functioin at $x=a$.

Now, we take an example which cannot be solved by the method of substitutions or method of factors.

Evaluate $\quad \lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Here, we do the following steps :

Step 1. Rationalise the factor containing square root.

Step 2. Simplify.

Step 3. Put the value of x and get the required result.

Solution :

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\frac{\sqrt{(1+x)^2}-\sqrt{(1-x)^2}}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$= \frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$= \frac{1+x-1+x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$= \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} \qquad [\because x \neq 0, \therefore \text{ It can be cancelled}]$$

$$= \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$\therefore \quad \lim_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x \to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{1+1} = 1$$

20.2 LEFT AND RIGHT HAND LIMITS

You have already seen that $x \to a$ means x takes values which are very close to 'a', i.e. either the value is greater than 'a' or less than 'a'.

In case x takes only those values which are less than 'a' and very close to 'a' then we say x is approaches 'a' from the left and we write it as $x \to a^-$. Similarly, if x takes values which are greater than 'a' and very close to 'a' then we say x is approaching 'a' from the right and we write it as $x \to a^+$.

Thus, if a function f (x) approaches a limit ℓ_1, as x approaches 'a' from left, we say that the left hand limit of $f(x)$ as $x \to a$ is ℓ_1.

We denote it by writing

$$\lim_{x \to a^-} f(x) = \ell_1 \qquad \text{or} \qquad \lim_{h \to 0} f(a-h) = \ell_1,\ h > 0$$

Similarly, if f (x) approaches the limit ℓ_2, as x approaches 'a' from right we say, that the right hand limit of $f(x)$ as $x \to a$ is ℓ_2.

We denote it by writing

$$\lim_{x \to a^+} f(x) = \ell_2 \qquad \text{or} \qquad \lim_{h \to 0} f(a+h) = \ell_2,\ h > 0$$

Working Rules

Finding the right hand limit i.e., $\lim\limits_{x \to a^+} f(x)$

Finding the left hand limit, i.e, $\lim\limits_{x \to a^-} f(x)$

Put $x = a + h$ Put $x = a - h$

Find $\lim_{h \to 0} f(a+h)$ Find $\lim_{h \to 0} f(a-h)$

Note : In both cases remember that h takes only positive values.

20.3 LIMIT OF FUNCTION y = f(x) AT x = a

Consider an example :

Find $\lim_{x \to 1} f(x)$, where $f(x) = x^2 + 5x + 3$

Here $$\lim_{x \to 1^+} f(x) = \lim_{h \to 0}\left[(1+h)^2 + 5(1+h) + 3\right]$$

$$= \lim_{h \to 0}\left[1 + 2h + h^2 + 5 + 5h + 3\right]$$

$$= 1 + 5 + 3 = 9 \qquad \text{.....(i)}$$

and $$\lim_{x \to 1^-} f(x) = \lim_{h \to 0}\left[(1-h)^2 + 5(1-h) + 3\right]$$

$$= \lim_{x \to 0}\left[1 - 2h + h^2 + 5 - 5h + 3\right]$$

$$= 1 + 5 + 3 = 9 \qquad \text{.....(ii)}$$

From (i) and (ii), $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$

Now consider another example :

Evaluate : $$\lim_{x \to 3} \frac{|x-3|}{x-3}$$

Here $$\lim_{x \to 3^+} \frac{|x-3|}{x-3} = \lim_{h \to 0} \frac{|(3+h)-3|}{[(3+h)-3]}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} \quad \text{(as } h>0\text{, so } |h| = h\text{)}$$

$$= 1 \qquad \text{.....(iii)}$$

and $$\lim_{x \to 3^-} \frac{|x-3|}{x-3} = \lim_{h \to 0} \frac{|(3-h)-3|}{[(3-h)-3]}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} \qquad \text{(as } h > 0\text{, so } |-h| = h\text{)}$$

$$= -1 \qquad .....(iv)$$

$\therefore$ From (iii) and (iv), $\lim_{x \to 3^+} \frac{|x-3|}{x-3} \neq \lim_{x \to 3^-} \frac{|x-3|}{x-3}$

Thus, in the first example right hand limit = left hand limit whereas in the second example right hand limit $\neq$ left hand limit.

Hence the left hand and the right hand limits may not always be equal.

We may conclude that

$\lim_{x \to 1} \left(x^2 + 5x + 3\right)$ exists (which is equal to 9) and $\lim_{x \to 3} \frac{|x-3|}{x-3}$ does not exist.

Note :

I $\left.\begin{array}{l} \lim_{x \to a^+} f(x) = \ell \\ \text{and } \lim_{x \to a^-} f(x) = \ell \end{array}\right\} \Rightarrow \lim_{x \to a} f(x) = \ell$

II $\left.\begin{array}{l} \lim_{x \to a^+} f(x) = \ell_1 \\ \text{and } \lim_{x \to a^-} f(x) = \ell_2 \end{array}\right\} \Rightarrow \lim_{x \to a} f(x)$ doesnotexist.

III $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ does not exist $\Rightarrow \lim_{x \to a} f(x)$ does not exist.

20.4 BASIC THEOREMS ON LIMITS

1. $\lim_{x \to a} cx = c \lim_{x \to a} x$, c being a constant.

To verify this, consider the function $f(x) = 5x$.

We observe that in $\lim_{x \to 2} 5x$, 5 being a constant is not affected by the limit.

$$\therefore \quad \lim_{x \to 2} 5x = 5 \lim_{x \to 2} x$$

$$= 5 \times 2 = 10$$

2. $\lim_{x \to a} \left[g(x) + h(x) + p(x) + ...\right] = \lim_{x \to a} g(x) + \lim_{x \to a} h(x) + \lim_{x \to a} p(x) + ...$

where $g(x), h(x), p(x), ...$ are any function.

3. $\lim_{x \to a} \left[f(x) \cdot g(x)\right] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

To verify this, consider $f(x) = 5x^2 + 2x + 3$

and g (x) = x + 2.

Then $$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(5x^2 + 2x + 3\right)$$

$$= 5\lim_{x\to 0} x^2 + 2\lim_{x\to 0} x + 3 = 3$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}(x+2) = \lim_{x\to 0} x + 2 = 2$$

$\therefore$ $$\lim_{x\to 0}(5x^2 + 2x + 3)\lim_{x\to 0}(x+2) = 6 \qquad .....(i)$$

Again $$\lim_{x\to 0}[f(x)\cdot g(x)] = \lim_{x\to 0}[(5x^2 + 2x + 3)(x+2)]$$

$$= \lim_{x\to 0}(5x^3 + 12x^2 + 7x + 6)$$

$$= 5\lim_{x\to 0} x^3 + 12\lim_{x\to 0} x^2 + 7\lim_{x\to 0} x + 6$$

$$= 6 \qquad .....(ii)$$

From (i) and (ii), $$\lim_{x\to 0}[(5x^2 + 2x + 3)(x+2)] = \lim_{x\to 0}(5x^2 + 2x + 3)\lim_{x\to 0}(x+2)$$

4. $$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$ provided $\lim_{x\to a} g(x) \neq 0$

To verify this, consider the function $f(x) = \dfrac{x^2 + 5x + 6}{x+2}$

we have $$\lim_{x\to -1}(x^2 + 5x + 6) = (-1)^2 + 5(-1) + 6$$

$$= 1 - 5 + 6$$

$$= 2$$

and $$\lim_{x\to -1}(x+2) = -1 + 2$$

$$= 1$$

$$\frac{\lim_{x\to -1}(x^2 + 5x + 6)}{\lim_{x\to -1}(x+2)} = \frac{2}{1} = 2 \qquad .....(i)$$

Also $\lim\limits_{x \to -1} \dfrac{(x^2+5x+6)}{x+2} = \lim\limits_{x \to -1} \dfrac{(x+3)(x+2)}{x+2}$ $\left[\begin{array}{l} \because x^2+5x+6 \\ = x^2+3x+2x+6 \\ = x(x+3)+2(x+3) \\ = (x+3)(x+2) \end{array}\right]$

$$= \lim_{x \to -1} (x+3)$$

$$= -1+3 = 2 \qquad .....(ii)$$

$\therefore$ From (i) and (ii),

$$\lim_{x \to -1} \frac{x^2+5x+6}{x+2} = \frac{\lim\limits_{x \to -1} \left(x^2+5x+6\right)}{\lim\limits_{x \to -1} (x+2)}$$

We have seen above that there are many ways that two given functions may be combined to form a new function. The limit of the combined function as $x \to a$ can be calculated from the limits of the given functions. To sum up, we state below some basic results on limits, which can be used to find the limit of the functions combined with basic operations.

If $\lim\limits_{x \to a} f(x) = \ell$ and $\lim\limits_{x \to a} g(x) = m$, then

(i) $\lim\limits_{x \to a} kf(x) = k \lim\limits_{x \to a} f(x) = k\ell$ where k is a constant.

(ii) $\lim\limits_{x \to a} \left[f(x) \pm g(x)\right] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = \ell \pm m$

(iii) $\lim\limits_{x \to a} \left[f(x) \cdot g(x)\right] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = \ell \cdot m$

(iv) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{\ell}{m}$, provided $\lim\limits_{x \to a} g(x) \neq 0$

The above results can be easily extended in case of more than two functions.

Example 20.1 Find $\lim\limits_{x \to 1} f(x)$, where

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

Solution : $f(x) = \dfrac{x^2-1}{x-1}$

$$= \frac{(x-1)(x+1)}{x-1}$$

$$= (x+1) \qquad [\because x \neq 1]$$

$$\therefore \quad \lim_{x\to 1} f(x) = \lim_{x\to 1}(x+1)$$

$$= 1 + 1 = 2$$

Note : $\frac{x^2-1}{x-1}$ is not defined at x=1. The value of $\lim_{x\to 1} f(x)$ is independent of the value of f (x) at x = 1.

Example 20.2 Evaluate : $\lim_{x\to 2} \frac{x^3-8}{x-2}$.

Solution : $\lim_{x\to 2} \frac{x^3-8}{x-2}$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)}$$

$$= \lim_{x\to 2} \left(x^2+2x+4\right) \qquad [\because x \neq 2]$$

$$= 2^2 + 2 \times 2 + 4$$

$$= 12$$

Example 20.3 Evaluate : $\lim_{x\to 2} \frac{\sqrt{3-x}-1}{2-x}$.

Solution : Rationalizing the numerator, we have

$$\frac{\sqrt{3-x}-1}{2-x} = \frac{\sqrt{3-x}-1}{2-x} \times \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}$$

$$= \frac{3-x-1}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$= \frac{2-x}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$\therefore \quad \lim_{x\to 2} \frac{\sqrt{3-x}-1}{2-x} = \lim_{x\to 2} \frac{2-x}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$= \lim_{x\to 2} \frac{1}{\left(\sqrt{3-x}+1\right)} \qquad [\because x \neq 2]$$

$$= \frac{1}{\left(\sqrt{3-2}+1\right)}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

Example 20.4 Evaluate : $\lim\limits_{x\to 3}\dfrac{\sqrt{12-x}-x}{\sqrt{6+x}-3}$.

Solution : Rationalizing the numerator as well as the denominator, we get

$$\lim_{x\to 3}\frac{\sqrt{12-x}-x}{\sqrt{6+x}-3}=\lim_{x\to 3}\frac{\left(\sqrt{12-x}-x\right)\left(\sqrt{12-x}+x\right)\left(\sqrt{6+x}+3\right)}{\left(\sqrt{6+x}-3\right)\left(\sqrt{6+x}+3\right)\left(\sqrt{12-x}+x\right)}$$

$$=\lim_{x\to 3}\frac{\left(12-x-x^2\right)}{6+x-9}\cdot\lim_{x\to 3}\frac{\sqrt{6+x}+3}{\sqrt{12-x}+x}$$

$$=\lim_{x\to 3}\frac{-(x+4)(x-3)}{(x-3)}\cdot\lim_{x\to 3}\frac{\sqrt{6+x}+3}{\sqrt{12-x}+x}\qquad [\because x\neq 3]$$

$$=-(3+4)\,\frac{6}{6}=-7$$

Note : Whenever in a function, the limits of both numerator and denominator are zero, you should simplify it in such a manner that the denominator of the resulting function is not zero.

However, if the limit of the denominator is 0 and the limit of the numerator is non zero, then the limit of the function does not exist.

Let us consider the example given below :

Example 20.5 Find $\lim\limits_{x\to 0}\dfrac{1}{x}$, if it exists.

Solution : We choose values of x that approach 0 from both the sides and tabulate the correspondling values of $\dfrac{1}{x}$.

x	−0.1	−.01	−.001	−.0001
$\frac{1}{x}$	−10	−100	−1000	−10000

x	0.1	.01	.001	.0001
$\frac{1}{x}$	10	100	1000	10000

We see that as $x\to 0$, the corresponding values of $\dfrac{1}{x}$ are not getting close to any number.

Hence, $\lim\limits_{x\to 0}\dfrac{1}{x}$ does not exist. This is illustrated by the graph in Fig. 20.2

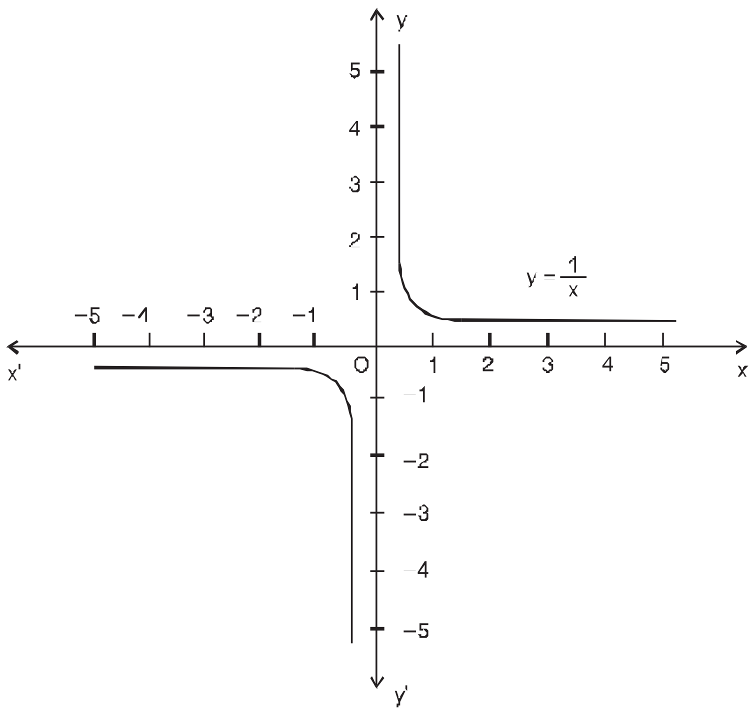


Fig. 20.2

Example 20.6 Evaluate : $\lim\limits_{x\to 0}(|x|+|-x|)$

Solution : Since |x| has different values for $x \geq 0$ and x<0, therefore we have to find out both left hand and right hand limits.

$$\lim_{x\to 0^-}(|x|+|-x|) = \lim_{h\to 0}(|0-h|+|-(0-h)|)$$

$$= \lim_{h\to 0}(|-h|+|-(-h)|)$$

$$= \lim_{h\to 0} h+h = \lim_{h\to 0} 2h = 0 \qquad \text{...(i)}$$

and $$\lim_{x\to 0^+}(|x|+|-x|) = \lim_{h\to 0}(|0+h|+|-(0+h)|)$$

$$= \lim_{x\to 0} h+h = \lim_{h\to 0} 2h = 0 \qquad \text{...(ii)}$$

From (i) and (ii),

$$\lim_{x\to 0^-}(|x|+|-x|) = \lim_{h\to 0^+}[|x|+|-x|]$$

Thus , $$\lim_{h\to 0}[|x|+|-x|] = 0$$

Note : We should remember that left hand and right hand limits are specially used when (a) the functions under consideration involve modulus function, and (b) function is defined by more than one rule.

Example 20.7 Find the vlaue of 'a' so that

$\lim\limits_{x \to 1} f(x)$ exist, where $f(x) = \begin{cases} 3x+5 & , x \le 1 \\ 2x+a, & x > 1 \end{cases}$

Solution :

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (3x+5) \qquad [\because f(x) = 3x+5 \text{ for } x \le 1]$$

$$= \lim_{h \to 0} [3(1-h)+5]$$

$$= 3 + 5 = 8 \qquad .....(i)$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (2x+a) \qquad [\because f(x) = 2x+a \text{ for } x > 1]$$

$$= \lim_{h \to 0} (2(1+h)+a)$$

$$= 2 + a \qquad .....(ii)$$

We are given that $\lim\limits_{x \to 1} f(x)$ will exists provided

$\Rightarrow \qquad \lim\limits_{x \to 1^-} = \lim\limits_{x \to 1^+} f(x)$

$\therefore$ From (i) and (ii),

$$2 + a = 8$$

$\therefore \qquad$ or, $a = 6$

Example 20.8 If a function f (x) is defined as

$$f(x) = \begin{cases} x & , \ 0 \le x < \frac{1}{2} \\ 0 & , \ x = \frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 \end{cases}$$

Examine the existence of $\lim\limits_{x \to \frac{1}{2}} f(x)$.

Solution : Here $\quad f(x) = \begin{cases} x & , \ 0 \le x < \frac{1}{2} & .....(i) \\ 0 & , \ x = \frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 & .....(ii) \end{cases}$

$$\lim_{x \to \left(\frac{1}{2}\right)^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$$

$$= \lim_{x \to 0} \left(\frac{1}{2} - h\right) \qquad \left[\because \frac{1}{2} - h < \frac{1}{2} \text{ and from(i)}, f\left(\frac{1}{2} - h\right) = \frac{1}{2} - h\right]$$

$$=\frac{1}{2}-0=\frac{1}{2} \qquad .....(iii)$$

$$\lim_{x\to\left(\frac{1}{2}\right)^+} f(x)=\lim_{h\to 0} f\left(\frac{1}{2}+h\right)$$

$$=\lim_{h\to 0}\left[\left(\frac{1}{2}+h\right)-1\right]\left[\because \frac{1}{2}+h>\frac{1}{2} \text{ and from(ii)}, f\left(\frac{1}{2}+h\right)=\left(\frac{1}{2}+h\right)-1\right]$$

$$=\frac{1}{2}+-1$$

$$=-\frac{1}{2} \qquad .....(iv)$$

From (iii) and (iv), left hand limit $\neq$ right hand limit

$\therefore$ $\lim_{x\to\frac{1}{2}} f(x)$ does not exist.

CHECK YOUR PROGRESS 20.1

1. Evaluate each of the following limits :

(a) $\lim_{x\to 2}[2(x+3)+7]$ (b) $\lim_{x\to 0}\left(x^2+3x+7\right)$ (c) $\lim_{x\to -1}\left[(x+3)^2-16\right]$

(d) $\lim_{x\to -1}\left[(x+1)^2+2\right]$ (e) $\lim_{x\to 0}\left[(2x+1)^3-5\right]$ (f) $\lim_{x\to 1}(3x+1)(x+1)$

2. Find the limits of each of the following functions :

(a) $\lim_{x\to 5}\frac{x-5}{x+2}$ (b) $\lim_{x\to 1}\frac{x+2}{x+1}$ (c) $\lim_{x\to -1}\frac{3x+5}{x-10}$

(d) $\lim_{x\to 0}\frac{px+q}{ax+b}$ (e) $\lim_{x\to 3}\frac{x^2-9}{x-3}$ (f) $\lim_{x\to -5}\frac{x^2-25}{x+5}$

(g) $\lim_{x\to 2}\frac{x^2-x-2}{x^2-3x+2}$ (h) $\lim_{x\to\frac{1}{3}}\frac{9x^2-1}{3x-1}$

3. Evaluate each of the following limits:

(a) $\lim_{x\to 1}\frac{x^3-1}{x-1}$ (b) $\lim_{x\to 0}\frac{x^3+7x}{x^2+2x}$ (c) $\lim_{x\to 1}\frac{x^4-1}{x-1}$

(d) $\lim_{x\to 1}\left[\frac{1}{x-1}-\frac{2}{x^2-1}\right]$

4. Evaluate each of the following limits :

(a) $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-\sqrt{4-x}}{x}$ (b) $\lim\limits_{x\to 0}\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ (c) $\lim\limits_{x\to 3}\dfrac{\sqrt{3+x}-\sqrt{6}}{x-3}$

(d) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+x}-1}$ (e) $\lim\limits_{x\to 2}\dfrac{\sqrt{3x-2}-x}{2-\sqrt{6-x}}$

5. (a) Find $\lim\limits_{x\to 0}\dfrac{2}{x}$, if it exists. (b) Find $\lim\limits_{x\to 2}\dfrac{1}{x-2}$, if it exists.

6. Find the values of the limits given below :

(a) $\lim\limits_{x\to 0}\dfrac{x}{5-|x|}$ (b) $\lim\limits_{x\to 2}\dfrac{1}{|x+2|}$ (c) $\lim\limits_{x\to 2}\dfrac{1}{|x-2|}$

(d) Show that $\lim\limits_{x\to 5}\dfrac{|x-5|}{x-5}$ does not exist.

7. (a) Find the left hand and right hand limits of the function

$$f(x)=\begin{cases}-2x+3, & x\le 1\\ 3x-5, & x>1\end{cases} \text{ as } x\to 1$$

(b) If $f(x)=\begin{cases}x^2, & x\le 1\\ 1, & x>1\end{cases}$, find $\lim\limits_{x\to 1} f(x)$

(c) Find $\lim\limits_{x\to 4} f(x)$ if it exists, given that $f(x)=\begin{cases}4x+3, & x<4\\ 3x+7, & x\ge 4\end{cases}$

8. Find the value of 'a' such that $\lim\limits_{x\to 2} f(x)$ exists,where $f(x)=\begin{cases}ax+5, & x<2\\ x-1, & x\ge 2\end{cases}$

9. Let $f(x)=\begin{cases}x, & x<1\\ 1, & x=1\\ x^2, & x>1\end{cases}$

Establish the existence of $\lim\limits_{x\to 1} f(x)$.

10. Find $\lim\limits_{x\to 2} f(x)$ if it exists, where

$$f(x)=\begin{cases}x-1, & x<2\\ 1, & x=2\\ x+1, & x>2\end{cases}$$

20.5 FINDING LIMITS OF SOME OF THE IMPORTANT FUNCTIONS

(i) Prove that $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$ where n is a positive integer.

Proof : $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = \lim\limits_{h\to 0}\dfrac{(a+h)^n - a^n}{a+h-a}$

$$= \lim_{h\to 0}\frac{\left(a^n + n\,a^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \dots + h^n\right) - a^n}{h}$$

$$= \lim_{h\to 0}\frac{h\left(n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \dots + h^{n-1}\right)}{h}$$

$$= \lim_{h\to 0}\left[n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \dots + h^{n-1}\right]$$

$$= n\,a^{n-1} + 0 + 0 + \dots + 0$$

$$= n\,a^{n-1}$$

$\therefore \qquad \lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = n\cdot a^{n-1}$

Note : However, the result is true for all n

(ii) Prove that (a) $\lim\limits_{x\to 0}\sin x = 0$ and (b) $\lim\limits_{x\to 0}\cos x = 1$

Proof : Consider a unit circle with centre B, in which $\angle C$ is a right angle and $\angle ABC = x$ radians.

Now sin x = A C and cos x = BC

As x decreases, A goes on coming nearer and nearer to C.

i.e., when $x \to 0, A \to C$

or when $x \to 0, AC \to 0$

and $BC \to AB$, i.e., $BC \to 1$

$\therefore$ When $x \to 0$ $\sin x \to 0$ and $\cos x \to 1$

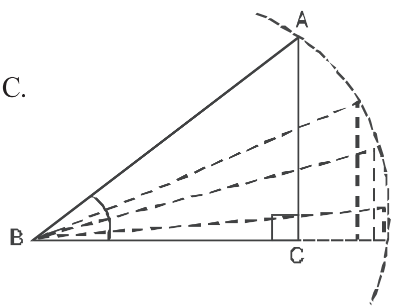


Fig. 20.3

Thus we have

$$\lim_{x\to 0}\sin x = 0 \text{ and } \lim_{x\to 0}\cos x = 1$$

(iii) Prove that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$

Proof : Draw a circle of radius 1 unit and with centre at the origin *O*. Let B (1,0) be a point on the circle. Let A be any other point on the circle. Draw $AC \perp OX$.

Let $\angle AOX = x$ radians, where $0<x<\frac{\pi}{2}$

Draw a tangent to the circle at B meeting OA produced at D. Then $BD \perp OX$.

Area of $\Delta AOC <$ area of sector $OBA <$ area of ΔOBD.

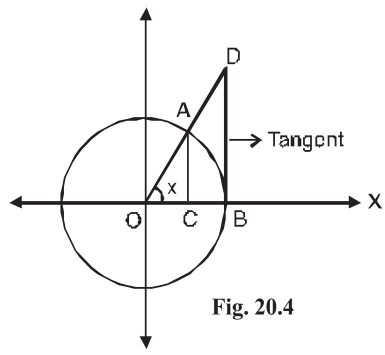


Fig. 20.4

or $\frac{1}{2} OC \times AC < \frac{1}{2} x(1)^2 < \frac{1}{2} OB \times BD$

$\left[\because \text{area of triangle} = \frac{1}{2} \text{base} \times \text{height and area of sector} = \frac{1}{2}\theta r^2\right]$

$\therefore \quad \frac{1}{2}\cos x \sin x < \frac{1}{2}x < \frac{1}{2} \cdot 1 \cdot \tan x$

$\left[\because \cos x = \frac{OC}{OA}, \sin x = \frac{AC}{OA} \text{ and } \tan x = \frac{BD}{OB}, OA = 1 = OB\right]$

i.e., $\quad \cos x < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$ [Dividing throughout by $\frac{1}{2}\sin x$]

or $\quad \cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$

or $\quad \frac{1}{\cos x} > \frac{\sin x}{x} < \cos x$

i.e., $\quad \cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$

Taking limit as $x \to 0$, we get

$$\lim_{x\to 0} \cos x < \lim_{x\to 0} \frac{\sin x}{x} < \lim_{x\to 0} \frac{1}{\cos x}$$

or $\quad 1 < \lim_{x\to 0} \frac{\sin x}{x} < 1$ $\left[\because \lim_{x\to 0} \cos x = 1 \text{ and } \lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1\right]$

Thus, $\quad \lim_{x\to 0} \frac{\sin x}{x} = 1$

Note : In the above results, it should be kept in mind that the angle x must be expressed in radians.

(iv) Prove that $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

Proof : By Binomial theorem, when $|x| < 1$, we get

$$(1+x)^{\frac{1}{x}} = \left[1 + \frac{1}{x}\cdot x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2 + + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3 + ...\infty\right]$$

$$= \left[1 + 1 + \frac{(1-x)}{2!} + \frac{(1-x)(1-2x)}{3!} + ...\infty\right]$$

$$\therefore \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0}\left[1 + 1 + \frac{1-x}{2!} + \frac{(1-x)(1-2x)}{3!} + ...\infty\right]$$

$$= \left[1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ...\infty\right]$$

$= e$ (By definition)

Thus $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

(v) Prove that

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{x\to 0}\frac{1}{x}\log(1+x) = \lim_{x\to 0}\log(1+x)^{1/x}$$

$$= \log e \qquad \left(\text{Using } \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e\right)$$

$$= 1$$

(vi) Prove that $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$

Proof : We know that $e^x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)$

$$\therefore \qquad e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...-1\right)$$

$$= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)$$

$$\therefore \qquad \frac{e^x - 1}{x} = \frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)}{x} \qquad \text{[Dividing throughout by x]}$$

$$= \frac{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + ...\right)}{x}$$

$$= \left(1+\frac{x}{2!}+\frac{x^2}{3!}+..........\right)$$

$$\therefore \quad \lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+..........\right)$$

$$= 1+0+0+........ = 1$$

Thus, $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}=1$

Example 20.9 Find the value of $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}}{x}$

Solution : We know that

$$\lim_{x\to 0}\frac{e^x-1}{x}=1 \qquad .....(i)$$

$\therefore$ Putting $x=-x$ in (i), we get

$$\lim_{x\to 0}\frac{e^{-x}-1}{-x}=1 \qquad .....(ii)$$

Given limit can be written as

$$\lim_{x\to 0}\frac{e^x-1+1-e^{-x}}{x} \qquad \text{[Adding (i) and (ii)]}$$

$$= \lim_{x\to 0}\left[\frac{e^x-1}{x}+\frac{1-e^{-x}}{x}\right]$$

$$= \lim_{x\to 0}\left[\frac{e^x-1}{x}+\frac{e^{-x}-1}{-x}\right]$$

$$= \lim_{x\to 0}\frac{e^x-1}{x}+\lim_{x\to 0}\frac{e^{-x}-1}{-x}$$

$$= 1+1 \qquad \text{[Using (i) and (ii)]}$$

$$= 2$$

Thus $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}}{x}=2$

Example 20.10 Evaluate : $\displaystyle\lim_{x\to 1}\frac{e^x-e}{x-1}$.

Solution : Put x = 1 + h, where $h\to 0$

$$\lim_{x\to 1}\frac{e^x-e}{x-1} = \lim_{h\to 0}\frac{e^{1+h}-e}{h}$$

$$= \lim_{h\to 0} \frac{e^1 \cdot e^h - e}{h}$$

$$= \lim_{h\to 0} \frac{e(e^h - 1)}{h}$$

$$= e\lim_{h\to 0} \frac{e^h - 1}{h}$$

$$= e \times 1 = e.$$

Thus $\quad \lim_{x\to 1} \frac{e^x - e}{x - 1} = e$

Example 20.11 Evaluate : $\lim_{x\to 0} \frac{\sin 3x}{x}$.

Solution : $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot 3$ [Multiplying and dividing by 3]

$$= 3 \lim_{3x\to 0} \frac{\sin 3x}{3x} \qquad [\because \text{ when } x \to 0, 3x \to 0]$$

$$= 3.1 \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

$$= 3$$

Thus, $\quad \lim_{x\to 0} \frac{\sin 3x}{x} = 3$

Example 20.12 Evaluate $\lim_{x\to 0} \frac{1 - \cos x}{2x^2}$.

Solution : $\lim_{x\to 0} \frac{1 - \cos x}{2x^2} = \lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{2x^2}$ $\quad \left[\begin{array}{l} \because \cos 2x = 1 - 2\sin^2 x, \\ \therefore 1 - \cos 2x = 2\sin^2 x \\ \text{or } 1 - \cos x = 2\sin^2 \frac{x}{2} \end{array}\right]$

$$= \lim_{x\to 0} \left(\frac{\sin \frac{x}{2}}{2 \times \frac{x}{2}} \right)^2 \qquad \text{[Multiplying and dividing the denominator by 2]}$$

$$= \frac{1}{4} \lim_{\frac{x}{2}\to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$

Notes

$$=\frac{1}{4}\times 1=\frac{1}{4}$$

$$\therefore \qquad \lim_{x\to 0}\frac{1-\cos x}{2x^2}=\frac{1}{4}$$

Example 20.13 Evaluate : $\lim_{\theta\to 0}\frac{1-\cos 4\theta}{1-\cos 6\theta}$

Solution : $\lim_{\theta\to 0}\frac{1-\cos 4\theta}{1-\cos 6\theta}=\lim_{\theta\to 0}\frac{2\sin^2 2\theta}{2\sin^2 3\theta}$

$$=\lim_{\theta\to 0}\left(\left(\frac{\sin 2\theta}{2\theta}\times 2\theta\right)^2\left(\frac{3\theta}{\sin 3\theta}\times\frac{1}{3\theta}\right)^2\right)$$

$$=\lim_{\theta\to 0}\left(\frac{\sin 2\theta}{2\theta}\right)^2\left(\frac{3\theta}{\sin 3\theta}\right)\frac{4\theta^2}{9\theta^2}$$

$$=\left(\frac{4}{9}\right)\lim_{2\theta\to 0}\left(\frac{\sin 2\theta}{2\theta}\right)^2\lim_{3\theta\to 0}\left(\frac{3\theta}{\sin 3\theta}\right)$$

$$=\frac{4}{9}\times 1\times 1=\frac{4}{9}$$

Example 20.14 Find the value of $\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}$.

Solution : Put $x=\frac{\pi}{2}+h$ $\qquad \because$ when $x\to\frac{\pi}{2}, h\to 0$

$$\therefore \qquad 2x=\pi+2h$$

$$\therefore \qquad \lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}=\lim_{h\to 0}\frac{1+\cos 2\left(\frac{\pi}{2}+h\right)}{[\pi-(\pi+2h)]^2}$$

$$=\lim_{h\to 0}\frac{1+\cos(\pi+2h)}{4h^2}$$

$$=\lim_{h\to 0}\frac{1-\cos 2h}{4h^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 h}{4h^2}$$

$$=\frac{1}{2}\lim_{h\to 0}\left(\frac{\sin h}{h}\right)^2$$

$$= \frac{1}{2} \times 1 = \frac{1}{2}$$

$$\therefore \qquad \lim_{x \to \frac{\pi}{2}} \frac{1+\cos 2x}{(\pi - 2x)^2} = \frac{1}{2}$$

Example 20.15 Evaluate $\lim_{x \to 0} \frac{\sin ax}{\tan bx}$

Solution : $\lim_{x \to 0} \frac{\sin ax}{\tan bx} = \lim_{x \to 0} \frac{\frac{\sin ax}{ax} \times a}{\frac{\tan bx}{bx} \times b}$

$$= \frac{a}{b} \frac{\lim_{x \to 0} \frac{\sin ax}{ax}}{\lim_{x \to 0} \frac{\tan bx}{bx}}$$

$$= \frac{a}{b} \cdot \frac{1}{1}$$

$$= \frac{a}{b}$$

$$\therefore \qquad \lim_{x \to 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$$

CHECK YOUR PROGRESS 20.2

1. Evaluate each of the following :

(a) $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$ (b) $\lim_{x \to 0} \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

2. Find the value of each of the following :

(a) $\lim_{x \to 1} \frac{e^{-x} - e^{-1}}{x - 1}$ (b) $\lim_{x \to 1} \frac{e - e^{x}}{x - 1}$

3. Evaluate the following :

(a) $\lim_{x \to 0} \frac{\sin 4x}{2x}$ (b) $\lim_{x \to 0} \frac{\sin x^2}{5x^2}$ (c) $\lim_{x \to 0} \frac{\sin x^2}{x}$

(d) $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$

4. Evaluate each of the following :

(a) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (b) $\lim_{x\to 0}\frac{1-\cos 8x}{x}$ (c) $\lim_{x\to 0}\frac{\sin 2x(1-\cos 2x)}{x^3}$

(d) $\lim_{x\to 0}\frac{1-\cos 2x}{3\tan^2 x}$

5. Find the values of the following :

(a) $\lim_{x\to 0}\frac{1-\cos ax}{1-\cos bx}$ (b) $\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}$ (c) $\lim_{x\to 0}\frac{\operatorname{cosec} x-\cot x}{x}$

6. Evaluate each of the following :

(a) $\lim_{x\to\pi}\frac{\sin x}{\pi-x}$ (b) $\lim_{x\to 1}\frac{\cos\frac{\pi}{2}x}{1-x}$ (c) $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)$

7. Evaluate the following :

(a) $\lim_{x\to 0}\frac{\sin 5x}{\tan 3x}$ (b) $\lim_{\theta\to 0}\frac{\tan 7\theta}{\sin 4\theta}$ (c) $\lim_{x\to 0}\frac{\sin 2x+\tan 3x}{4x-\tan 5x}$

20.5 CONTINUITY OF A FUNCTION AT A POINT

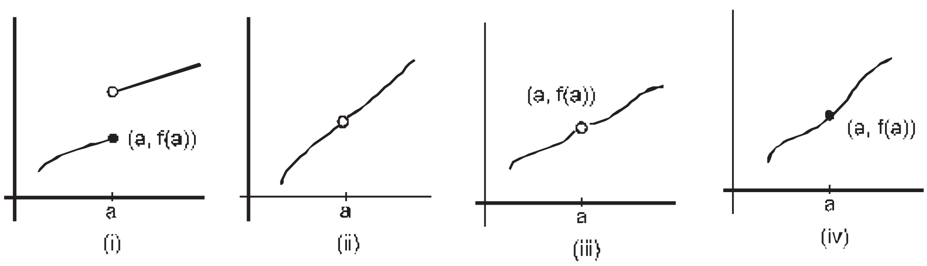


Fig. 20.5

Let us observe the above graphs of a function.

We can draw the graph (iv) without lifting the pencil but in case of graphs (i), (ii) and (iii), the pencil has to be lifted to draw the whole graph.

In case of (iv), we say that the function is continuous at x = a. In other three cases, the function is not continuous at x = a. i.e., they are discontinuous at x = a.

In case (i), the limit of the function does not exist at x = a.

In case (ii), the limit exists but the function is not defined at x = a.

In case (iii), the limit exists, but is not equal to value of the function at x = a.

In case (iv), the limit exists and is equal to value of the function at x = a.

Example 20.16 Examine the continuity of the function f (x) = $x - a$ at x = a.

Solution : $\lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} [(a+h) - a]$$

$$= 0 \qquad \text{.....(i)}$$

Also $\qquad f(a) = a - a = 0 \qquad$ .....(ii)

From (i) and (ii),

$$\lim_{x \to a} f(x) = f(a)$$

Thus f (x) is continuous at x = a.

Example 20.17 Show that f (x) = c is continuous.

Solution : The domain of constant function c is R.Let 'a' be any arbitrary real number.

$$\therefore \qquad \lim_{x \to a} f(x) = c \text{ and } f(a) = c$$

$$\therefore \qquad \lim_{x \to a} f(x) = f(a)$$

$\therefore$ f (x) is continuous at x = a. But 'a' is arbitrary. Hence f(x) = c is a constant function.

Example 20.18 Show that $f(x) = cx + d$ is a continuous function.

Solution : The domain of linear function $f(x) = cx + d$ is R; and let 'a' be any arbitrary real number.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to a} [c(a+h) + d]$$

$$= ca + d \qquad \text{.....(i)}$$

Also $\qquad f(a) = ca + d \qquad$ .....(ii)

From (i) and (ii), $\lim\limits_{x \to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a

and since a is any arbitrary , f (x) is a continuous function.

Example 20.19 Prove that $f(x) = \sin x$ is a continuous function.

Solution : Let f (x) = sin x

The domain of sin x is R. let 'a' be any arbitrary real number.

$$\therefore \qquad \lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to 0} \sin(a+h)$$

$$= \lim_{h \to 0} [\sin a . \cos h + \cos a . \sin h]$$

$$= \sin a \lim_{h \to 0} \cos h + \cos a \lim_{h \to 0} \sin h \qquad \left[\because \lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) \text{ where k is a constant}\right]$$

$$= \sin a \times 1 + \cos a \times 0 \qquad \left[\because \lim_{x \to 0} \sin x = 0 \text{ and } \lim_{x \to 0} \cos x = 1\right]$$

$$= \sin a \qquad .....(i)$$

Also $f(a) = \sin a$ .....(ii)

From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ sin x is continuous at x = a

$\because$ sin x is continuous at x = a and 'a' is an aribitary point.

Therefore, $f(x) = \sin x$ is continuous.

Example 20.20 Given that the function f (x) is continuous at x = 1. Find a when,

$f(x) = ax + 5$ and f (1) = 4.

Solution : $\lim_{x \to 1} f(x) = \lim_{h \to 0} (1+h)$

$$= \lim_{h \to 0} a(1+h) + 5$$

$$= a + 5 \qquad .....(i)$$

Also $f(1) = 4$ .....(ii)

As f (x) is continuous at x = 1, therefore $\lim_{x \to 1} f(x) = f(1)$

$\therefore$ From (1) and (ii), $a + 5 = 4$

or $a = 4 - 5$ or $a = -1$

Definition :

1. A function f (x) is said to be continuous in an open inteval]a,b[if it is continuous at every point of]a,b[*.
2. A function f (x) is said to be continuous in the closed interval [a,b] if it is continuous at every point of the open interval]a,b[and is continuous at the point a from the right and continuous at b from the left.

   i.e. $\lim_{x \to a^+} f(x) = f(a)$

   and $\lim_{x \to b^-} f(x) = f(b)$

---

* *In the open interval]a,b[we do not consider the end points a and b.*

Example 20.21 Prove that tan x is continuous when $0 \le x < \frac{\pi}{2}$

Solution : Let f (x) = tan x

The domain of tan x is $R - (2n+1)\frac{\pi}{2}, n \in I$

Let $a \in R - (2n+1)\frac{\pi}{2}$, be arbitrary.

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to 0} \tan(a+h)$$

$$= \lim_{h\to 0} \frac{\sin(a+h)}{\cos(a+h)}$$

$$= \lim_{h\to 0} \frac{\sin a \cos h + \cos a \sin h}{\cos a \cos h - \sin a \sin h}$$

$$= \frac{\sin a \lim_{h\to 0} \cos h + \cos a \lim_{h\to 0} \sin h}{\cos a \lim_{h\to 0} \cos h - \sin a \lim_{h\to 0} \sin h}$$

$$= \frac{\sin a \times 1 + \cos a \times 0}{\cos a \times 1 - \sin a \times 0}$$

$$= \frac{\sin a}{\cos a}$$

$= \tan a$ .....(i) $[\because \forall a \in \text{Domain of } \tan x, \cos a \ne 0]$

Also f (a) = tan a .....(ii)

$\therefore$ From (i) and (ii), $\lim_{h\to a} f(x) = f(a)$

$\therefore$ f(x) is continuous at x = a. But 'a' is arbitrary.

$\therefore$ tan x is continuous for all x in the interval $0 \le x < \frac{\pi}{2}$.

CHECK YOUR PROGRESS 20.3

1. Examine the continuity of the functions given below :

   (a) $f(x) = x - 5$ at $x = 2$ (b) $f(x) = 2x + 7$ at $x = 0$

   (c) $f(x) = \frac{5}{3}x + 7$ at $x = 3$ (d) $f(x) = px + q$ at $x = -q$

2. Show that f (x)=2a+3b is continuous, where a and b are constants.

3. Show that 5 x + 7 is a continuous function

4. (a) Show that cos x is a continuous function.

(b) Show that cot x is continuous at all points of its domain.

5. Find the value of the constants in the functions given below :

(a) $f(x) = px - 5 \text{ and } f(2) = 1$ such that f (x) is continuous at x = 2.

(b) $f(x) = a + 5x \text{ and } f(0) = 4$ such that f (x) is continuous at x = 0.

(c) $f(x) = 2x + 3b \text{ and } f(-2) = \frac{2}{3}$ such that f(x) is continuous at $x = -2$.

20.7 CONTINUITY OR OTHERWISE OF A FUNCTION AT A POINT

So far, we have considered only those functions which are continuous. Now we shall discuss some examples of functions which may or may not be continuous.

Example 20.22 Show that the function $f(x) = e^x$ is a continuous function.

Solution : Domain of e^x is R. Let $a \in R$. where 'a ' is arbitrary.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$, where h is a very small number.

$$= \lim_{h \to 0} e^{a+h}$$

$$= \lim_{h \to 0} e^a \cdot e^h$$

$$= e^a \lim_{h \to 0} e^h$$

$$= e^a \times 1 \qquad .....(i)$$

$$= e^a \qquad .....(ii)$$

Also $f(a) = e^a$

$\therefore$ From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a

Since a is arbitary, e^x is a continuous function.

Example 20.23 By means of graph discuss the continuity of the function $f(x) = \frac{x^2 - 1}{x - 1}$.

Solution : The grah of the function is shown in the adjoining figure. The function is discontinuous as there is a gap in the graph at x = 1.

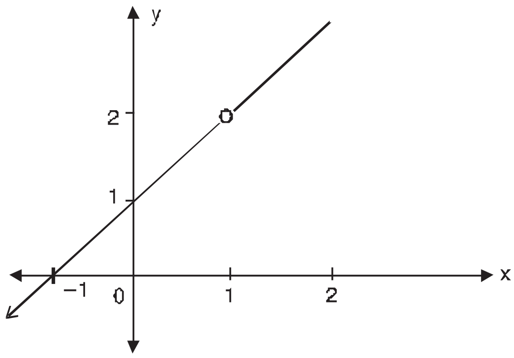


Fig. 20.6

CHECK YOUR PROGRESS 20.4

1. (a) Show that $f(x) = e^{5x}$ is a continuous function.

   (b) Show that $f(x) = e^{\frac{-2}{3}x}$ is a continuous function.

   (c) Show that $f(x) = e^{3x+2}$ is a continuous function.

   (d) Show that $f(x) = e^{-2x+5}$ is a continuous function.

2. By means of graph, examine the continuity of each of the following functions :

   (a) $f(x) = x+1$. (b) $f(x) = \dfrac{x+2}{x-2}$

   (c) $f(x) = \dfrac{x^2-9}{x+3}$ (d) $f(x) = \dfrac{x^2-16}{x-4}$

20.6 PROPERTIES OF CONTINUOUS FUNCTIONS

(i) Consider the function $f(x) = 4$. Graph of the function $f(x) = 4$ is shown in the Fig. 20.7. From the graph, we see that the function is continuous. In general, all constant functions are continuous.

(ii) If a function is continuous then the constant multiple of that function is also continuous.

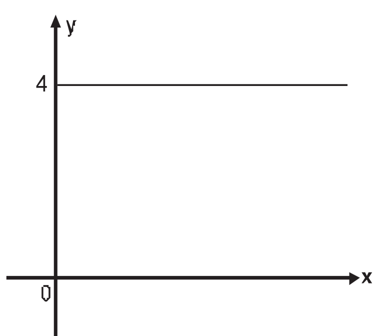


Fig. 20.7

Consider the function $f(x) = \frac{7}{2}x$. We know that x is a constant function. Let 'a' be an arbitrary real number.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

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$$= \lim_{h \to 0} \frac{7}{2}(a+h)$$

$$= \frac{7}{2}a \qquad .....(i)$$

Also $\quad f(a) = \frac{7}{2}a \qquad .....(ii)$

$\therefore$ From (i) and (ii),

$$\lim_{x \to a} f(x) = f(a)$$

$\therefore$ $f(x) = \frac{7}{2}x$ is continuous at $x = a$.

As $\frac{7}{2}$ is constant, and x is continuous function at $x = a$, $\frac{7}{2}x$ is also a continuous function at $x = a$.

(iii) Consider the function $f(x) = x^2 + 2x$. We know that the function x^2 and $2x$ are continuous.

Now $\quad \lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} \left[(a+h)^2 + 2(a+h) \right]$$

$$= \lim_{h \to 0} \left[a^2 + 2ah + h^2 + 2a + 2ah \right]$$

$$= a^2 + 2a \qquad .....(i)$$

Also $\quad f(a) = a^2 + 2a \qquad .....(ii)$

$\therefore$ From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ $f(x)$ is continuous at $x = a$.

Thus we can say that if x^2 and $2x$ are two continuous functions at $x = a$ then $(x^2 + 2x)$ is also continuous at $x = a$.

(iv) Consider the function $f(x) = (x^2 + 1)(x + 2)$. We know that $(x^2 + 1)$ and $(x + 2)$ are two continuous functions.

Also $\quad f(x) = (x^2 + 1)(x + 2)$

$$= x^3 + 2x^2 + x + 2$$

As $x^3, 2x^2, x$ and 2 are continuous functions, therefore.

$x^3 + 2x^2 + x + 2$ is also a continuous function.

$\therefore$ We can say that if $(x^2 + 1)$ and $(x+2)$ are two continuous functions then $(x^2 + 1)(x + 2)$ is also a continuous function.

(v) Consider the function $f(x)=\frac{x^2-4}{x+2}$ at x = 2. We know that $\left(x^2-4\right)$ is continuous at x = 2. Also (x + 2) is continuous at x = 2.

Again $$\lim_{x\to 2}\frac{x^2-4}{x+2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x+2}$$

$$=\lim_{x\to 2}(x-2)$$

$$=2-2=0$$

Also $$f(2)=\frac{(2)^2-4}{2+2}$$

$$=\frac{0}{4}=0$$

$\therefore$ $\lim_{x\to 2}f(x)=f(2)$. Thus f (x) is continuous at x = 2.

$\therefore$ If x^2-4 and $x+2$ are two continuous functions at $x=2$, then $\frac{x^2-4}{x+2}$ is also continuous.

(vi) Consider the function $f(x)=|x-2|$. The function can be written as

$$f(x)=\begin{cases}-(x-2), & x<2\\ (x-2), & x\geq 2\end{cases}$$

$$\lim_{x\to 2^-}f(x)=\lim_{h\to 0}f(2-h),\ h>0$$

$$=\lim_{h\to 0}\left[(2-h)-2\right]$$

$$=2-2=0$$

$$\lim_{x\to 2^+}f(x)=\lim_{h\to 0}f(2+h),\ h>0 \qquad \text{......(i)}$$

$$=\lim_{x\to 2}\left[(2+h)-2\right]$$

$$=2-2=0 \qquad \text{.....(ii)}$$

Also $$f(2)=(2-2)=0 \qquad \text{.....(iii)}$$

$\therefore$ From (i), (ii) and (iii), $\lim_{x\to 2}f(x)=f(2)$

Thus, $|x-2|$ is continuous at x = 2.

After considering the above results, we state below some properties of continuous functions.

If f (x) and g (x) are two functions which are continuous at a point x = a, then

(i) C f (x) is continuous at x = a, where C is a constant.

(ii) $f(x)\pm g(x)$ is continuous at x = a.

(iii) $f(x) \cdot g(x)$ is continuous at x = a.

(iv) f (x)/g (x) is continuous at x = a, provided g (a) $\neq$ 0.

(v) |f(x)| is continuous at x = a.

Note : Every constant function is continuous.

20.9 IMPORTANT RESULTS ON CONTINUITY

By using the properties mentioned above, we shall now discuss some important results on continuity.

(i) Consider the function $f(x) = px + q, x \in R$ (i)

The domain of this functions is the set of real numbers. Let a be any arbitary real number. Taking limit of both sides of (i), we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} (px+q) = pa+q \quad [= \text{value of } p\,x + q \text{ at } x = a.]$$

$\therefore$ px +q is continuous at x = a.

Similarly, if we consider $f(x) = 5x^2 + 2x + 3$, we can show that it is a continuous function.

In general $f(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n$

where $a_0, a_1, a_2 .....a_n$ are constants and n is a non-negative integer,

we can show that $a_0, a_1x, a_2x^2, ...a_nx^n$ are all continuos at a point x = c (where c is any real number) and by property (ii), their sum is also continuous at x = c.

$\therefore$ f (x) is continuous at any point c.

Hence every polynomial function is continuous at every point.

(ii) Consider a function $f(x) = \dfrac{(x+1)(x+3)}{(x-5)}$, $f(x)$ is not defined when $x - 5 = 0$ i.e, at x = 5.

Since (x + 1) and (x + 3) are both continuous, we can say that (x + 1) (x + 3) is also continuous. [Using property iii]

$\therefore$ Denominator of the function f (x), i.e., $(x - 5)$ is also continuous.

$\therefore$ Using the property (iv), we can say that the function $\dfrac{(x+1)(x+3)}{(x-5)}$ is continuous at all points except at x = 5.

In general if $f(x) = \dfrac{p(x)}{q(x)}$, where p (x) and q (x) are polynomial functions and q (x) $\neq$ 0, then f (x) is continuous if p (x) and q (x) both are continuous.

Example 20.24 Examine the continuity of the following function at x = 2.

$$f(x) = \begin{cases} 3x - 2 & \text{for } x < 2 \\ x + 2 & \text{for } x \geq 2 \end{cases}$$

Solution: Since f(x) is defined as the polynomial function 3x–2 on the left hand side of the point *x* = 2 and by another polynomial function x +2 on the right hand side of x = 2, we shall find the left hand limit and right hand limit of the function at x = 2 separately.

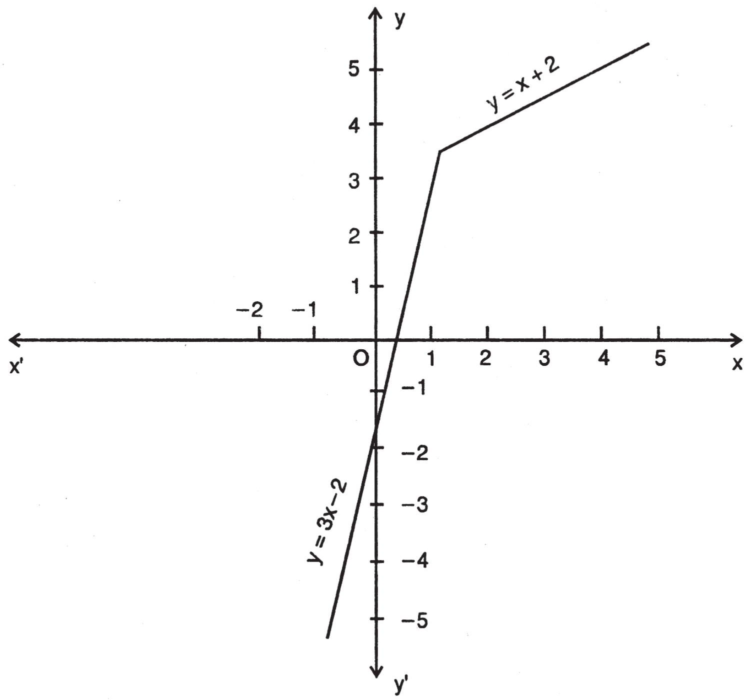

Fig. 20.8

$$\text{Left hand limit} = \lim_{x\to 2^-} f(x)$$

$$= \lim_{x\to 2}(3x-2)$$

$$= 3 \times 2 - 2 = 4$$

Right hand limit at x = 2;

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(x+2) = 4$$

Since the left hand lirmt and the right hand limit at x = 2 are equal, the limit 0 the function f (x) exists at x = 2 and is equal to 4 i.e., $\lim_{x\to 2} f(x) = 4$.

Also f(x) is defined by (x +2) at x = 2

$$f(2) = 2 + 2 = 4$$

Thus $\lim_{x\to 2} f(x) = 2$

Hence (x) is continuous at x = 2.

Example 20.25

(i) Draw the graph of $f(x) = |x|$.

(ii) Discusss the continuity of f(x) at x = 0.

Solution: We know that for $x \geq 0$, $|x| = x$ and for $x < 0$, $|x| = -x$. Hence f(x) can be written as.

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

(i) The graph of the function is given in Fig 20.9

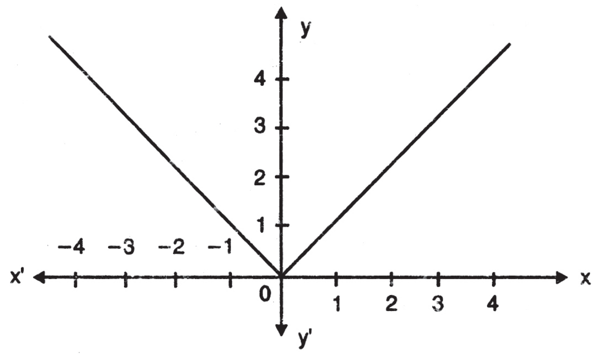

Fig. 20.9

(ii) Left hand limit $= \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0} (-x) = 0$$

Right hand limit $= \lim_{x \to 0^+} f(x)$

$$= \lim_{x \to 0} x = 0$$

Thus, $\lim_{x \to 0} f(x) = 0$

A1so, $f(x) = 0$

$\therefore$ $\lim_{x \to 0} f(x) = f(0)$

Hence the function f (x) is continuous at x = 0.

Example 20.26 Examine the continuity of $f(x) = |x - b|$ at x = b.

Solution: We have $f(x) = |x - b|$. This function can be written as

$$f(x) = \begin{cases} -(x-b), & x < b \\ (x-b), & x \geq b \end{cases}$$

Left hand limit $= \lim_{x \to b^-} f(x) = \lim_{h \to 0} f(b-h)$

$= \lim_{h \to 0} [-(b-h-b)]$

$= \lim_{h \to 0} h = 0$ ...(i)

Righthandlimit $= \lim_{x \to b^+} f(x) = \lim_{h \to 0} f(b+h)$

$= \lim_{h \to 0} f(b+h)$

$= \lim_{h \to 0} h = 0$ ...(ii)

Also, $f(b) = b - b = 0$

From (i), (ii) and (iii), $\lim_{x \to b} f(x) = f(b)$ ...(iii)

Thus, f(x) is continuous at x = b.

Example 20.27 If $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

find whether f(x) is continuous at x = 0 or not.

Solution: Here $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

Left hand limit $= \lim_{x \to 0^-} \dfrac{\sin 2x}{x}$

$= \lim_{h \to 0} \dfrac{\sin 2(0-h)}{0-h}$

$= \lim_{h \to 0} \dfrac{-\sin 2h}{-h}$

$= \lim_{h \to 0} \left(\dfrac{\sin 2h}{h} \times \dfrac{2}{1} \right)$

$= 1 \times 2 = 2$ ...(i)

Right hand limit $= \lim_{x \to 0^+} \dfrac{\sin 2x}{x}$

$= \lim_{h \to 0} \dfrac{\sin 2(0+h)}{0+h}$

$= \lim_{h \to 0} \dfrac{\sin 2h}{2h} \times \dfrac{2}{1}$

$= 1 \times 2 = 2$

Also $\quad f(0) = 2$

From (i) to (iii),

$$\lim_{x\to 0} f(x) = 2 = f(0)$$

Hence f(x) is continuous at x = 0.

Example 20.28 If $f(x) = \dfrac{x^2-1}{x-1}$ for $x \neq 1$ and f(x) = 2 when x = 1, show that the function f(x) is continuous at x = 1.

Solution : Here $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 2 \quad , & x = 1 \end{cases}$

Left hand limit $\lim\limits_{x\to 1^-} f(x)$

$$= \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0} \frac{(1-h)^2-1}{(1-h)-1}$$

$$= \lim_{h\to 0} \frac{1-2h+h^2-1}{-h}$$

$$= \lim_{h\to 0} \frac{h(h-2)}{-h}$$

$$= \lim_{h\to 0} -(h-2)$$

$$= 2$$

Right hand limit $= \lim\limits_{x\to 1^+} f(x)$ ...(i)

$$= \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0} \frac{(1+h)^2-1}{(1+h)-1}$$

$$= \lim_{h\to 0} \frac{1+2h+h^2-1}{h}$$

$$= \lim_{h\to 0} \frac{h(h+2)}{h}$$

$= 2$ ...(ii)

Also $\quad f(1) = 2$ (Given) ...(iii)

$\therefore$ From (i) to (iii),

$$\lim_{x \to 1} f(x) = f(1)$$

Thus f(x) is continuous at x = 1.

Example 20.29 Find whether f (x) is continuous at x = 0 or not, where

$$f(x) = \begin{cases} \dfrac{x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

Solution : $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{x}{x}$

$$= \lim_{x \to 0} 1 = 1$$

and $\qquad f(0) = 2$

$\therefore \qquad \lim_{x \to 0} f(x) \neq f(0)$

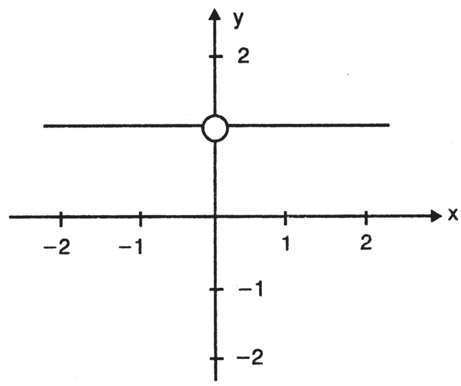

Fig. 20.10

Hence f (x) is not continuous at x = 0. The graph of the function is given in Fig. 20.10.

Clearly, the point (0,1) does not lie on the graph. Therefore, the function is discontinuous at x = 0.

Signum Function : The function f (x) = sgn(x) (read as signum x) is defined as

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Find the left hand limit and right hand limit of the function from its graph given below:

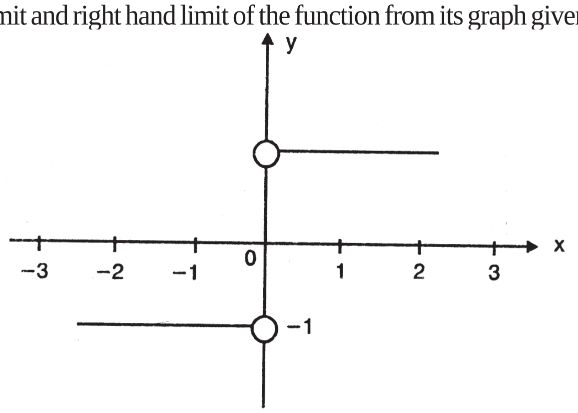

Fig. 20.11

From the graph, we see that as $x \to 0^+$, $f(x) \to 1$ and as $(x) \to 0^-$, $f(x) \to -1$

Hence, $\lim_{x \to 0^+} f(x) = 1, \lim_{x \to 0^-} f(x) = -1$

As these limits are not equal, $\lim_{x \to 0} f(x)$ does not exist. Hence f(x) is discontinuous at x =0.

Greatest Integer Function: Let us consider the function f(x) = [x] where [x] denotes the greatest integerles than or equal to x. Find whether f (x) is continuous at

(i) $x = \frac{1}{2}$ (ii) x =1

To solve this, let us take some arbitrary values ofx say 1.3, 0.2, –0.2 ..... By the defmitionof greatest integer function,

In general :

for $-3 \le x < -2$, $[x] = -3$

for $-2 \le x < -1$, $[x] = -2$

for $-1 \le x < 0$, $[x] = -1$

for $0 \le x < 1$, $[x] = 0$

for $1 \le x < 2$, $[x] = 1$ and so on.

The graph of the function f(x) = [x] is given in Fig. 20.12

(i) From graph

$$\lim_{x \to \frac{1}{2}^-} f(x) = 0, \quad \lim_{x \to \frac{1}{2}^+} f(x) = 0,$$

$$\therefore \quad \lim_{x \to \frac{1}{2}} f(x) = 0$$

Also $f\left(\frac{1}{2}\right) = [0.5] = 0$

Thus $\lim_{x \to \frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)$

Hence f(x) is continuous at

$$x = \frac{1}{2}$$

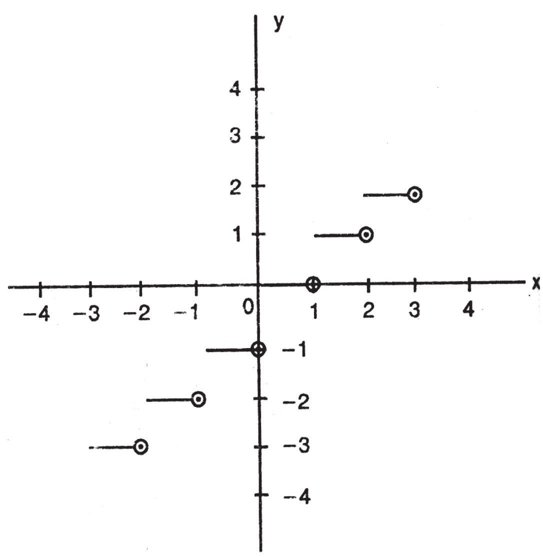

Fig. 20.12

(ii) $\lim_{x \to 1^-} f(x) = 0, \lim_{x \to 1^+} f(x) = 1$

Thus $\lim_{x \to 1} f(x)$, does not exist.

Hence, f (x) is discontinuous at x = 1.

Note : The function f(x) = [x] is also known as Step Function.

Example 20.30 At what points is the function $\frac{x-1}{(x+4)(x-5)}$ continuous ?

Solution: Here $f(x) = \dfrac{x-1}{(x+4)(x-5)}$

The function in the numerator i.e., x –1 is continuous. lne function in the demoninator is (x + 4) (x – 5) which is also continuous.

But f(x) is notdefmed attheyoints – 4 and 5.

$\therefore$ The function f(x) is continuous at all points except – 4 and 5 at which it is not defined.

In other words, f (x) is continuous at all points of its domain.

Example 20.31 Find the points of discountinuity of the function $f(x) = \dfrac{x^2+2x+5}{x^2-8x+12}$.

Solution: Here f (x) is a rational function.

Denominator $= x^2 - 8x + 12$

$= (x - 2)(x - 6)$ is zero at x = 2 and x = 6

$\therefore$ f(x) is not defined at x = 2 and x = 6.

Also we know that a rational function is continuous at all points of its domain.

$\therefore$ f(x) is continuous for all values ofx except x = 2 and x = 6.

CHECK YOUR PROGRESS 20.5

1. (a) If f(x) = 2x + 1, when x1 and f(x)=3 when x = 1, show that the function f(x) continuous at x = 1.

   (b) If $f(x) = \begin{cases} 4x+3, & x \neq 2 \\ 3x+5, & x = 2 \end{cases}$, find whether the function f is continuous at x = 2.

   (c) Determine whether f (x) is continuous at x = 2, where

   $$f(x) = \begin{cases} 4x+3, & x \leq 2 \\ 8-x, & x > 2 \end{cases}$$

   (d) Examine the continuity of f (x) at x = 1, where

   $$f(x) = \begin{cases} x^2, & x \leq 1 \\ x+5, & x > 1 \end{cases}$$

   (e) Determine the values of k so that the function

   $$f(x) = \begin{cases} kx^2, & x \leq 2 \\ 3, & x > 2 \end{cases}$$ is continuous at x = 2.

2. Examine the continuity of the following functions :

   (a) f(x) = |x–2| at x = 2 (b) f(x) = |x + 5| at x = –5

   (c) f(x) =| a – x| at x = a

(d) $f(x)=\begin{cases}\dfrac{|x-2|}{x-2}, & x\neq 2\\ 1, & x=2\end{cases}$ at $x = 2$

(e) $f(x)=\begin{cases}\dfrac{|x-a|}{x-a}, & x\neq a\\ 1, & x=a\end{cases}$ at $x = a$

3. (a) If $f(x)=\begin{cases}\sin 4x, & x\neq 0\\ 2, & x=0\end{cases}$ at $x = 0$

(b) If $f(x)=\begin{cases}\dfrac{\sin 7x}{x}, & x\neq 0\\ 7, & x=0\end{cases}$ at $x = 0$

(c) For what value of a is the function

$f(x)=\begin{cases}\dfrac{\sin 5x}{3x}, & x\neq 0\\ a, & x=0\end{cases}$ continuous at $x = 0$?

4. (a) Show that the function f (x) is continuous at x = 2, where

$$f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2}, & \text{for } x\neq 0\\ 3, & \text{for } x=2\end{cases}$$

(b) Test the continuity of the function f(x) at x = 1, where

$$f(x)=\begin{cases}\dfrac{x^2-4x+3}{x-1}, & \text{for } x\neq 1\\ -2, & \text{for } x=1\end{cases}$$

(c) For what value of k is the following function continuous at x = 1 ?

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & \text{when } x\neq 1\\ k, & \text{when } x=1\end{cases}$$

(d) Discuss the continuity of the function f(x) at x = 2, when

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & \text{for } x\neq 2\\ 7, & x=2\end{cases}$$

5. (a) If $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$, find hether f is continuous at x = 0.

(b) Test the continuity of the function f (x) at the origin.

where $$f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 1, & x=0\end{cases}$$

6. Find whether the function f (x)=[x] is continuous at

(a) $x=\frac{4}{3}$ (b) $x=3$ (c) $x=-1$ (d) $x=\frac{2}{3}$

7. At what points is the function f (x) continuous in each of the following cases ?

(a) $f(x)=\dfrac{x+2}{(x-1)(x-4)}$ (b) $f(x)=\dfrac{x-5}{(x+2)(x-3)}$ (c) $f(x)=\dfrac{x-3}{x^2+5x-6}$

(d) $f(x)=\dfrac{x^2+2x+5}{x^2-8x+16}$

LET US SUM UP

- If a function f(x) approaches l when x approches a, we say that l is the limit of f (x). Symbolically, it is written as

$$\lim_{x\to a} f(x)=\ell$$

- If $\lim_{x\to a} f(x)=\ell$ and $\lim_{x\to a} g(x)=m$, then

(i) $\lim_{x\to a} kf(x)=k\lim_{x\to a} f(x)=k\ell$

(ii) $\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)=\ell\pm m$

(iii) $\lim_{x\to a}\left[f(x)g(x)\right]=\lim_{x\to a} f(x)\lim_{x\to a} g(x)=\ell m$

(iv) $\lim_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\dfrac{\ell}{m}$, provided $\lim_{x\to a} g(x)\neq 0$

- **LIMIT OF IMPORTANT FUNCTIONS**

(i) $\lim_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$ (ii) $\lim_{x\to 0}\sin x=0$

(iii) $\lim_{x\to 0}\cos x=1$ (iv) $\lim_{x\to 0}\dfrac{\sin x}{x}=1$

(v) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$ (vi) $\lim_{x\to 0}\dfrac{\log(1+x)}{x}=1$

(vii) $\lim_{x\to 0}\dfrac{e^x-1}{x}=1$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

Evaluate the following limits :

1. $\lim\limits_{x \to 1} 5$

2. $\lim\limits_{x \to 0} \sqrt{2}$

3. $\lim\limits_{x \to 1} \dfrac{4x^5 + 9x + 7}{3x^6 + x^3 + 1}$

4. $\lim\limits_{x \to 2} \dfrac{x^2 + 2x}{x^3 + x^2 - 2x}$

5. $\lim\limits_{x \to 0} \dfrac{(x+k)^4 - x^4}{k(k+2x)}$

6. $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$

7. $\lim\limits_{x \to -1} \left[\dfrac{1}{x+1} + \dfrac{2}{x^2 - 1}\right]$

8. $\lim\limits_{x \to 1} \dfrac{(2x-3)\sqrt{x} - 1}{(2x+3)(x-1)}$

9. $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{\sqrt{x+2} - \sqrt{3x-2}}$

10. $\lim\limits_{x \to 1} \left[\dfrac{1}{x-1} - \dfrac{2}{x^2 - 1}\right]$

11. $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x}$

12. $\lim\limits_{x \to a} \dfrac{x^2 - (a+1)x + a^2}{x^2 - a^2}$

Find the left hand and right hand limits of the following functions :

13. $f(x) = \begin{cases} -2x + 3 & \text{if } x \le 1 \\ 3x - 5 & \text{if } x > 1 \end{cases}$ as $x \to 1$

14. $f(x) = \dfrac{x^2 - 1}{|x+1|}$ as $x \to 1$

Evaluate the following limits :

15. $\lim\limits_{x \to 1^-} \dfrac{|x+1|}{x+1}$

16. $\lim\limits_{x \to 2^+} \dfrac{|x-2|}{x-2}$

17. $\lim\limits_{x \to 2^-} \dfrac{x-2}{|x-2|}$

18. If $f(x) = \dfrac{(x+2)^2 - 4}{x}$, prove that $\lim\limits_{x \to 0} f(x) = 4$ though f (0) is not defined.

19. Find k so that $\lim\limits_{x \to 2} f(x)$ may exist where $f(x) = \begin{cases} 5x + 2, x \le 2 \\ 2x + k, x > 2 \end{cases}$

20. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 7x}{2x}$

21. Evauate $\lim\limits_{x \to 0} \left[\dfrac{e^x + e^{-x} - 2}{x^2}\right]$

22. Evaluate $\lim\limits_{x\to 0}\dfrac{1-\cos 3x}{x^2}$ 23.Find the value of $\lim\limits_{x\to 0}\dfrac{\sin 2x+3x}{2x+\sin 3x}$

24. Evaluate $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$ 25. Evaluate $\lim\limits_{\theta\to 0}\dfrac{\sin 5\theta}{\tan 8\theta}$

Examine the continuity of the following :

26. $f(x)\begin{cases} 1+3x \text{ if } x>-1 \\ 2 \text{ if } x\le -1 \end{cases}$

at $x=-1$

27. $f(x)=\begin{cases} \dfrac{1}{x}-x, 0<x<\dfrac{1}{2} \\ \dfrac{1}{2}, x=\dfrac{1}{2} \\ \dfrac{3}{2}-x, \dfrac{1}{2}<x<1 \end{cases}$

at $x=\dfrac{1}{2}$

28. For what value of k, will the function

$$f(x)=\begin{cases} \dfrac{x^2-16}{x-4} \text{ if } x\ne 4 \\ k \text{ if } x=4 \end{cases}$$

be continuous at x = 4 ?

29. Determine the points of discontinuty, if any, of the following functions :

(a) $\dfrac{x^2+3}{x^2+x+1}$ (b) $\dfrac{4x^2+3x+5}{x^2-2x+1}$

(b) $\dfrac{x^2+x+1}{x^2-3x+1}$ (d) $f(x)=\begin{cases} x^4-16, & x\ne 2 \\ 16, & x=2 \end{cases}$

30. Show that the function $f(x)=\begin{cases} \dfrac{\sin x}{x}+\cos, & x\ne 0 \\ 2, & x=0 \end{cases}$ is continuous at $x=0$

31. Determine the value of 'a', so that the function f (x) defined by

$$f(x)=\begin{cases} \dfrac{a\cos x}{\pi-2x}, & x\ne\dfrac{\pi}{2} \\ 5, & x=\dfrac{\pi}{2} \end{cases}$$ is continuous.

MODULE - V
Calculus

Notes

ANSWERS

CHECK YOUR PROGRESS 20.1

1. (a) 17 (b) 7 (c) 0 (d) 2
(e)- 4 (f) 8

2. (a) 0 (b) $\frac{3}{2}$ (c) $-\frac{2}{11}$ (d) $\frac{q}{b}$ (e) 6
(f) -10 (g) 3 (h) 2

3. (a) 3 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{1}{2}$

4. (a) $\frac{1}{2}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{1}{2\sqrt{6}}$ (d) 2 (e) -1

5. (a) Does not exist (b) Does not exist

6. (a) 0 (b) $\frac{1}{4}$ (c) does not exist

7. (a) $1, -2$ (b) 1 (c) 19

8. $a = -2$

10. limit does not exist

CHECK YOUR PROGRESS 20.2

1. (a) 2 (b) $\frac{e^2 - 1}{e^2 + 1}$

2. (a) $-\frac{1}{e}$ (b) $-e$

3. (a) 2 (b) $\frac{1}{5}$ (c) 0 (d) $\frac{a}{b}$

4. (a) $\frac{1}{2}$ (b) 0 (c) 4 (d) $\frac{2}{3}$

5. (a) $\frac{a^2}{b^2}$ (b) 2 (c) $\frac{1}{2}$

6. (a) 1 (b) $\frac{\pi}{2}$ (c) 0

7. (a) $\frac{5}{3}$ (b) $\frac{7}{4}$ (c) -5

CHECK YOUR PROGRESS 20.3

1. (a) Continuous (b) Continuous
   (c) Continuous (d) Continuous

5. (a) p =3 (b) a = 4 (c) $b = \frac{14}{9}$

CHECK YOUR PROGRESS 20.4

2. (a) Continuous
   (b) Discontinuous at x = 2
   (c) Discontinuous at $x = -3$
   (d) Discontinuous at x = 4

CHECK YOUR PROGRESS 20.5

1. (b) Continuous (c) Discontinuous
   (d) Discontinuous (e) $k = \frac{3}{4}$

2 (a) Continuous (c) Continuous,
  (d) Discontinuous (e) Discontinuous

3 (a) Discontinuous (b) Continuous (c) $\frac{5}{3}$

4 (b) Continuous (c) k = 2
  (d) Discontinuous

5. (a) Discontinuous (b) Discontinuous

6 (a) Continuous (b) Discontinuous
  (c) Discontinuous (d) Continuous

7. (a) All real number except 1 and 4
   (b) All real numbers except −2 and 3
   (c) All real number except −6 and 1
   (d) All real numbers except 4

TERMINAL EXERCISE

1. 5 2. $\sqrt{2}$ 3. 4 4. $-\frac{1}{3}$

5. $2x^2$ 6. 1 7. $-\frac{1}{2}$ 8. $-\frac{1}{10}$

9. -8 10. $\frac{1}{2}$ 11. 1 12. $\frac{a-1}{2a}$

13. $1,-2$ 14. $-2,2$ 15. -1 16. 1

17. -1 19. $k=8$ 20. $\frac{7}{2}$ 21. 1

22. $\frac{9}{2}$ 23. 1 24. $\frac{2}{\pi}$ 25. $\frac{5}{8}$

26. Discontinuous 27. Discontinuous

28. $k=8$

29. (a) No (b) $x=1$

(c) $x=1,\ x=2$ (d) $x=2$

31. 10

21

DIFFERENTIATION

The differential calculus was introduced sometime during 1665 or 1666, when Isaac Newton first concieved the process we now know as differentiation (a mathematical process and it yields a result called derivative). Among the discoveries of Newton and Leibnitz are rules for finding derivatives of sums, products and quotients of composite functions together with many other results. In this lesson we define derivative of a function, give its geometrical and physical interpretations, discuss various laws of derivatives and introduce notion of second order derivative of a function.

OBJECTIVES

After studying this lesson, you will be able to :

- define and interpret geometrically the derivative of a function y = f(x) at x = a;
- prove that the derivative of a constant function f(x) = c, is zero;
- find the derivative of $f(x) = x^n, n \in Q$ from first principle and apply to find the derivatives of various functions;
- find the derivatives of the functions of the form cf(x), $[f(x) \pm g(x)]$ and polynomial functions;
- state and apply the results concerning derivatives of the product and quotient of two functions;
- state and apply the chain rule for the derivative of a function;
- find the derivative of algebraic functions (including rational functions); and
- find second order derivative of a function.

EXPECTED BACKGROUND KNOWLEDGE

- Binomial Theorem for any index
- Functions and their graphs
- Notion of limit of a function

21.1 DERIVATIVE OF A FUNCTION

Consider a function and a point say (5,25) on its graph. If x changes from 5 to 5.1, 5.01, 5.001..... etc., then correspondingly, y changes from 25 to 26.01, 25.1001, 25.010001,....A small change in x causes some small change in the value of y. We denote this change in the value of x by a symbol δx and the corresponding change caused in y by δy and call these respectively as an increment in x and increment in y, irrespective of sign of increment. The ratio $\frac{\delta x}{\delta y}$ of increment is termed as incrementary ratio. Here, observing the following table for $y = x^2$ at (5,25), we have for $\delta x = 0.1, 0.01, 0.001, 0.0001, ...$ $\delta y = 1.01, .1001, .010001, .00100001, ...$

x	5.1	5.01	5.001	5.0001
δx	.1	.01	.001	.0001
y	26.01	25.1001	25.010001	25.00100001
δy	1.01	.1001	.010001	.00100001
$\frac{\delta y}{\delta x}$	10.1	10.01	10.001	10.0001

We make the following observations from the above table :

(i) δy varies when δx varies.

(ii) $\delta y \to 0$ when $\delta x \to 0$.

(iii) The ratio $\frac{\delta y}{\delta x}$ tends to a number which is 10.

Hence, this example illustrates that $\delta y \to 0$ when $\delta x \to 0$ but $\frac{\delta y}{\delta x}$ tends to a finite number, not necessarily zero. The limit, $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ is equivalently represented by $\frac{dy}{dx}$. $\frac{dy}{dx}$ is called the derivative of y with respect to x and is read as differential coefficient of y with respect to x.

That is, $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = 10$ in the above example and note that while δx and δy are small numbers (increments), the ratio $\frac{\delta y}{\delta x}$ of these small numbers approaches a definite value 10.

In general, let us consider a function

$$y = f(x) \qquad \text{(i)}$$

To find its derivative, consider δx to be a small change in the value of x, so $x + \delta x$ will be the new value of x where f (x) is defined. There shall be a corresponding change in the value of y. Denoting this change by δy ; y + δy will be the resultant value of y, thus,

$$y + \delta y = f(x + \delta x) \qquad \text{...(ii)}$$

Subtracting (i) from (ii), we have,

$$(y + \delta y) - y = f(x + \delta x) - f(x)$$

or

$$\delta y = f(x + \delta x) - f(x) \qquad \text{...(iii)}$$

To find the rate of change, we divide (iii) by δx

$$\therefore \qquad \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x} \qquad \text{...(iv)}$$

Lastly, we consider the limit of the ratio $\frac{\delta y}{\delta x}$ as $\delta x \to 0$.

If
$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \qquad \text{...(v)}$$

is a finite quantity, then f(x) is called derivable and the limit is called derivative of f(x) with respect to (w.r.t.) x and is denoted by the symbol $f'(x)$ or by $\frac{d}{dx}$ of f(x)

i.e. $\frac{d}{dx} f(x)$ or $\frac{dy}{dx}$ (read as $\frac{d}{dx}$ of y).

Thus,
$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x)$$

Remarks

(1) The limiting process indicated by equation (v) is a mathematical operation. This mathematical process is known as differentiation and it yields a result called a derivative.

(2) A function whose derivative exists at a point is said to be derivable at that point.

(3) It may be verified that if f (x) is derivabale at a point x = a, then, it must be continuous at that point. However, the converse is not necessarily true.

(4) The symbols Δx and h are also used in place of δx i.e.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{or} \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(5) If y = f (x), then $\frac{dy}{dx}$ is also denoted by y_1 or y'.

21.2 VELOCITY AS LIMIT

Let a particle initially at rest at 0 moves along a straignt line OP., The distance s

Fig. 21.1

covered by it in reaching P is a function of time t, We may write distance

$$OP = s = f(t) \qquad \text{...(i)}$$

In the same way in reaching a point Q close to P covering PQ

i.e., δs is a fraction of time δt so that

$$OQ = OP + PQ$$

$= s + \delta s$

$= f(t + \delta t)$ ...(ii)

The average velocity of the particle in the interval δt is given by

$$= \frac{\text{Change in distance}}{\text{Change in time}}$$

$$= \frac{(s+\delta s) - s}{(t+\delta t) - t}, \qquad \text{[From (i) and (ii)]}$$

$$= \frac{f(t + \delta t) - f(t)}{\delta t}$$

(average rate at which distance is travelled in the interval δt).

Now we make δt smaller to obtain average velocity in smaller interval near P. The limit of average velocity as $\delta t \to 0$ is the instantaneous velocity of the particle at time t (at the point P).

$\therefore$ Velocity at time $t = \lim\limits_{\delta t \to 0} \dfrac{f(t + \delta t) - f(t)}{\delta t}$

It is denoted by $\dfrac{ds}{dt}$.

Thus, if f (t) gives the distance of a moving particle at time t, then the derivative of 'f' at $t = t_0$ represents the instantaneous speed of the particle at the point P i.e. at time $t = t_0$.

This is also referred to as the physical interpretation of a derivative of a function at a point.

Note : The derivative $\dfrac{dy}{dx}$ represents instantaneous rate of change of y w.r.t. x.

Example 21.1 The distance 's' meters travelled in time t seconds by a car is given by the relation

$$s = 3t^2$$

Find the velocity of car at time t =4 seconds.

Solution : Here, $f(t) = s = 3t^2$

$\therefore$ $f(t + \delta t) = s + \delta s = 3(t + \delta t)^2$

Velocity of car at any time $t = \lim\limits_{\delta t \to 0} \dfrac{f(t + \delta t) - f(t)}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{3(t + \delta t)^2 - 3t^2}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{3(t^2 + 2t.\delta t + \delta t^2) - 3t^2}{\delta t}$$

$$= \lim_{\delta t \to 0} (6t + 3\delta t)$$

$$= 6t$$

$\therefore$ Velocity of the car at t = 4 sec

$= (6 \times 4)$ m/sec

= 24 m/sec.

CHECK YOUR PROGRESS 21.1

1. Find the velocity of particles moving along a straight line for the given time-distance relations at the indicated values of time t :

(a) $s = 2 + 3t; t = \frac{1}{3}.$ (b) $s = 8t - 7; t = 4.$

(c) $s = t^2 + 3t; t = \frac{3}{2}.$ (d) $s = 7t^2 - 4t + 1; t = \frac{5}{2}.$

2. The distance s metres travelled in t seconds by a particle moving in a straight line is given by $s = t^4 - 18t^2$. Find its speed at t = 10 seconds.

3. A particle is moving along a horizontal line. Its distance s meters from a fixed point O at t seconds is given by $s = 10 - t^2 + t^3$. Determine its instantaneous speed at the end of 3 seconds.

21.3 GEOMETRICAL INTERPRETATION OF dy/dx

Let y = f (x) be a continuous function of x, draw its graph and denote it by APQB.

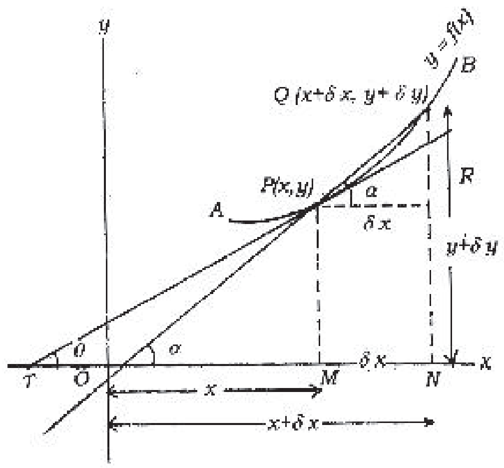

Fig. 21.2

Let P (x,y) be any point on the graph of y = f(x) or curve represented by y = f(x). Let $Q(x + \delta x, y + \delta y)$ be another point on the same curve in the neighbourhood of point P.

Draw PM and QN perpendiculars to x-axis and PR parallel to x-axis such that PR meets QN at R.Join QP and produce the secant line to any point S. Secant line QPS makes angle say α with the positive direction of x-axis. Draw PT tangent to the curve at the point P, making angle θ with the x-axis.

Now, In $\Delta QPR, \angle QPR = \alpha$

$$\tan\alpha = \frac{QR}{PR} = \frac{QN - RN}{MN} = \frac{QN - PM}{ON - OM} = \frac{(y+\delta y) - y}{(x+\delta x) - x} = \frac{\delta y}{\delta x} \qquad \text{(i)}$$

Now, let the point Q move along the curve towards P so that Q approaches nearer and nearer the point P.

Thus, when $Q \to P$, $\delta x \to 0$, $\delta y \to 0$, $\alpha \to 0, (\tan\alpha \to \tan\theta)$ and consequently, the secant QPS tends to coincide with the tangent PT.

From (i). $\tan\alpha = \dfrac{\delta y}{\delta x}$

In the limiting case, $\lim\limits_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \alpha \to \theta}} \tan\alpha = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

or $$\tan\theta = \frac{dy}{dx} \qquad \text{...(ii)}$$

Thus the derivative $\dfrac{dy}{dx}$ of the function $y = f(x)$ at any point P (x,y) on the curve represents the slope or gradient of the tangent at the point P.

This is called the geometrical interpretation of $\dfrac{dy}{dx}$.

It should be noted that $\dfrac{dy}{dx}$ has different values at different points of the curve.

Therefore, in order to find the gradient of the curve at a particular point, find $\dfrac{dy}{dx}$ from the equation of the curve $y = f(x)$ and substitute the coordinates of the point in $\dfrac{dy}{dx}$.

Corollary 1

If tangent to the curve at P is parallel to x-axis, then $\theta = 0°$ or $180°$, i.e., $\dfrac{dy}{dx} = \tan 0°$ or tan $180°$ i.e., $\dfrac{dy}{dx} = 0$.

That is tangent to the curve represented by $y = f(x)$ at P is parallel to x-axis.

Corollary 2

If tangent to the curve at P is perpendicular to x-axis, $\theta = 90°$ or $\dfrac{dy}{dx} = \tan 90° = \infty$.

That is, the tangent to the curve represented by $y = f(x)$ at P is parallel to y-axis.

21.4 DERIVATIVE OF CONSTANT FUNCTION

Statement : The derivative of a constant is zero.

Proof : Let y = c be a constant function. Then y = c can be written as

$$y = cx^0 \qquad [\because x^0 = 1] \qquad ..(i)$$

Let δx be a small increment in x. Corresponding to this increment, let δy be the increment in the value of y so that

$$y + \delta y = c(x + \delta x)^0 \qquad ..(ii)$$

Subtracting (i) from (ii),

$$(y + \delta y) - y = c(x + \delta x)^0 - cx^0, \qquad (\because x^0 = 1)$$

or $\quad \delta y = c - c \quad$ or $\quad \delta y = 0$

Dividing by $\quad \delta x, \quad \dfrac{\delta y}{\delta x} = \dfrac{0}{\delta x} \quad$ or $\quad \dfrac{\delta y}{\delta x} = 0$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 0 \quad \text{or} \quad \frac{dy}{dx} = 0$$

or $\quad \dfrac{dc}{dx} = 0 \qquad$ [y = c from (i)]

This proves that rate of change of constant quantity is zero. Therefore, derivative of a constant quantity is zero.

21.5 DERIVATIVE OF A FUNCTION FROM FIRST PRINCIPLE

Recalling the definition of derivative of a function at a point, we have the following working rule for finding the derivative of a function from first principle:

Step I. Write down the given function in the form of y = f (x) ....(i)

Step II. Let dx be an increment in x, δy be the corresponding increment in y so that

$$y + \delta y = f(x + \delta x) \qquad ....(ii)$$

Step III. Subtracting (i) from (ii), we get

$$\delta y = f(x + \delta x) - f(x) \qquad ..(iii)$$

Step IV. Dividing the result obtained in step (iii) by δx, we get,

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Step V. Proceeding to limit as $\delta x \to 0$.

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Note : The method of finding derivative of function from first principle is also called **delta** or **ab-ininitio** method.

Next, we find derivatives of some standard and simple functions by first principle.

21.6 DERIVATIVES OF THE FUNCTIONS FROM THE FIRST PRINCIPLE

$$\text{Let } y = x^n \qquad .....(i)$$

For a small increment δx in x, let the corresponding increment in y be δy.

Then $$y + \delta y = (x + \delta x)^n. \qquad ...(ii)$$

Subtracing (i) from (ii) we have,

$$(y + \delta y) - y = (x + \delta x)^n - x^n$$

$$\therefore \qquad \delta y = x^n \left(1 + \frac{\delta x}{x}\right)^n - x^n$$

$$= x^n \left[\left(1 + \frac{\delta x}{x}\right)^n - 1\right]$$

Since $\frac{\delta x}{x} < 1$, as δx is a small quantity compared to x, we can expand $\left(1 + \frac{\delta x}{x}\right)^n$ by Binomial theorem for any index.

Expanding $\left(1 + \frac{\delta x}{x}\right)^n$ by Binomial theorem, we have

$$\delta y = x^n \left[1 + n\left(\frac{\delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{\delta x}{x}\right)^3 + \ldots - 1\right]$$

$$= x^n (\delta x)\left[\frac{n}{x} + \frac{n(n-1)}{2}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = x^n\left[\frac{n}{x} + \frac{n(n-1)}{2!}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

Proceeding to limit when $\delta x \to 0$, $(\delta x)^2$ and higher powers of δx will also tend to zero.

$$\therefore \qquad \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^n\left[\frac{n}{x} + \frac{n(n-1)}{2!}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

$$\text{or} \qquad \operatorname*{lt}_{\delta x \to 0}\frac{\delta y}{\delta x} = \frac{dy}{dx} = x^n\left[\frac{n}{x} + 0 + 0 + \ldots\right]$$

$$\text{or} \qquad \frac{dy}{dx} = x^n \cdot \frac{n}{x} = nx^{n-1}$$

or $$\frac{d}{dx}(x^n) = nx^{n-1}, \qquad \left[\because\ y = x^n\right]$$

This is known as Newton's Power Formula or Power Rule

Note : We can apply the above formula to find derivative of functions like $x, x^2, x^3, ...$

i.e. when n = 1,2,3,...

e.g. $$\frac{d}{dx}x = \frac{d}{dx}x^1 = 1x^{1-1} = 1x^0 = 1.1 = 1$$

$$\frac{d}{dx}x^2 = 2x^{2-1} = 2x$$

$$\frac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2, \text{ and so on.}$$

Example 21.2 Find the derivative of each of the following :

(i) x^{10} (ii) x^{50} (iii) x^{91}

Solution :

(i) $$\frac{d}{dx}\left(x^{10}\right) = 10x^{10-1} = 10x^9$$

(ii) $$\frac{d}{dx}\left(x^{50}\right) = 50x^{50-1} = 50x^{49}$$

(iii) $$\frac{d}{dx}\left(x^{91}\right) = 91x^{91-1} = 91x^{90}$$

Remark : Newton's power formula is also applicable when n is a rational number.

i.e. say when n=$\frac{1}{2}$. For example,

Let $$y = x^{\frac{1}{2}} = \sqrt{x}$$

Then $$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2}\frac{1}{x^{1/2}} \quad \text{or} \quad \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

We shall now find the derivatives of some simple functions from definition or first principles.

Example 21.3 Find the derivative of x^2 from the first principles.

Solution : Let $$y = x^2 \qquad \text{(i)}$$

For a small increment δx in x let the corresponding increment in y be δy.

$$y + \delta y = (x + \delta x)^2 \qquad \text{(ii)}$$

Subtracting (i) from (ii), we have

$$(y+\delta y)-y=(x+\delta x)^2-x^2$$

or $$\delta y = x^2+2x(\delta x)+(\delta x)^2-x^2$$

or $$\delta y = 2x(\delta x)+(\delta x)^2$$

Divide by δx, we have

$$\frac{\delta y}{\delta x}=2x+\delta x$$

Proceeding to limit when $\delta x \to 0$, we have

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(2x+\delta x)$$

or $$\frac{dy}{dx}=2x+\lim_{\delta x\to 0}(\delta x)$$
$$=2x+0$$
$$=2x$$

or $$\frac{dy}{dx}=2x \qquad \text{or} \qquad \frac{d}{dx}\left(x^2\right)=2x$$

Example 21.4 Find the derivative of $\frac{1}{x}, x\neq 0$ by delta method.

Solution : Let $y=\frac{1}{x}$ (i)

For a small increment δx in x, let the corresponding increment in y be δy.

$\therefore$ $$y+\delta y=\frac{1}{x+\delta x}$$ (ii)

Subtracting (i) from (ii), we have,

$$(y+\delta y)-y=\frac{1}{x+\delta x}-\frac{1}{x}$$

or $$\delta y=\frac{x-(x+\delta x)}{(x+\delta x)x}$$

$$=\frac{-\delta x}{x(x+\delta x)}$$ (iii)

Dividing (iii) by δx, we have

$$\frac{\delta y}{\delta x}=\frac{-1}{x(x+\delta x)}$$

Proceeding to limit as $\delta x\to 0$, we have

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}-\frac{1}{x(x+\delta x)}$$

or $\quad \dfrac{dy}{dx} = \dfrac{-1}{x(x+0)}$ or $\quad \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$

Example 21.5 Find the derivative of $\sqrt{x}$ by ab-initio method.

Solution : Let $y = \sqrt{x}$ ..(i)

For a small increment δx in x, let δy be the corresponding increment in y.

$\therefore \quad y + \delta y = \sqrt{x + \delta x}$ ...(ii)

Subtracting (i) from (ii), we have

$$(y+\delta y) - y = \sqrt{x+\partial x} - \sqrt{x} \qquad \text{...(iii)}$$

or $\quad \delta y = \sqrt{x+\delta x} - \sqrt{x}$

Rationalising the numerator of the right hand side of (iii), we have

$$\delta y = \frac{\sqrt{x+\delta x} - \sqrt{x}}{\sqrt{x+\delta x} + \sqrt{x}}\left(\sqrt{x+\delta x} + \sqrt{x}\right)$$

$$= \frac{(x+\delta x) - x}{\sqrt{x+\delta x} + \sqrt{x}} \quad \text{or} \quad \delta y = \frac{\delta x}{\sqrt{x+\delta x} + \sqrt{x}}$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x+\delta x} + \sqrt{x}}$$

Proceeding to limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left[\frac{1}{\sqrt{x+\delta} + \sqrt{x}}\right]$$

or $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x} + \sqrt{x}}$ or $\quad \dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{1}{2\sqrt{x}}$

Example 21.6 If f (x) is a differentiable function and c is a constant, find the derivative of

$$\phi(x) = cf(x)$$

Solution : We have to find derivative of function

$$\phi(x) = cf(x) \qquad \text{...(i)}$$

For a small increment δx in x, let the values of the functions $\phi(x)$ be $\phi(x+\delta x)$ and that of f (x)be $f(x+\delta x)$

$\therefore \quad \phi(x+\delta x) = cf(x+\delta x)$ ...(ii)

Subtracting (i) from (ii), we have

$$\phi(x+\delta x) - \phi(x) = c\left[f(x+\delta x) - f(x)\right]$$

Dividing by δx, we have

$$\frac{\phi(x+\delta x) - \phi(x)}{\delta x} = c\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right]$$

Proceeding to limit as $\delta x \to 0$, we have

$$\lim_{\delta x\to 0}\frac{\phi(x+\delta x)-\phi(x)}{\delta x}=\lim_{\delta x\to 0} c\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

or $$\phi'(x)=c\lim_{\delta x\to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

or $$\phi'(x)=cf'(x)$$

Thus, $$\frac{d}{dx}\left[cf(x)\right]=c\frac{df}{dx}$$

CHECK YOUR PROGRESS 21.2

1. Find the derivative of each of the following functions by delta method :

(a) $10x$ (b) $2x+3$ (c) $3x^2$

(d) x^2+5x (e) $7x^3$

2. Find the derivative of each of the following functions using ab-initio method:

(a) $\frac{1}{x}, x\neq 0$ (b) $\frac{1}{ax}, x\neq 0$ (c) $x+\frac{1}{x}, x\neq 0$

(d) $\frac{1}{ax+b}, x\neq\frac{-b}{a}$ (e) $\frac{ax+b}{cx+d}, x\neq\frac{-d}{c}$ (f) $\frac{x+2}{3x+5}, x\neq\frac{-5}{3}$

3. Find the derivative of each of the following functions from first principles :

(a) $\frac{1}{\sqrt{x}}, x\neq 0$ (b) $\frac{1}{\sqrt{ax+b}}, x\neq\frac{-b}{a}$ (c) $\sqrt{x}+\frac{1}{\sqrt{x}}, x\neq 0$

(d) $\frac{1+x}{1-x}, x\neq 1$

4. Find the derivative of each of the following functions by using delta method :

(a) $f(x)=3\sqrt{x}$. Also find $f'(2)$. (b) $f(r)=\pi r^2$. Also find $f'(2)$.

(c) $f(r)=\frac{4}{3}\pi r^3$. Also find $f'(3)$.

21.7 ALGEBRA OF DERIVATIVES

Many functions arise as combinations of other functions. The combination could be sum, difference, product or quotient of functions. We also come across situations where a given function can be expressed as a function of a function.

In order to make derivative as an effective tool in such cases, we need to establish rules for finding derivatives of sum, difference, product, quotient and function of a function. These, in turn, will enable one to find derivatives of polynomials and algebraic (including rational) functions.

21.8 DERIVATIVES OF SUM AND DIFFERENCE OF FUNCTIONS

If f (x) and g (x) are both derivable functions and h (x) = f (x) + g (x), then what is h' (x) ?

Here $\qquad h(x) = f(x) + g(x)$

Let δx be the increment in x and δy be the correponding increment in y.

$\therefore \qquad h(x+\delta x) = f(x+\delta x) + g(x+\delta x)$

Hence $$h'(x) = \lim_{\delta x \to 0} \frac{[f(x+\delta x)+g(x+\delta x)]-[f(x)+g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[f(x+\delta x)-f(x)]+[g(x+\delta x)-g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{f(x+\delta x)-f(x)}{\delta x}+\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}+\lim_{\delta x \to 0} \frac{g(x+\delta x)-g(x)}{\delta x}$$

or $$h'(x) = f'(x) + g'(x)$$

Thus we see that the *derivative of sum of two functions is sum of their derivatives*.

This is called the **SUM RULE**.

e.g. $\qquad y = x^2 + x^3$

Then $$y' = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(x^3\right)$$

$$= 2x + 3x^2$$

Thus $\qquad y' = 2x + 3x^2$

This sum rule can easily give us the difference rule as well, because

if $\qquad h(x) = f(x) - g(x)$

then $\qquad h(x) = f(x) + [-g(x)]$

$\therefore \qquad h'(x) = f'(x) + [-g'(x)]$

$$= f'(x) - g'(x)$$

i.e. *the derivative of difference of two functions is the difference of their derivatives*.

This is called **DIFFERENCE RULE.**

Thus we have

Sum rule $\qquad : \qquad \frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

Difference rule : $\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}\left[f(x)\right]-\frac{d}{dx}\left[g(x)\right]$

Example 21.7 Find the derivative of each of the following functions :

(i) $y=10t^2+20t^3$

(ii) $y=2x^3-3x^2$

(iii) $y=x^3+\frac{1}{x^2}-\frac{1}{x},\quad x\neq 0$

Solution :

(i) We have, $y=10t^2+20t^3$

$\therefore \quad \frac{dy}{dt}=10(2t)+20\left(3t^2\right)$

$=20t+60t^2$

(ii) $y=2x^3-3x^2$

$\therefore \quad \frac{dy}{dx}=6x^2-6x$

(iii) $y=x^3+\frac{1}{x^2}-\frac{1}{x} \qquad x\neq 0$

$=x^3+x^{-2}-x^{-1}$

$\therefore \quad \frac{dy}{dx}=3x^2+(-2)x^{-3}-(-1)x^{-2}$

$=3x^2-\frac{2}{x^3}+\frac{1}{x^2}$

Example 21.8 Evaluate the derivative of each of the following functions at the indicated value (s) :

(i) $s=4.9t^2+2.4,\quad t=1, t=5$ (ii) $y=x^3+3x^2+4x+5,\quad x=1$

Solution : (i) We have $s=4.9t^2+2.4$

$\therefore \quad \frac{ds}{dt}=4.9(2t)$

$=9.8\,t$

$\left.\frac{ds}{dt}\right]_{t=1}=9.8(1)=9.8$

$\left.\frac{ds}{dt}\right]_{t=5}=9.8(5)=49$

(ii) We have $y = x^3 + 3x^2 + 4x + 5$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left[x^3 + 3x^2 + 4x + 5\right] = 3x^2 + 6x + 4$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=1} = 3(1)^2 + 6(1) + 4 = 13$$

CHECK YOUR PROGRESS 21.3

1. Find y' when :
   (a) $y = 12$ (b) $y = 12x$ (c) $y = 12x + 12$

2. Find the derivatives of each of the following functions :

   (a) $f(x) = 20x^9 + 5x$ (b) $f(x) = -50x^4 - 20x^2 + 4$

   (c) $f(x) = 4x^3 - 9 - 6x^2$ (d) $f(x) = \frac{5}{9}x^9 + 3x$

   (e) $f(x) = x^3 - 3x^2 + 3x - \frac{2}{5}$ (f) $f(x) = \frac{x^8}{8} - \frac{x^6}{6} + \frac{x^4}{4} - 2$

   (g) $f(x) = \frac{2}{5}x^{\frac{2}{3}} - x^{\frac{-4}{5}} + \frac{3}{x^2}$ (h) $f(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$

3. (a) If $f(x) = 16x + 2$, find $f'(0), f'(3), f'(8)$

   (b) If $f(x) = \frac{x^3}{3} - \frac{x^2}{2} + x - 16$, find $f'(-1),\ f'(0),\ f'(1)$

   (c) If $f(x) = \frac{x^4}{4} + \frac{3}{7}x^7 + 2x - 5$, find $f'(-2)$

   (d) Given that $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$ and hence $\left.\frac{dV}{dr}\right]_{r=2}$

21.9 DERIVATIVE OF PRODUCT OF FUNCTIONS

You are all familiar with the four fundamental operations of Arithmetic : addition, subtraction, multiplication and division. Having dealt with the sum and the difference rules, we now consider the derivative of product of two functions.

Consider $y = (x^2 + 1)^2$

This is same as $y = (x^2 + 1)(x^2 + 1)$

So we need now to derive the way to find the derivative in such situation.

We write $y = (x^2 + 1)(x^2 + 1)$

Let δx be the increment in x and δy the correrponding increment in y. Then

$$y + \delta y = [(x+\delta x)^2 + 1][(x+\delta x)^2 + 1)]$$

$$\Rightarrow \qquad \delta y = [(x+\delta x)^2 + 1][(x+\delta x)^2 + 1)] - (x^2+1)(x^2+1)$$

$$= [(x+\delta x)^2 + 1][(x+\delta x)^2 + 1 - x^2 - 1)] + (x^2+1)[(x+\delta x)^2 + 1] - (x^2+1)(x^2+1)$$

$$= [(x+\delta x)^2 + 1][(x+\delta x)^2 - x^2] + (x^2+1)[(x+\delta x)^2 + 1 - (x^2+1)]$$

$$= [(x+\delta x)^2 + 1][(x+\delta x)^2 - x^2] + (x^2+1)[(x+\delta x)^2 - x^2]$$

$$\therefore \qquad \frac{\delta y}{\delta x} = \left[(x+\delta x)^2 + 1\right]\cdot\left[\frac{(x+\delta x)^2 - x^2}{\delta x}\right] + \left(x^2+1\right)\left[\frac{(x+\delta x)^2 - x^2}{\delta x}\right]$$

$$= \left[(x+\delta x)^2 + 1\right]\cdot\left[\frac{2x\delta x + (\delta x)^2}{\delta x}\right] + \left(x^2+1\right)\left[\frac{2x\delta x + (\delta x)^2}{\delta x}\right]$$

$$= [(x+\delta x)^2 + 1](2x + \delta x) + (x^2+1)(2x+\delta x)$$

$$\therefore \qquad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} [(x+\delta x)^2 + 1]\cdot[2x + \delta x] + \lim_{\delta x \to 0} (x^2+1)(2x+\delta x)$$

or $$\frac{dy}{dx} = \left(x^2+1\right)\left(2x\right) + \left(x^2+1\right)\cdot\left(2x\right)$$

$$= 4x\left(x^2+1\right)$$

Let us analyse : $$\frac{dy}{dx} = \left(x^2+1\right)\underbrace{(2x)}_{\text{derivative of } x^2+1} + \left(x^2+1\right)\underbrace{(2x)}_{\text{derivative of } x^2+1}$$

Consider $y = x^3 \cdot x^2$

Is $\dfrac{dy}{dx} = x^3 \cdot (2x) + x^2 \cdot \left(3x^2\right)$?

Let us check $x^3(2x) + x^2\left(3x^2\right)$

$$= 2x^4 + 3x^4$$

$$= 5x^4$$

We have $y = x^3 \cdot x^2$

$$= x^5$$

$$\therefore \qquad \frac{dy}{dx} = 5x^4$$

In general, if f (x) and g (x) are two functions of x then the derivative of their product is defined by

$$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)$$

$$=[\text{Ist function}]\left[\frac{d}{dx}(\text{Secondfunction})\right]+[\text{Secondfunction}]\left[\frac{d}{dx}(\text{Istfunction})\right]$$

which is read as derivative of product of two functions is equal to

= [Ist function] [Derivative of Second function] +

[Second function] [Derivative of Ist function]

This is called the **PRODUCT RULE.**

Example 21.9 Find $\frac{dy}{dx}$, if $y=5x^6\left(7x^2+4x\right)$

Method I. Here y is a product of two functions.

$$\therefore \qquad \frac{dy}{dx}=\left(5x^6\right)\cdot\frac{d}{dx}\left(7x^2+4x\right)+\left(7x^2+4x\right)\frac{d}{dx}\left(5x^6\right)$$

$$=\left(5x^6\right)\left(14x+4\right)+\left(7x^2+4x\right)\left(30x^5\right)$$

$$=70x^7+20x^6+210x^7+120x^6$$

$$=280x^7+140x^6$$

Method II $\qquad y=5x^6\left(7x^2+4x\right)$

$$=35x^8+20x^7$$

$$\therefore \qquad \frac{dy}{dx}=35\times 8x^7+20\times 7x^6=280x^7+140x^6$$

which is the same as in Method I.

This rule can be extended to find the derivative of two or more than two functions.

Remark : If f (x), g (x) and h(x) are three given functions of x, then

$$\frac{d}{dx}\left[f(x)g(x)h(x)\right]=f(x)g(x)\frac{d}{dx}h(x)+g(x)h(x)\frac{d}{dx}f(x)+h(x)f(x)\frac{d}{dx}g(x)$$

Example 21.10 Find the derivative of $\left[f(x)g(x)h(x)\right]$ if

$$f(x)=x,\ g(x)=(x-3),\ \text{and} \quad h(x)=x^2+x$$

Solution : Let $y=x(x-3)\left(x^2+x\right)$

To find the derivative of y, we can combine any two functions, given on the R.H.S. and apply the product rule or use result mentioned in the above remark.

In other words, we can write

$$y=\left[x(x-3)\right]\left(x^2+x\right)$$

Let $u(x) = f(x)g(x)$

$$= x(x-3)$$

$$= x^2 - 3x$$

Also $h(x) = x^2 + x$

$\therefore$ $y = u(x) \times h(x)$

Hence $$\frac{dy}{dx} = x(x-3)\frac{d}{dx}\left(x^2 + x\right) + \left(x^2 + x\right)\frac{d}{dx}\left(x^2 - 3x\right)$$

$$= x(x-3)(2x+1) + \left(x^2 + x\right)(2x-3)$$

$$= x(x-3)(2x+1) + \left(x^2 + x\right)(x-3) + x\left(x^2 + x\right)$$

$$= [f(x)g(x)] \cdot h'(x) + [g(x)h(x)]f'(x) + [h(x)f(x)].g'(x)$$

Hence $$\frac{d}{dx}[f(x)g(x)h(x)] = [f(x)g(x)]\frac{d}{dx}[h(x)]$$

$$+[g(x)h(x)]\frac{d}{dx}[f(x)] + h(x)f(x)\frac{d}{dx}[g(x)]$$

Alternatively, we can directly find the derivative of product of the given three functions.

$$\frac{dy}{dx} = [x(x-3)]\frac{d}{dx}(x^2 + x) + [(x-3)(x^2 + x)]\frac{d}{dx}(x) + [(x^2 + x)\, x]\frac{d}{dx}(x-3)$$

$$= x(x-3)(2x+1) + (x-3)(x^2 + x)\cdot 1 + (x^2 + x)\cdot x \cdot 1$$

$$= 4x^3 - 6x^2 - 6x$$

CHECK YOUR PROGRESS 21.4

1. Find the derivative of each of the following functions by product rule :

(a) $f(x) = (3x+1)(2x-7)$ (b) $f(x) = (x+1)(-3x-2)$

(c) $f(x) = (x+1)(-2x-9)$ (d) $y = (x-1)(x-2)$

(e) $y = x^2(2x^2 + 3x + 8)$ (f) $y = (2x+3)(5x^2 - 7x + 1)$

(g) $u(x) = (x^2 - 4x + 5)(x^3 - 2)$

2. Find the derivative of each of the functions given below :

(a) $f(r) = r(1-r)(\pi r^2 + r)$ (b) $f(x) = (x-1)(x-2)(x-3)$

(c) $f(x) = (x^2 + 2)(x^3 - 3x^2 + 4)(x^4 - 1)$

(d) $f(x) = (3x^2 + 7)(5x - 1)\left(3x^2 + 9x + 8\right)$

Notes

21.10 QUOTIENT RULE

You have learnt sum Rule, Difference Rule and Product Rule to find derivative of a function expressed respectively as either the sum or difference or product of two functions. Let us now take a step further and learn the "Quotient Rule for finding derivative of a function which is the quotient of two functions.

Let $\qquad g(x)=\dfrac{1}{r(x)}, \qquad [r(x)\neq 0]$

Let us find the derivative of g (x) by first principles

$$g(x)=\frac{1}{r(x)}$$

$$\therefore \qquad g'(x)=\lim_{\delta x\to 0}\left[\frac{\dfrac{1}{r(x+\delta x)}-\dfrac{1}{r(x)}}{\delta x}\right]$$

$$=\lim_{\delta x\to 0}\left[\frac{r(x)-r(x+\delta x)}{\delta(x)r(x)r(x+\delta x)}\right]$$

$$=\lim_{\delta x\to 0}\left[\frac{r(x)-r(x+\delta x)}{\delta x}\right]\lim_{\delta x\to 0}\frac{1}{r(x).r(x+\delta x)}$$

$$=-r'(x)\cdot\frac{1}{[r(x)]^2}=-\frac{r'(x)}{[r(x)]^2}$$

Consider any two functions f (x) and g (x) such that $\phi(x)=\dfrac{f(x)}{g(x)}, \qquad g(x)\neq 0$

We can write $\phi(x)=f(x)\cdot\dfrac{1}{g(x)}$

$$\therefore \qquad \phi(x)=f'(x)\cdot\frac{1}{g(x)}+f(x)\frac{d}{dx}\left[\frac{1}{g(x)}\right]$$

$$=\frac{f'(x)}{g(x)}+f(x)\left[\frac{-g'(x)}{[g(x)]^2}\right]$$

$$=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

$$=\frac{\text{(Denominator)(Derivative of Numerator)} - \text{(Numerator)(Derivative of Denominator)}}{\text{(Denominator)}^2}$$

Hence $\qquad \dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

This is called the quotient Rule.

Example 21.11 Find $f'(x)$ if $f(x)=\dfrac{4x+3}{2x-1}$, $\quad x \neq \dfrac{1}{2}$

Solution :

$$f'(x)=\frac{(2x-1)\dfrac{d}{dx}(4x+3)-(4x+3)\dfrac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$=\frac{(2x-1).4-(4x+3).2}{(2x-1)^2}$$

$$=\frac{-10}{(2x-1)^2}$$

Let us consider the following example:

Let $\quad f(x)=\dfrac{1}{2x-1}, \qquad x \neq \dfrac{1}{2}$

$$\frac{d}{dx}\left[\frac{1}{2x-1}\right]=\frac{(2x-1)\dfrac{d}{dx}(1)-1\dfrac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$=\frac{(2x-1)\times 0-2}{(2x-1)^2} \qquad \left[\because \frac{d}{dx}(1)=0\right]$$

i.e. $\quad \dfrac{d}{dx}\left[\dfrac{1}{2x-1}\right]=-\dfrac{2}{(2x-1)^2}$

CHECK YOUR PROGRESS 21.5

1. Find the derivative of each of the following :

(a) $y=\dfrac{2}{5x-7}, \; x\neq\dfrac{7}{5}$ $\quad$ (b) $y=\dfrac{3x-2}{x^2+x-1}$ $\quad$ (c) $y=\dfrac{x^2-1}{x^2+1}$

(d) $f(x)=\dfrac{x^4}{x^2-3}$ $\quad$ (e) $f(x)=\dfrac{x^5-2x}{x^7}$ $\quad$ (f) $f(x)=\dfrac{x}{x^2+x+1}$

(g) $f(x)=\dfrac{\sqrt{x}}{x^3+4}$

2. Find $f'(x)$ if

(a) $f(x)=\dfrac{x(x^2+3)}{x-2}$, $\qquad [x\neq 2]$

(b) $f(x)=\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$, $\qquad [x\neq 3, \quad x\neq 4]$

21.11 CHAIN RULE

Earlier, we have come across functions of the type $\sqrt{x^4 + 8x^2 + 1}$. This function can not be expressed as a sum, difference, product or a quotient of two functions. Therefore, the techniques developed so far do not help us find the derivative of such a function. Thus, we need to develop a rule to find the derivative of such a function.

Let us write : $y = \sqrt{x^4 + 8x^2 + 1}$ or $y = \sqrt{t}$ where $t = x^4 + 8x^2 + 1$

That is, y is a function of t and t is a function of x. Thus y is a function of a function. We proceed to find the derivative of a function of a function.

Let δt be the increment in t and δy, the corresponding increment in y.

Then $\delta y \to 0$ as $\delta t \to 0$

$$\therefore \qquad \frac{dy}{dt} = \lim_{\delta t \to 0} \frac{\delta y}{\delta t} \qquad \text{(i)}$$

Similarly t is a function of x.

$$\therefore \qquad \delta t \to 0 \qquad \text{as} \qquad \delta x \to 0$$

$$\therefore \qquad \frac{dt}{dx} = \lim_{\delta x \to 0} \frac{\delta t}{\delta x} \qquad \text{(ii)}$$

Here y is a function of t and t is a function of x. Therefore $\delta y \to 0$ as $\delta x \to 0$

From (i) and (ii), we get

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \left[\lim_{\delta t \to 0} \frac{\delta y}{\delta t}\right]\left[\lim_{\delta x \to 0} \frac{\delta t}{\delta x}\right]$$

$$= \frac{dy}{dt}\,\frac{dt}{dx}$$

Thus $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

This is called the **Chain Rule.**

Example 21.12 If $y = \sqrt{x^4 + 8x^2 + 1}$, find $\frac{dy}{dx}$

Solution : We are given that

$$y = \sqrt{x^4 + 8x^2 + 1}$$

which we may write as

$$y = \sqrt{t}, \text{ where } t = x^4 + 8x^2 + 1 \qquad \text{(i)}$$

$$\therefore \qquad \frac{dy}{dt} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = 4x^3 + 16x$$

Here $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{2\sqrt{t}} \cdot (4x^3 + 16x)$$

$$= \frac{4x^3 + 16x}{2\sqrt{x^4 + 8x^2 + 1}} = \frac{2x^3 + 8x}{\sqrt{x^4 + 8x^2 + 1}} \qquad \text{(Using (i))}$$

Example 21.13 Find the derivative of $y = \sqrt[3]{x^2 + 5x - 7}$

Solution : We have

$$y = \sqrt[3]{x^2 + 5x - 7}$$

$$\frac{d}{dx}\sqrt[3]{x^2 + 5x - 7} = \frac{d}{dx}t^{1/3} \quad \text{where } t = x^2 + 5x - 7 \qquad \text{(i)}$$

$$= \frac{d}{dt}t^{1/3}\,\frac{dt}{dx} \qquad \text{(Using chain rule)}$$

$$= \frac{1}{3}t^{-2/3}\cdot(2x + 5)$$

$$\therefore \quad \frac{d}{dx}\sqrt[3]{x^2 + 5x - 7} = \frac{1}{3}[x^2 + 5x - 7]^{-2/3}\cdot(2x + 5) \qquad \text{[Using (i)]}$$

Example 21.14 Find the derivative of the function $y = \dfrac{5}{(x^2 - 3)^7}$

Solution : $$\frac{dy}{dx} = \frac{d}{dx}\left\{5(x^2 - 3)^{-7}\right\}$$

$$= 5[(-7)(x^2 - 3)^{-8}]\cdot\frac{d}{dx}(x^2 - 3) \qquad \text{(Using chain Rule)}$$

$$= -35(x^2 - 3)^{-8}\cdot(2x)$$

$$= \frac{-70x}{(x^2 - 3)^8}$$

Example 21.15 Find $\dfrac{dy}{dx}$ where $y = \dfrac{1}{4}v^4$ and $v = \dfrac{2}{3}x^3 + 5$

Solution : We have $y = \dfrac{1}{4}v^4$ and $v = \dfrac{2}{3}x^3 + 5$

$$\frac{dy}{dv} = \frac{1}{4}(4v^3) = v^3 = \left(\frac{2}{3}x^3 + 5\right)^3 \qquad \text{...(i)}$$

and $$\frac{dv}{dx} = \frac{2}{3}(3x^2) = 2x^2 \qquad \text{..(ii)}$$

Thus $$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx}$$

$$= \left(\frac{2}{3}x^3 + 5\right)^3 (2x^2) \qquad \text{[Using (i) and (ii)]}$$

Remark

We have seen in the previous examples that by using various rules of derivatives we can find derivatives of algebraic functions.

CHECK YOUR PROGRESS 21.6

1. Find the derivative of each of the following functions :

(a) $f(x) = (5x-3)^7$ (b) $f(x) = (3x^2 - 15)^{35}$

(c) $f(x) = (1-x^2)^{17}$ (d) $f(x) = \frac{(3-x)^5}{7}$

(e) $y = \frac{1}{x^2 + 3x + 1}$ (f) $y = \sqrt[3]{(x^2+1)^5}$

(g) $y = \frac{1}{\sqrt{7-3x^2}}$ (h) $y = \left[\frac{1}{6}x^6 + \frac{1}{2}x^4 + \frac{1}{16}\right]^5$

(i) $y = (2x^2 + 5x - 3)^{-4}$ (j) $y = x + \sqrt{x^2 + 8}$

2. Find $\frac{dy}{dx}$ if

(a) $y = \frac{3-v}{2+v}, v = \frac{4x}{1-x^2}$ (b) $y = at^2, t = \frac{x}{2a}$

21.12 DERIVATIVES OF A FUNCTION OF SECOND ORDER

Second Order Derivative : Given y is a function of x, say f (x). If the derivative $\frac{dy}{dx}$ is a derivable function of x, then the derivative of $\frac{dy}{dx}$ is known as the second derivative of y = f (x) with respect to x and is denoted by $\frac{d^2y}{dx^2}$. Other symbols used for the second derivative of y are D^2, f'', y'', y_2 etc.

Remark

Thus the value of f '' at x is given by

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(h)}{h}$$

Notes

The derivatives of third, fourth, ...orders can be similarly defined.

Thus the second derivative, or second order derivative of y with respect to x is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$$

Example 21.16 Find the second order derivative of

(i) x^2 (ii) x^3+1 (iii) $(x^2+1)(x-1)$ (iv) $\frac{x+1}{x-1}$

Solution : (i) Let $y=x^2$, then $\frac{dy}{dx}=2x$

and $\frac{d^2y}{dx^2}=\frac{d}{dx}(2x)=2\cdot\frac{d(x)}{dx}$

$= 2.1 = 2$

$\therefore$ $\frac{d^2y}{dx^2}=2$

(ii) Let $y=x^3+1$, then

$\frac{dy}{dx}=3x^2$ (by sum rule and derivative of a constant is zero)

and $\frac{d^2y}{dx^2}=\frac{d}{dx}(3x^2)=3.2x=6x$

$\therefore$ $\frac{d^2y}{dx^2}=6x$

(iii) Let $y=(x^2+1)(x-1)$, then

$\frac{dy}{dx}=(x^2+1)\frac{d}{dx}(x-1)+(x-1),\frac{d}{dx}(x^2+1)$

$=(x^2+1)\cdot 1+(x-1)\cdot 2x$ or $\frac{dy}{dx}=x^2+1+2x^2-2x=3x^2-2x+1$

and $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(3x^2-2x+1\right)=6x-2$

$\therefore$ $\frac{d^2y}{dx^2}=6x-2$

(iv) Let $y=\frac{x+1}{x-1}$, then

$$\frac{dy}{dx}=\frac{(x-1)\cdot 1-(x+1).1}{(x-1)^2}=\frac{-2}{(x-1)^2}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{-2}{(x-1)^2}\right] = -2 \cdot -2 \cdot \frac{1}{(x-1)^3} = \frac{4}{(x-1)^3}$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{4}{(x-1)^3}$$

CHECK YOUR PROGRESS 21.7

Find the derivatives of second order for each of the following functions :

(a) x^3 (b) $x^4 + 3x^3 + 9x^2 + 10x + 1$

(c) $\frac{x^2+1}{x+1}$ (d) $\sqrt{x^2+1}$

LET US SUM UP

- The derivative of a function f (x) with respect to x is defined as

$$f'(x) = \underset{\delta x \to 0}{\text{lt}} \frac{f(x+\delta x) - f(x)}{\delta x}, \ \delta x > 0$$

- The derivative of a constant is zero i.e., $\frac{dc}{dx} = 0$, where c a is constant.
- Newton's Power Formula

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

- Geometrically, the derivative $\frac{dy}{dx}$ of the function $y = f(x)$ at point $P(x, y)$ is the slope or gradient of the tangent on the curve represented by $y = f(x)$ at the point P.
- The derivative of y with respect to x is the instantaneous rate of change of y with respect to x.
- If f (x) is a derivable function and c is a constant, then

$\frac{d}{dx}\left[cf(x)\right] = cf'(x)$, where $f'(x)$ denotes the derivative of $f(x)$.
- 'Sum or difference rule' of functions :

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Derivative of the sum or difference of two functions is equal to the sum or diference of their derivatives respectively.

- **Product rule :**

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$= (\text{Ist function})\left(\frac{d}{dx}\text{IInd function}\right) + (\text{IIndfunction})\left(\frac{d}{dx}\text{Ist function}\right)$$

- **Quotient rule :** If $\phi(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$, then

$$\phi'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

$$= \frac{(\text{Denominator})\left(\frac{d}{dx}(\text{Numerator})\right) - \text{Numerator}\left(\frac{d}{dx}(\text{Denominator})\right)}{(\text{Denominator})^2}$$

- **Chain Rule :** $\frac{d}{dx}\left[f\{g(x)\}\right] = f'[g(x)]\cdot\frac{d}{dx}[g(x)]$

=derivative of f(x) w.r.t g (x)×derivative of g (x) w.r.t.x

- The derivative of second order of y w.r.t. to x is $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. The distance s meters travelled in time t seconds by a car is given by the relation $s = t^2$. Caclulate.

   (a) the rate of change of distance with respect to time t.

   (b) the speed of car at time t = 3 seconds.

2. Given $f(t) = 3 - 4t^2$. Use delta method to find $f'(t), f\left(\frac{1}{3}\right)$.

3. Find the derivative of $f(x) = x^4$ from the first principles. Hence find

$$f'(0), f'\left(-\frac{1}{2}\right)$$

4. Find the derivative of the function $\sqrt{2x+1}$ from the first principles.

5. Find the derivatives of each of the following functions by the first principles :

(a) ax + b, where a and b are constants (b) $2x^2 + 5$

(c) $x^3 + 3x^2 + 5$ (d) $(x-1)^2$

6. Find the derivative of each of the following functions :

(a) $f(x) = px^4 + qx^2 + 7x - 11$ (b) $f(x) = x^3 - 3x^2 + 5x - 8$

(c) $f(x) = x + \dfrac{1}{x}$ (d) $f(x) = \dfrac{x^2 - a}{a - 2}, a \neq 2$

7. Find the derivative of each of the functions given below by two ways, first by product rule, and then by expanding the product. Verify that the two answers are the same.

(a) $y = \sqrt{x}\left(1 + \dfrac{1}{\sqrt{x}}\right)$ (b) $y = x^{\frac{3}{2}}\left(2 + 5x + \dfrac{1}{x}\right)$

8. Find the derivative of the following functions :

(a) $f(x) = \dfrac{x}{x^2 - 1}$ (b) $f(x) = \dfrac{3}{(x-1)^2} + \dfrac{10}{x^3}$

(c) $f(x) = \dfrac{1}{(1 + x^4)}$ (d) $f(x) = \dfrac{(x+1)(x-2)}{\sqrt{x}}$

(e) $f(x) = \dfrac{3x^2 + 4x - 5}{x}$ (f) $f(x) = \dfrac{x - 4}{2\sqrt{x}}$

(g) $f(x) = \dfrac{(x^3 + 1)(x - 2)}{x^2}$

9. Use chain rule, to find the derivative of each of the functions given below :

(a) $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$ (b) $\sqrt{\dfrac{1+x}{1-x}}$ (c) $\sqrt[3]{x^2(x^2+3)}$

10. Find the derivatives of second order for each of the following :

(a) $\sqrt{x+1}$ (b) $x\sqrt{x-1}$

Notes

(a) $\sqrt{x+1}$ (b) $x \cdot \sqrt{x-1}$

[redacted]

CHECK YOUR PROGRESS 21.1

1. (a) 3 (b) 8 (c) 6 (d) 31
2. 3640 m/s
3. 21 m/s

CHECK YOUR PROGRESS 21.2

1. (a) 10 (b) 2 (c) 6x (d) 2x+5 (e) $21x^2$
2. (a) $-\frac{1}{x^2}$ (b) $-\frac{1}{ax^2}$ (c) $1-\frac{1}{x^2}$ (d) $\frac{-a}{(ax+b)^2}$ (e) $\frac{ad-bc}{(cx+d)^2}$

   (f) $-\frac{1}{(3x+5)^2}$
3. (a) $-\frac{1}{2x\sqrt{x}}$ (b) $\frac{-a}{2(ax+b)(\sqrt{ax+b})}$ (c) $\frac{1}{2\sqrt{x}}\left(1-\frac{1}{x}\right)$

   (d) $\frac{2}{(1-x)^2}$
4. (a) $\frac{3}{2\sqrt{x}};\frac{3}{2\sqrt{2}}$ (b) $2\pi r$; 4π (c) $2\pi r^2$; 36π

CHECK YOUR PROGRESS 21.3

1. (a) 0 (b) 12 (c) 12
2. (a) $180x^8+5$ (b) $-200x^3-40x$ (c) $12x^2-12x$

   (d) $5x^8+3$ (e) $3x^2-6x+3$ (f) $x^7-x^5+x^3$

   (g) $\frac{4}{15}x^{\frac{-1}{3}}+\frac{4}{5}x^{\frac{-9}{5}}-6x^{-3}$ (h) $\frac{1}{2\sqrt{x}}+\frac{1}{2x^{\frac{3}{2}}}$
3. (a) 16, 16, 16 (b) 3,1 ,1 (c) 186

   (d) $4\pi r^2, 16\pi$

CHECK YOUR PROGRESS 21.4

1. (a) $12x-19$ (b) $-6x-5$ (c) $4x-11$

   (d) $2x-3$ (e) $8x^3+9x^2+16x$ (f) $30x^2+2x-19$

(g) $5x^4 - 16x^3 + 15x^2 - 4x + 8$

2. (a) $-4\pi r^3 + 3(\pi - 1)r^2 + 2r$ (b) $3x^2 - 12x + 11$

(c) $9x^8 - 28x^7 + 14x^6 - 12x^5 - 5x^4 + 44x^3 - 6x^2 + 4x$

(d) $(5x-1)(3x^2+9x+8).6x + 5(3x^2+7)(3x^2+9x+8) + (3x^2+7)(5x-1)(6x+9)$

CHECK YOUR PROGRESS 21.5

1. (a) $\frac{-10}{(5x-7)^2}$ (b) $\frac{-3x^2+4x-1}{(x^2+x+1)^2}$ (c) $\frac{4x}{(x^2+1)^2}$

(d) $\frac{2x^5-12x^3}{(x^2-3)^2}$ (e) $\frac{-2x^4+12}{x^7}$ (f) $\frac{1-x^2}{(x^2+x+1)^2}$ (g) $\frac{4-5x^3}{2\sqrt{x}(x^3+4)^2}$

2. (a) $\frac{2x^3-6x^2-6}{(x-2)^2}$ (b) $\frac{-4x^2+20x-22}{(x-3)^2(x-4)^2}$

CHECK YOUR PROGRESS 21.6

1. (a) $35(5x-6)^6$ (b) $210x(3x^2-15)^{34}$

(c) $-34x(1-x^2)^{16}$ (d) $\frac{-5}{7}(3-x)^4$

(e) $-(2x+3)(x^2+3x+1)^{-2}$ (f) $\frac{10x}{3}\left(x^2+1\right)^{\frac{2}{3}}$

(g) $3x(7-3x^2)^{-3/2}$ (h) $5(x^5+2x^3)\left(\frac{x^6}{6}+\frac{x^4}{2}+\frac{1}{16}\right)^4$

(i) $-4(4x+5)(2x^2+5x-3)^{-5}$ (j) $1+\frac{x}{\sqrt{x^2+8}}$

2. (a) $\frac{-5(1+x^2)}{(1+2x-x^2)^2}$ (b) $\frac{x}{2a}$

CHECK YOUR PROGRESS 21.7

1. (a) $6x$ (b) $12x^2+18x+18$ (c) $\frac{4}{(x+1)^3}$ (d) $\frac{1}{(1+x^2)^{3/2}}$

TERMINAL EXERCISE

1. (a) 2 t (b) 6 seconds

2 $-8t, -\frac{8}{3}$ 3. $0, \frac{-1}{2}$ 4. $\frac{1}{\sqrt{2x+1}}$

5. (a) a   (b) $4x$ (c) $3x^2+6x$ (d) $2(x-1)$

6. (a) $4px^3+2qx+7$ (b) $3x^2-6x+5$

(c) $1-\frac{1}{x^2}$ (d) $\frac{2x}{a-2}$

7. (a) $\frac{1}{2\sqrt{x}}$ (b) $3\sqrt{x}+\frac{25}{2}x\sqrt{x}+\frac{1}{2\sqrt{x}}$

8. (a) $\frac{-(x^2+1)}{(x^2-1)^2}$ (b) $\frac{-6}{(x-1)^3}-\frac{30}{x^4}$

(c) $\frac{-4x^3}{(1+x^4)^2}$ (d) $\frac{3}{2}\sqrt{x}-\frac{1}{2\sqrt{x}}+\frac{1}{x^{3/2}}$

(e) $3+\frac{5}{x^2}$ (f) $\frac{1}{4\sqrt{x}}+\frac{1}{x\sqrt{x}}$

(g) $3x^2-2-\frac{1}{x^2}+\frac{4}{x^3}$

9. (a) $1-\frac{1}{x^2}$ (b) $\frac{1}{\sqrt{1+x}\cdot(1-x)^{\frac{3}{2}}}$ (c) $\frac{4x^3+6x}{3(x^4+3x^2)^{\frac{2}{3}}}$

10. (a) $-\frac{1}{4(x+1)^{\frac{3}{2}}}$ (b) $\frac{2+x-x^2}{4(x-1)^{\frac{1}{2}}}$

22

DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

Trigonometry is the branch of Mathematics that has made itself indispensable for other branches of higher Mathematics may it be calculus, vectors, three dimensional geometry, functions-harmonic and simple and otherwise just cannot be processed without encountering trigonometric functions. Further within the specific limit, trigonometric functions give us the inverses as well.

The question now arises : Are all the rules of finding the derivatives studied by us so far appliacable to trigonometric functions ?

This is what we propose to explore in this lesson and in the process, develop the formulae or results for finding the derivatives of trigonometric functions and their inverses . In all discussions involving the trignometric functions and their inverses, radian measure is used, unless otherwise specifically mentioned.

OBJECTIVES

After studying this lesson, you will be able to :

- find the derivative of trigonometric functions from first principle;
- find the derivative of inverse trigonometric functions from first principle;
- apply product, quotient and chain rule in finding derivatives of trigonometric and inverse trigonometric functions; and
- find second order derivative of a function.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of trigonometric ratios as functions of angles.
- Standard limits of trigonometric functions namely.

(i) $\lim\limits_{x\to 0}\sin x = 0$ (ii) $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ (iii) $\lim\limits_{x\to 0}\cos x = 1$ (iv) $\lim\limits_{x\to 0}\frac{\tan x}{x} = 1$

- Definition of derivative, and rules of finding derivatives of function.

22.1 DERIVATIVE OF TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLE

(i) Let y = sin x

For a small increment δx in x, let the corresponding increment in y be δy.

$\therefore \quad y + \delta y = \sin(x + \delta x)$

and $\quad \delta y = \sin(x+\delta x) - \sin x$

$$= 2\cos\left[x + \frac{\delta x}{2}\right]\sin\frac{\delta x}{2} \qquad \left[\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$\therefore \quad \frac{\delta y}{\delta x} = 2\cos\left(x + \frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\cos\left(x + \frac{\delta x}{2}\right)\cdot\lim_{\delta x \to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}} = \cos x.1 \qquad \left[\because \lim_{\delta x \to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}} = 1\right]$$

Thus, $\quad \dfrac{dy}{dx} = \cos x$

i.e., $\quad \dfrac{d}{dx}(\sin x) = \cos x$

(ii) Let $\quad y = \cos x$

For a small increment δx in x, let the corresponding increment in y be δy.

$\therefore \quad y + \delta y = \cos(x + \delta x)$

and $\quad \delta y = \cos(x+\delta x) - \cos x$

$$= -2\sin\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

$$\therefore \quad \frac{\delta y}{\delta x} = -2\sin\left(x + \frac{\delta x}{2}\right)\cdot\frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -\lim_{\delta x \to 0}\sin\left(x + \frac{dx}{2}\right)\cdot\lim_{\delta x \to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$= -\sin x \cdot 1$$

Thus, $\quad \dfrac{dy}{dx} = -\sin x$

i.e., $\frac{d}{dx}(\cos x) = -\sin x$

(iii) Let $y = \tan x$

For a small increament δx in x, let the corresponding increament in y be δy .

$\therefore \quad y + \delta y = \tan(x + \delta x)$

and $\quad \delta y = \tan(x + \delta x) - \tan x$

$$= \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+\delta x)\cdot\cos x - \sin x.\cos(x+\delta x)}{\cos(x+\delta x)\cos x}$$

$$= \frac{\sin[(x+\delta x)-x]}{\cos(x+\delta x)\cos x}$$

$$= \frac{\sin\delta x}{\cos(x+\delta x)\cdot\cos x}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{\sin\delta x}{\delta x}\cdot\frac{1}{\cos(x+\delta x)\cos x}$$

or
$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}\cdot\lim_{\delta x\to 0}\frac{1}{\cos(x+\delta x)\cos x}$$

$$= 1\cdot\frac{1}{\cos^2 x} \qquad \left[\because \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x} = 1\right]$$

$$= \sec^2 x$$

Thus, $\frac{dy}{dx} = \sec^2 x$

i.e. $\quad \frac{d}{dx}(\tan x) = \sec^2 x$

(iv) Let $\quad y = \sec x$

For a small increament δx in x, let the corresponding increament in y be δy .

$\therefore \quad y + \delta y = \sec(x+\delta x)$

and $\quad \delta y = \sec(x+\delta x) - \sec x$

$$= \frac{1}{\cos(x+\delta x)} - \frac{1}{\cos x}$$

$$= \frac{\cos x - \cos(x+\delta x)}{\cos(x+\delta x)\cos x}$$

$$= \frac{2\sin\left[x+\frac{\delta x}{2}\right]\sin\frac{\delta x}{2}}{\cos(x+\delta x)\cos x}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sin\left(x + \frac{\delta x}{2}\right)}{\cos(x + \delta x)\cos x} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sin\left(x + \frac{\delta x}{2}\right)}{\cos(x + \delta x)\cos x} \lim_{\delta x \to 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$= \frac{\sin x}{\cos^2 x} \cdot 1$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x . \sec x$$

Thus, $$\frac{dy}{dx} = \sec x . \tan x$$

i.e. $$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

Similarly, we can show that

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

and $$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$$

Example 22.1 Find the derivative of $\cot x^2$ from first principle.

Solution : $y = \cot x^2$

For a small increament δx in x, let the corresponding increament in y be δy .

$\therefore$ $$y + \delta y = \cot(x + \delta x)^2$$

and $$\delta y = \cot(x + \delta x)^2 - \cot x^2$$

$$= \frac{\cos(x + \delta x)^2}{\sin(x + \delta x)^2} - \frac{\cos x^2}{\sin x^2}$$

$$= \frac{\cos(x + \delta x)^2 \sin x^2 - \cos x^2 \sin(x + \delta x)^2}{\sin(x + \delta x)^2 \sin x^2}$$

$$= \frac{\sin[x^2 - (x + \delta x)^2]}{\sin(x + \delta x)^2 \sin x^2}$$

$$= \frac{\sin[-2x\delta x - (\delta x)^2]}{\sin(x + \delta x)^2 \sin x^2}$$

$$= \frac{-\sin[(2x+\delta x)\delta x]}{\sin(x+\delta x)^2 \sin x^2}$$

$$\therefore \qquad \frac{\delta y}{\delta x} = \frac{-\sin[(2x+\delta x)\delta x]}{\delta x \sin(x+\delta x)^2 \sin x^2}$$

and $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = -\lim_{\delta x \to 0} \frac{\sin[(2x+\delta x)\delta x]}{\delta x(2x+\delta x)} \lim_{\delta x \to 0} \frac{2x+\delta x}{\sin(x+\delta x)^2 \sin x^2}$$

or $$\frac{dy}{dx} = -1 \cdot \frac{2x}{\sin x^2 . \sin x^2} \qquad \left[\because \lim_{\delta x \to 0} \frac{\sin[(2x+\delta x)\delta x]}{\delta x(2x+\delta x)} = 1\right]$$

$$= \frac{-2x}{(\sin x^2)^2} = \frac{-2x}{\sin^2 x^2}$$

$$= -2x . \operatorname{cosec}^2 x^2$$

Hence $$\frac{d}{dx}(\cot x^2) = -2x \cdot \operatorname{cosec}^2 x^2$$

Example 22.2 Find the derivative of $\sqrt{\operatorname{cosec} x}$ from first principle.

Solution : Let $y = \sqrt{\operatorname{cosec} x}$

and $$y + \delta y = \sqrt{\operatorname{cosec}(x+\delta x)}$$

$$\therefore \qquad \delta y = \frac{\left[\sqrt{\operatorname{cosec}(x+\delta x)} - \sqrt{\operatorname{cosec} x}\right]\left[\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}\right]}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}}$$

$$= \frac{\operatorname{cosec}(x+\delta x) - \operatorname{cosec} x}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}}$$

$$= \frac{\dfrac{1}{\sin(x+\delta x)} - \dfrac{1}{\sin x}}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}}$$

$$= \frac{\sin x - \sin(x+\delta x)}{\left[\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}\right][\sin(x+\delta x)\sin x]}$$

$$= -\frac{2\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\left(\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}\right)[\sin(x+\delta x)\sin x]}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = -\lim_{\delta x \to 0} \frac{\cos\left(x+\frac{\delta x}{2}\right)}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}]} \times \frac{\dfrac{\sin \delta x/2}{\delta x/2}}{[\sin(x+\delta x).\sin x]}$$

or $$\frac{dy}{dx} = \frac{-\cos x}{(2\sqrt{(\operatorname{cosec} x)}(\sin x)^2}$$

MODULE - V
Calculus

Notes

$$= -\frac{1}{2}(\text{cosec}x)^{-\frac{1}{2}}(\text{cosec}\,x\cot x)$$

Thus, $$\frac{d}{dx}\left(\sqrt{\text{cosec}x}\right) = -\frac{1}{2}(\text{cosec}x)^{-\frac{1}{2}}(\text{cosec}x\cot x)$$

Example 22.3 Find the derivative of $\sec^2 x$ from first principle.

Solution : Let $y = \sec^2 x$

and $$y + \delta y = \sec^2(x + \delta x)$$

then, $$\delta y = \sec^2(x + \delta x) - \sec^2 x$$

$$= \frac{\cos^2 x - \cos^2(x + \delta x)}{\cos^2(x + \delta x)\cos^2 x}$$

$$= \frac{\sin[(x + \delta x + x]\sin[(x + \delta x - x)]}{\cos^2(x + \delta x)\cos^2 x}$$

$$= \frac{\sin(2x + \delta x)\sin\delta x}{\cos^2(x + \delta x)\cos^2 x}$$

$$\frac{\delta y}{\delta x} = \frac{\sin(2x + \delta x)\sin\delta x}{\cos^2(x + \delta x)\cos^2 x.\delta x}$$

Now, $$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{\sin(2x + \delta x)\sin\delta x}{\cos^2(x + \delta x)\cos^2 x.\delta x}$$

$$\frac{dy}{dx} = \frac{\sin 2x}{\cos^2 x\cos^2 x}$$

$$= \frac{2\sin x\cos x}{\cos^2 x\cos^2 x} = 2\tan x.\sec^2 x$$

$$= 2\sec x(\sec x.\tan x)$$

$$= 2\sec x\ (\sec x \tan x)$$

CHECK YOUR PROGRESS 22.1

1. Find the derivative from first principle of the following functions with respect to x :

(a) cosec x (b) cot x (c) cos 2 x

(d) cot 2 x (e) $\text{cosec}x^2$ (f) $\sqrt{\sin x}$

2. Find the derivative of each of the following functions :

(a) $2\sin^2 x$ (b) $\text{cosec}^2 x$ (c) $\tan^2 x$

22.2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

You have learnt how we can find the derivative of a trigonometric function from first principle and also how to deal with these functions as a function of a function as shown in the alternative method. Now we consider some more examples of these derivatives.

Example 22.4 Find the derivative of each of the following functions :

(i) $\sin 2x$ (ii) $\tan\sqrt{x}$ (iii) $\text{cosec}(5x^3)$

Solution : (i) Let $y = \sin 2x,$

$= \sin t,$ where $t = 2x$

$$\frac{dy}{dt} = \cos t \quad \text{and} \quad \frac{dt}{dx} = 2$$

By chain Rule, $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$, we have

$$\frac{dy}{dx} = \cos t\,(2) = 2.\cos t = 2\cos 2x$$

Hence, $$\frac{d}{dx}(\sin 2x) = 2\cos 2x$$

(ii) Let $y = \tan\sqrt{x}$

$= \tan t$ where $t = \sqrt{x}$

$\therefore$ $$\frac{dy}{dt} = \sec^2 t \quad \text{and} \quad \frac{dt}{dx} = \frac{1}{2\sqrt{x}}$$

By chain rule, $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$, we have

$$\frac{dy}{dx} = \sec^2 t\cdot\frac{1}{2\sqrt{x}} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

Hence, $$\frac{d}{dx}\left(\tan\sqrt{x}\right) = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

Alternatively : Let $y = \tan\sqrt{x}$

$$\frac{dy}{dx} = \sec^2\sqrt{x}\,\frac{d}{dx}\sqrt{x} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

(iii) Let $y = \text{cosec}(5x^3)$

$\therefore$ $$\frac{dy}{dx} = -\text{cosec}(5x^3)\cot(5x^3)\cdot\frac{d}{dx}[5x^3]$$

$$= -15x^2\,\text{cosec}(5x^3)\cot(5x^3)$$

or you may solve it by substituting $t = 5x^3$

Example 22.5 Find the derivative of each of the following functions :

(i) $y = x^4 \sin 2x$ (ii) $y = \dfrac{\sin x}{1+\cos x}$

Solution : $y = x^4 \sin 2x$

(i) $\therefore$ $\dfrac{dy}{dx} = x^4 \dfrac{d}{dx}(\sin 2x) + \sin 2x \dfrac{d}{dx}(x^4)$ (Using product rule)

$$= x^4 (2\cos 2x) + \sin 2x (4x^3)$$

$$= 2x^4 \cos 2x + 4x^3 \sin 2x$$

$$= 2x^3 [x\cos 2x + 2\sin 2x]$$

(ii) $y = \dfrac{\sin x}{1+\cos x}$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$= \tan\frac{x}{2}$$

$\therefore$ $\dfrac{dy}{dx} = \sec^2\dfrac{x}{2}\cdot\dfrac{d}{dx}\left(\dfrac{x}{2}\right) = \dfrac{1}{2}\sec^2\dfrac{x}{2}$

Alternatively : You may find the derivative by using quotient rule

Let $y = \dfrac{\sin x}{1+\cos x}$

$\therefore$ $$\frac{dy}{dx} = \frac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2}$$

$$= \frac{1}{(1+\cos x)}$$

$$= \frac{1}{2\cos^2\frac{x}{2}} = \frac{1}{2}\sec^2\frac{x}{2}$$

Example 22.6 Find the derivative of each of the following functions w.r.t. x :

(i) $\cos^2 x$ (ii) $\sqrt{\sin^3 x}$

Solution : (i) Let $y = \cos^2 x$

$= t^2$ where $t = \cos x$

$\therefore \quad \frac{dy}{dt} = 2t \quad$ and $\quad \frac{dt}{dx} = -\sin x$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}, \text{ we have}$$

$$\frac{dy}{dx} = 2 \cos x. (-\sin x)$$

$$= -2\cos x \sin x = -\sin 2x$$

(ii) Let $y = \sqrt{\sin^3 x}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(\sin^3 x)^{-1/2} \cdot \frac{d}{dx}(\sin^3 x)$$

$$= \frac{1}{2\sqrt{\sin^3 x}} \cdot 3\sin^2 x \cdot \cos x$$

$$= \frac{3}{2}\sqrt{\sin x} \cos x$$

Thus, $\frac{d}{dx}\left(\sqrt{\sin^3 x}\right) = \frac{3}{2}\sqrt{\sin x} \cos x$

Example 22.7 Find $\frac{dy}{dx}$, when

(i) $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$ (ii) $y = a(1-\cos t), x = a(t+\sin t)$

Solution : We have,

(i) $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left[\frac{1-\sin x}{1+\sin x}\right]^{-\frac{1}{2}} \cdot \frac{d}{dx}\left[\frac{1-\sin x}{1+\sin x}\right]$$

$$=\frac{1}{2}\sqrt{\frac{1+\sin x}{1-\sin x}}\cdot\frac{(-\cos x)(1+\sin x)-(1-\sin x)(\cos x)}{(1+\sin x)^2}$$

$$=\frac{1}{2}\sqrt{\frac{1+\sin x}{1-\sin x}}\cdot\left(\frac{-2\cos x}{(1+\sin x)^2}\right)$$

$$=-\frac{\sqrt{1+\sin x}}{\sqrt{1-\sin x}}\cdot\frac{\sqrt{1-\sin^2 x}}{(1+\sin x)^2}$$

$$=-\frac{\sqrt{1+\sin x}\sqrt{1+\sin x}}{(1+\sin x)^2}=\frac{-1}{1+\sin x}$$

Thus, $\quad dy/dx \;=-\dfrac{1}{1+\sin x}$

Alternatively, it is more convenient to find the derivative of such square root function by rationalising the denominator.

$$y=\frac{\sqrt{1-\sin x}}{\sqrt{1+\sin x}}\times\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}$$

$$=\frac{1-\sin x}{\sqrt{1-\sin^2 x}}$$

$$=\frac{1-\sin x}{\cos x}$$

$$=\sec x-\tan x$$

$\therefore$ $$\frac{dy}{dx}=\sec x\tan x-\sec^2 x=\frac{\sin x}{\cos^2 x}-\frac{1}{\cos^2 x}$$

$$=\frac{\sin x-1}{1-\sin^2 x}=-\frac{1}{1+\sin x}$$

(ii) $\qquad x=a(t+\sin t),\quad y=a(1-\cos t)$

$\therefore$ $$\frac{dx}{dt}=a(1+\cos t),\quad \frac{dy}{dt}=a(\sin t)$$

Using chain rule, $\qquad \dfrac{dy}{dx}=\dfrac{dy}{dt}\cdot\dfrac{dt}{dx}$, we have

$$\frac{dy}{dx}=\frac{a(\sin t)}{a(1+\cos t)}$$

$$=\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}}=\tan\frac{t}{2}$$

Example 22.8 Find the derivative of each of the following functions at the indicated points :

(i) $y = \sin 2x + (2x-5)^2$ at $x = \dfrac{\pi}{2}$

(ii) $y = \cot x + \sec^2 x + 5$ at $x = \pi/6$

Solution :

(i) $$y = \sin 2x + (2x-5)^2$$

$$\therefore \quad \frac{dy}{dx} = \cos 2x \frac{d}{dx}(2x) + 2(2x-5)\frac{d}{dx}(2x-5)$$

$$= 2\cos 2x + 4(2x-5)$$

At $x = \dfrac{\pi}{2}$, $$\frac{dy}{dx} = 2\cos\pi + 4(\pi - 5)$$

$$= -2 + 4\pi - 20$$

$$= 4\pi - 22$$

(ii) $$y = \cot x + \sec^2 x + 5$$

$$\therefore \quad \frac{dy}{dx} = -\operatorname{cosec}^2 x + 2\sec x(\sec x \tan x)$$

$$= -\operatorname{cosec}^2 x + 2\sec^2 x \tan x$$

At $x = \dfrac{\pi}{6}$, $$\frac{dy}{dx} = -\operatorname{cosec}^2 \frac{\pi}{6} + 2\sec^2\frac{\pi}{6}\tan\frac{\pi}{6}$$

$$= -4 + 2 \cdot \frac{4}{3}\frac{1}{\sqrt{3}}$$

$$= -4 + \frac{8}{3\sqrt{3}}$$

Example 22.9 If sin y = x sin (a+y), prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Solution : It is given that

$$\sin y = x \sin(a+y) \quad \text{or} \quad x = \frac{\sin y}{\sin(a+y)} \qquad .....(1)$$

Differentiating w.r.t. x on both sides of (1) we get

$$1 = \left[\frac{\sin(a+y)\cos y - \sin y \cos(a+y)}{\sin^2(a+y)}\right]\frac{dy}{dx}$$

or $$1 = \left[\frac{\sin(a+y-y)}{\sin^2(a+y)}\right]\frac{dy}{dx}$$

or $$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$$

Example 22.10 If $y=\sqrt{\sin x+\sqrt{\sin x+....\text{to infinity}}}$,

prove that $$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

Solution : We are given that

$$y=\sqrt{\sin x+\sqrt{\sin x+...\text{to infinity}}}$$

or $$y=\sqrt{\sin x+y} \qquad \text{or} \qquad y^2=\sin x+y$$

Differentiating with respect to x , we get

$$2y\frac{dy}{dx}=\cos x+\frac{dy}{dx} \qquad \text{or} \qquad (2y-1)\frac{dy}{dx}=\cos x$$

Thus, $$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

CHECK YOUR PROGRESS 22.2

1. Find the derivative of each of the following functions w.r.t x :

(a) $y=3\sin 4x$ (b) $y=\cos 5x$ (c) $y=\tan\sqrt{x}$

(d) $y=\sin\sqrt{x}$ (e) $y=\sin x^2$ (f) $y=\sqrt{2}\tan 2x$

(g) $y=\pi\cot 3x$ (h) $y=\sec 10x$ (i) $y=\operatorname{cosec} 2x$

2. Find the derivative of each of the following functions :

(a) $f(x)=\dfrac{\sec x-1}{\sec x+1}$ (b) $f(x)=\dfrac{\sin x+\cos x}{\sin x-\cos x}$ (c) $f(x)=x\sin x$

(d) $f(x)=(1+x^2)\cos x$ (e) $f(x)=x\operatorname{cosec} x$ (f) $f(x)=\sin 2x\cos 3x$

(g) $f(x)=\sqrt{\sin 3x}$

3. Find the derivative of each of the following functions :

(a) $y=\sin^3 x$ (b) $y=\cos^2 x$ (c) $y=\tan^4 x$

(d) $y=\cot^4 x$ (e) $y=\sec^5 x$ (f) $y=\operatorname{cosec}^3 x$

(g) $y=\sec\sqrt{x}$ (h) $y=\sqrt{\dfrac{\sec x+\tan x}{\sec x-\tan x}}$

4. Find the derivative of the following functions at the indicated points :

(a) $y = \cos(2x + \pi/2), x = \frac{\pi}{3}$ (b) $y = \frac{1+\sin x}{\cos x}, x = \frac{\pi}{4}$

5. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + ...,}}}$ to infinity

Show that $(2y-1)\frac{dy}{dx} = \sec^2 x$.

6. If $\cos y = x\cos(a+y)$,

prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

22.3 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLE

We now find derivatives of standard inverse trignometric functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, by first principle.

(i) We will show that by first principle the derivative of $\sin^{-1} x$ w.r.t. x is given by

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}$$

Let $y = \sin^{-1} x$. Then $x = \sin y$ and so $x + \delta x = \sin(y + \delta y)$

As $\delta x \to 0$, $\delta y \to 0$.

Now, $\delta x = \sin(y+\delta y) - \sin y$

$\therefore$ $1 = \frac{\sin(y+\delta y) - \sin y}{\delta x}$ [On dividing both sides by δx]

or $1 = \frac{\sin(y+\delta y) - \sin y}{\delta y} \cdot \frac{\delta y}{\delta x}$

$\therefore$ $1 = \lim_{\delta y \to 0} \frac{\sin(y+\delta y) - \sin y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ [$\because \delta y \to 0$ when $\delta x \to 0$]

$$= \left[\lim_{\delta y \to 0} \frac{2\cos\left(y + \frac{1}{2}\delta y\right)\sin\left(\frac{1}{2}\delta y\right)}{\delta y}\right] \cdot \frac{dy}{dx}$$

$$= (\cos y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}$$

$$\therefore \quad \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{\left(1-x^2\right)}}$$

(ii) $$\frac{\mathbf{d}}{\mathbf{dx}}\left(\mathbf{cos^{-1}\,x}\right) = -\frac{\mathbf{1}}{\sqrt{\left(\mathbf{1-x^2}\right)}}.$$

For proof proceed exactly as in the case of $\sin^{-1} x$.

(iii) Now we show that,

$$\frac{\mathbf{d}}{\mathbf{dx}}\left(\mathbf{tan^{-1}\,x}\right) = \frac{\mathbf{1}}{\mathbf{1+x^2}}$$

Let $y = \tan^{-1} x$ .Then $x = \tan y$ and so $x + \delta x = \tan(y + \delta y)$

As $\delta x \to 0$, also $\delta y \to 0$

Now, $\qquad \delta x = \tan(y+\delta y) - \tan y$

$$\therefore \qquad 1 = \frac{\tan(y+\delta y) - \tan y}{\delta y}\cdot\frac{\delta y}{\delta x}.$$

$$\therefore \qquad 1 = \lim_{\delta y \to 0}\frac{\tan(y+\delta y) - \tan y}{\delta y}\cdot\lim_{\delta x \to 0}\frac{\delta y}{\delta x} \qquad [\because \delta y \to 0 \text{ when } \delta x \to 0]$$

$$= \left[\lim_{\delta y \to 0}\left\{\frac{\sin(y+\delta y)}{\cos(y+\delta y)} - \frac{\sin y}{\cos y}\right\}\Big/\delta y\right]\cdot\frac{dy}{dx}$$

$$= \frac{dy}{dx}\cdot\lim_{\delta y \to 0}\frac{\sin(y+\delta y)\cos y - \cos(y+\delta y)\sin y}{\delta y.\cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx}\cdot\lim_{\delta y \to 0}\frac{\sin(y+\delta y - y)}{\delta y.\cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx}\cdot\lim_{\delta y \to 0}\left[\frac{\sin\delta y}{\delta y}\cdot\frac{1}{\cos(y+\delta y)\cos y}\right]$$

$$= \frac{dy}{dx}\,\frac{1}{\cos^2 y} = \frac{dy}{dx}\cdot\sec^2 y$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\therefore \qquad \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

(iv) $$\frac{\mathbf{d}}{\mathbf{dx}}\left(\mathbf{cot^{-1}\,x}\right) = -\frac{\mathbf{1}}{\mathbf{1+x^2}}$$

For proof proceed exactly as in the case of $\tan^{-1} x$.

(v) We have by first principle $\mathbf{\frac{d}{dx}(sec^{-1} x) = \frac{1}{x\sqrt{(x^2-1)}}}$

Let $y = \sec^{-1} x$. Then $x = \sec y$ and so $x + \delta x = \sec(y + \delta y)$.

As $\delta x \to 0$, also $\delta y \to 0$.

Now $\delta x = \sec(y+\delta y) - \sec y.$

$$\therefore \qquad 1 = \frac{\sec(y+\delta y) - \sec y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

$$1 = \lim_{\delta y \to 0} \frac{\sec(y+\delta y) - \sec y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \qquad [\because \delta y \to 0 \text{ when } \delta x \to 0]$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{2\sin\left(y + \frac{1}{2}\delta y\right)\sin\left(\frac{1}{2}\delta y\right)}{\delta y . \cos y \cos(y + \delta y)}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{\sin\left(y + \frac{1}{2}\delta y\right)}{\cos y \cos(y+\delta y)} \cdot \frac{\sin\left(\frac{1}{2}\delta y\right)}{\frac{1}{2}\delta y} \right]$$

$$= \frac{dy}{dx} \cdot \frac{\sin y}{\cos y \cos y} = \frac{dy}{dx} \cdot \sec y \tan y$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\sqrt{(\sec^2 y - 1)}} = \frac{1}{x\sqrt{(x^2-1)}}$$

$$\therefore \qquad \frac{d}{dx} = (\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

(vi) $\mathbf{\frac{d}{dx}\left(cosec^{-1} x\right) = \frac{1}{x\sqrt{(x^2-1)}}}$.

For proof proceed as in the case of $\sec^{-1} x$.

Example 22.11 Find derivative of $\sin^{-1}(x^2)$ from first principle.

Solution : Let $y = \sin^{-1} x^2$

$$\therefore \qquad x^2 = \sin y$$

Now, $(x + \delta x)^2 = \sin(y + \delta y)$

$$\frac{(x+\delta x)^2 - x^2}{\delta x} = \frac{\sin(y+\delta y) - \sin y}{\delta x}$$

Notes

$$\lim_{\delta x\to 0}\frac{(x+\delta x)^2-x^2}{(x+\delta x)-x}=\lim_{\delta y\leftarrow 0}\frac{2\cos\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2}}{2\,\frac{\delta y}{2}}\cdot\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$\Rightarrow$ $$2x=\cos y\cdot\frac{dy}{dx}$$

$\Rightarrow$ $$\frac{dy}{dx}=\frac{2x}{\cos y}=\frac{2x}{\sqrt{1-\sin^2 y}}=\frac{2x}{\sqrt{1-x^4}}.$$

Example 22.12 Find derivative of $\sin^{-1}\sqrt{x}$ w.r.t. x by delta method.

Solution : Let $y=\sin^{-1}\sqrt{x}$

$\Rightarrow$ $$\sin y=\sqrt{x} \quad ..(1)$$

Also $$\sin(y+\delta y)=\sqrt{x+\delta x} \quad ..(2)$$

From (1) and (2), we get

$$\sin(y+\delta y)-\sin y=\sqrt{x+\delta x}-\sqrt{x}$$

or $$2\cos\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right)=\frac{\left(\sqrt{x+\delta x}-\sqrt{x}\right)\left(\sqrt{x+\delta x}+\sqrt{x}\right)}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$=\frac{\delta x}{\sqrt{x+\delta x}+\sqrt{x}}$$

$\therefore$ $$\frac{2\cos\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right)}{\delta x}=\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$$

or $$\frac{\delta y}{\delta x}\cdot\cos\left(y+\frac{\delta y}{2}\right)\cdot\frac{\sin\left(\frac{\delta y}{2}\right)}{\frac{\delta y}{2}}=\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$$

$\therefore$ $$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}\cdot\lim_{\delta y\to 0}\cos\left(y+\frac{\delta y}{2}\right)\cdot\lim_{\delta y\to 0}\frac{\sin\left(\frac{\delta y}{2}\right)}{\frac{\delta y}{2}}$$

$$=\lim_{\delta x\to 0}\frac{1}{\sqrt{x+\delta x}+\sqrt{x}} \qquad (\because\ \delta y\to 0\ \text{as}\ \delta x\to 0)$$

or $$\frac{dy}{dx}\cos y=\frac{1}{2\sqrt{x}}$$ or $$\frac{dy}{dx}=\frac{1}{2\sqrt{x}\cos y}=\frac{1}{2\sqrt{x}\sqrt{1-\sin^2 y}}=\frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$\therefore$ $$\frac{dy}{dx}=\frac{1}{2\sqrt{x}\,\sqrt{1-x}}$$

CHECK YOUR PROGRESS 22.3

1. Find by first principle that derivative of each of the following :

(i) $\cos^{-1} x^2$ (ii) $\dfrac{\cos^{-1} x}{x}$ (iii) $\cos^{-1}\sqrt{x}$

(iv) $\tan^{-1} x^2$ (v) $\dfrac{\tan^{-1} x}{x}$ (vi) $\tan^{-1}\sqrt{x}$

22.4 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

In the previous section, we have learnt to find derivatives of inverse trignometric functions by first principle. Now we learn to find derivatives of inverse trigonometric functions by alternative methods. We start with standard inverse trignometric functions $\sin^{-1} x, \cos^{-1} x, ...$

(i) Derivative of $\sin^{-1} x$

Solution : Let $y = \sin^{-1} x$

$\therefore \qquad x = \sin y \qquad$ (i)

Differentiating w.r.t. y

$$\frac{dx}{dy} = \cos y$$

$$\therefore \quad \frac{dx}{dy} = \sqrt{1-\sin^2 y} \qquad \text{...[Using (i)]}$$

or

$$\frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \qquad \left[\because \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}\right]$$

Hence, $\dfrac{d}{dx}[\sin^{-1} x] = \dfrac{1}{\sqrt{1-x^2}}$

Similarly we can show that

$$\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$$

(ii) Derivative of $\tan^{-1} x$

Solution : Let $\tan^{-1} x = y$

$\therefore \qquad x = \tan y$

Differentiating w.r.t. y, $\dfrac{dx}{dy} = \sec^2 y$

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and
$$\frac{dy}{dx}=\frac{1}{\sec^2 y}$$

$$=\frac{1}{1+\tan^2 y} \qquad [\because \text{ We have written } \tan y \text{ in terms of } x]$$

$$=\frac{1}{1+x^2}$$

Hence, $$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

Similarly, $$\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}.$$

(iii) Derivative of $\sec^{-1}x$

Solution : Let $\sec^{-1}x=y$

$\therefore$ $x=\sec y$ and $\frac{dx}{dy}=\sec y\tan y$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\sec y\tan y}$$

$$=\frac{1}{\sec y\,[\pm\sqrt{\sec^2 y-1}]}$$

$$=\frac{1}{\pm\sec y\sqrt{\sec^2 y-1}}$$

$$=\frac{1}{|x|\sqrt{\sec^2 y-1}}$$

Note : (i) When $x>1$, sec y is + ve and tan y is + ve, $y\in\left(0,\frac{\pi}{2}\right)$

(ii) When $x<-1$, sec y is −ve and tan y is −ve, $y\in\left(\frac{\pi}{2},\pi\right)$

Hence, $$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$$

Similarly $$\frac{d}{dx}(\operatorname{cosec}^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}}$$

Example 22.13 Find the derivative of each of the following :

(i) $\sin^{-1}\sqrt{x}$ (ii) $\cos^{-1}x^2$ (iii) $(\operatorname{cosec}^{-1}x)^2$

Solution :

(i) Let $y=\sin^{-1}\sqrt{x}$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}} \frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$\therefore \qquad \frac{d}{dx} \sin^{-1} x = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

(ii) Let $y = \cos^{-1} x^2$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2)$$

$$= \frac{-1}{\sqrt{1-x^4}} \cdot (2x)$$

$$\therefore \qquad \frac{d}{dx}\left(\cos^{-1} x^2\right) = \frac{-2x}{\sqrt{1-x^4}}$$

(iii) Let $y = (\operatorname{cosec}^{-1} x)^2$

$$\frac{dy}{dx} = 2(\operatorname{cosec}^{-1} x) \cdot \frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right)$$

$$= 2(\operatorname{cosec}^{-1} x) \cdot \frac{-1}{|x|\sqrt{x^2-1}}$$

$$= \frac{-2\operatorname{cosec}^{-1} x}{|x|\sqrt{x^2-1}}$$

$$\therefore \qquad \frac{d}{dx}(\operatorname{cosec}^{-1} x)^2 = \frac{-2\operatorname{cosec}^{-1} x}{|x|\sqrt{x^2-1}}$$

Example 22.14 Find the derivative of each of the following :

(i) $\tan^{-1} \frac{\cos x}{1+\sin x}$ (ii) $\sin(2\sin^{-1} x)$

Solution :

Let (i) $y = \tan^{-1} \frac{\cos x}{1+\sin x}$

$$= \tan^{-1} \frac{\sin\left(\frac{\pi}{2} - x\right)}{1+\cos\left(\frac{\pi}{2} - x\right)}$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$$

$$= \frac{\pi}{4}-\frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = -1/2$$

(ii) $y = \sin(2\sin^{-1}x)$

Let $y = \sin(2\sin^{-1}x)$

$$\therefore \quad \frac{dy}{dx} = \cos(2\sin^{-1}x)\cdot\frac{d}{dx}(2\sin^{-1}x)$$

$$\therefore \quad \frac{dy}{dx} = \cos(2\sin^{-1}x)\cdot\frac{2}{\sqrt{1-x^2}}$$

$$= \frac{2\cos(2\sin^{-1}x)}{\sqrt{1-x^2}}$$

Example 22.15 Show that the derivative of $\tan^{-1}\frac{2x}{1-x^2}$ w.r.t $\sin^{-1}\frac{2x}{1+x^2}$ is 1.

Solution : Let $y = \tan^{-1}\frac{2x}{1-x^2}$ and $z = \sin^{-1}\frac{2x}{1+x^2}$

Let $x = \tan\theta$

$$\therefore \quad y = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} \text{ and } z = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$$

$$= \tan^{-1}(\tan 2\theta) \text{ and } z = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta \qquad \text{and} \qquad z = 2\theta$$

$$\frac{dy}{d\theta} = 2 \qquad \text{and} \qquad \frac{dz}{d\theta} = 2$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dz} = 2\cdot\frac{1}{2} = 1$$ (By chain rule)

CHECK YOUR PROGRESS 22.4

Find the derivative of each of the following functions w.r.t. x and express the result in the simplest form (1-3) :

1. (a) $\sin^{-1}x^2$ (b) $\cos^{-1}\frac{x}{2}$ (c) $\cos^{-1}\frac{1}{x}$

2. (a) $\tan^{-1}(\operatorname{cosec}x - \cot x)$ (b) $\cot^{-1}(\sec x + \tan x)$ (c) $\tan^{-1}\frac{\cos x - \sin x}{\cos x + \sin x}$

3. (a) $\sin(\cos^{-1}x)$ (b) $\sec(\tan^{-1}x)$ (c) $\sin^{-1}(1-2x^2)$

(d) $\cos^{-1}(4x^3-3x)$ (e) $\cot^{-1}\left(\sqrt{1+x^2}+x\right)$

4. Find the derivative of :

$\dfrac{\tan^{-1}x}{1+\tan^{-1}x}$ w.r.t. $\tan^{-1}x$.

22.5 SECOND ORDER DERIVATIVES

We know that the second order derivative of a function is the derivative of the first derivative of that function. In this section, we shall find the second order derivatives of trigonometric and inverse trigonometric functions. In the process, we shall be using product rule, quotient rule and chain rule.

Let us take some examples.

Example 22.16 Find the second order derivative of

(i) $\sin x$ (ii) $x\cos x$ (iii) $\cos^{-1}x$

Solution : (i) Let y = sin x

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx}=\cos x$$

Differentiating w.r.t x both sides again, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(\cos x)=-\sin x$$

$$\therefore \quad \frac{d^2y}{dx^2}=-\sin x$$

(ii) Let y = x cos x

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx}=x(-\sin x)+\cos x.1$$

$$\frac{dy}{dx}=-x\sin x+\cos x$$

Differentiating w.r.t. x both sides again, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(-x\sin x+\cos x\right)$$

$$=-\left(x.\cos x+\sin x\right)-\sin x$$

$$=-x.\cos x-2\sin x$$

$$\therefore \quad \frac{d^2y}{dx^2}=-\left(x.\cos x+2\sin x\right)$$

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(iii) Let $y = \cos^{-1} x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\left(1-x^2\right)^{1/2}} = -\left(1-x^2\right)^{-\frac{1}{2}}$$

Differentiating w.r.t. x both sides, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{-1}{2}\cdot\left(1-x^2\right)^{-3/2}\cdot(-2x)\right]$$

$$= -\frac{x}{\left(1-x^2\right)^{3/2}}$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{-x}{\left(1-x^2\right)^{3/2}}$$

Example 22.17 If $y = \sin^{-1} x$, show that $\left(1-x^2\right)y_2 - xy_1 = 0$, where y_2 and y_1 respectively denote the second and first, order derivatives of y w.r.t. x.

Solution : We have, $y = \sin^{-1} x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

or $$\left(\frac{dy}{dx}\right)^2 = \frac{1}{1-x^2} \qquad \text{(squaring both sides)}$$

or $$\left(1-x^2\right)y_1^2 = 0$$

Differentiating w.r.t. x both sides, we get

$$\left(1-x^2\right)\cdot 2y_1\frac{d}{dx}(y_1) + (-2x)\cdot y_1^2 = 0$$

or $$\left(1-x^2\right)\cdot 2y_1y_2 - 2x\,y_1^2 = 0$$

or $$\left(1-x^2\right)y_2 - x\,y_1 = 0$$

CHECK YOUR PROGRESS 22.5

1. Find the second order derivative of each of the following :

   (a) $\sin(\cos x)$ (b) $x^2 \tan^{-1} x$

2. If $y = \frac{1}{2}\left(\sin^{-1} x\right)^2$, show that $\left(1 - x^2\right)y_2 - xy_1 = 1$.

3. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0$.

4. If $y = x + \tan x$, show that $\cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$

LET US SUM UP

- (i) $\frac{d}{dx}(\sin x) = \cos x$ (ii) $\frac{d}{dx}(\cos x) = -\sin x$

  (iii) $\frac{d}{dx}(\tan x) = \sec^2 x$ (iv) $\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$

  (v) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (vi) $\frac{d}{dx}(\mathrm{cosec} x) = -\mathrm{cosec} x \cot x$

- If u is a derivabale function of x, then

  (i) $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$ (ii) $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$

  (iii) $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$ (iv) $\frac{d}{dx}(\cot u) = -\mathrm{cosec}^2 u \frac{du}{dx}$

  (v) $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$ (vi) $\frac{d}{dx}(\mathrm{coseu}) = -\mathrm{cosec}\, u \cot u \frac{du}{dx}$

- (i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$ (ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$

  (iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$ (iv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}$

  (v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$ (vi) $\frac{d}{dx}(\mathrm{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2 - 1}}$

- If u is a derivable function of x, then

  (i) $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \cdot \frac{du}{dx}$ (ii) $\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1 - u^2}} \cdot \frac{du}{dx}$

  (iii) $\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2} \cdot \frac{du}{dx}$ (iv) $\frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1 + u^2} \cdot \frac{du}{dx}$

  (v) $\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}} \cdot \frac{du}{dx}$ (vi) $\frac{d}{dx}(\mathrm{cosec}^{-1} u) = \frac{-1}{|u|\sqrt{u^2 - 1}} \cdot \frac{du}{dx}$

The second order derivative of a trignometric function is the derivative of their first order derivatives.

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. If $y = x^3 \tan^2 \frac{x}{2}$, find $\frac{dy}{dx}$.

2. Evaluate, $\frac{d}{dx}\sqrt{\sin^4 x + \cos^4 x}$ at $x = \frac{\pi}{2}$ and 0.

3. If $y = \frac{5x}{\sqrt[3]{(1-x)^2}} + \cos^2(2x+1)$, find $\frac{dy}{dx}$.

4. If $y = \sec^{-1}\frac{\sqrt{x}+1}{\sqrt{x}-1} + \sin^{-1}\frac{\sqrt{x}-1}{\sqrt{x}+1}$, then show that $\frac{dy}{dx} = 0$

5. If $x = a\cos^3\theta, y = a\sin^3\theta$, then find $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

6. If $y = \sqrt{x + \sqrt{x + \sqrt{x + ...}}}$, find $\frac{dy}{dx}$.

7. Find the derivative of $\sin^{-1} x$ w.r.t. $\cos^{-1}\sqrt{1-x^2}$

8. If $y = \cos(\cos x)$, prove that

$$\frac{d^2y}{dx^2} - \cot x \cdot \frac{dy}{dx} + y.\sin^2 x = 0.$$

9. If $y = \tan^{-1} x$ show that

$$(1+x)^2 y_2 + 2xy_1 = 0.$$

10. If $y = (\cos^{-1} x)^2$, show that

$$(1-x^2)y_2 - xy_1 - 2 = 0.$$

ANSWERS

CHECK YOUR PROGRESS 22.1

(1) (a) $-\operatorname{cosec} x \cot x$ (b) $-\operatorname{cosec}^2 x$ (c) $-2\sin 2x$

(d) $-2\operatorname{cosec}^2 2x$ (e) $-2x \operatorname{cosec} x^2 \cot x^2$ (f) $\dfrac{\cos x}{2\sqrt{\sin x}}$

2. (a) $2\sin 2x$ (b) $-2\operatorname{cosec}^2 x \cot x$ (c) $2\tan x \sec^2 x$

CHECK YOUR PROGRESS 22.2

1. (a) $12\cos 4x$ (b) $-5\sin 5x$ (c) $\dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$ (d) $\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$

(e) $2x\cos x^2$ (f) $2\sqrt{2}\sec^2 2x$ (g) $-3\pi \operatorname{cosec}^2 3x$

(h) $10\sec 10x \tan 10x$ (i) $-2\operatorname{cosec} 2x \cot 2x$

2. (a) $\dfrac{2\sec x \tan x}{(\sec x + 1)^2}$ (b) $\dfrac{-2}{(\sin x - \cos x)^2}$ (c) $x\cos x + \sin x$

(d) $2x\cos x - (1+x^2)\sin x$

(e) $\operatorname{cosec} x(1 - x\cot x)$ (f) $2\cos 2x \cos 3x - 3\sin 2x \sin 3x$ (g) $\dfrac{3\cos 3x}{2\sqrt{\sin 3x}}$

3. (a) $3\sin^2 x \cos x$ (b) $-\sin 2x$ (c) $4\tan^3 x \sec^2 x$ (d) $-4\cot^3 x \operatorname{cosec}^2 x$

(e) $5\sec^5 x \tan x$ (f) $-3\operatorname{cosec}^3 x \cot x$ (g) $\dfrac{\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}}$

(h) $\sec x(\sec x + \tan x)$

4. (a) 1 (b) $\sqrt{2} + 2$

CHECK YOUR PROGRESS 22.3

1. (i) $\dfrac{-2x}{\sqrt{1-x^4}}$ (ii) $\dfrac{-1}{x\sqrt{1-x^2}} - \dfrac{-\cos^{-1} x}{x^2}$ (iii) $\dfrac{-1}{2x^{\frac{1}{2}}\sqrt{(1-x)}}$

(iv) $\dfrac{2x}{1+x^4}$ (v) $\dfrac{1}{x(1+x^2)} - \dfrac{\tan^{-1} x}{x^2}$ (vi) $\dfrac{1}{2x^{\frac{1}{2}}(1+x)}$

CHECK YOUR PROGRESS 22.4

1. (a) $\dfrac{2x}{\sqrt{1-x^4}}$ (b) $\dfrac{-1}{\sqrt{4-x^2}}$ (c) $\dfrac{1}{x\sqrt{x^2-1}}$

2. (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) -1

3. (a) $-\frac{\cos\left(\cos^{-1} x\right)}{\sqrt{1-x^2}}$ (b) $\frac{x}{1+x^2}\cdot\sec\left(\tan^{-1} x\right)$

(c) $\frac{-2}{\sqrt{1-x^2}}$ (d) $\frac{-3}{\sqrt{1-x^2}}$ (e) $\frac{-1}{2(1+x^2)}$

4. $\frac{1}{\left(1+\tan^{-1} x\right)^2}$

CHECK YOUR PROGRESS 22.5

1. (a) $-\cos x\cos(\cos x)-\sin^2 x\sin(\cos x)$

(b) $\frac{2x(2+x^2)}{(1+x^2)^2}+2\tan^{-1} x$

TERMINAL EXERCISE

1. $x^3\tan\frac{x}{2}\sec^2\frac{x}{2}+3x^2\tan^2\frac{x}{2}$ 2. 0, 0

3. $\frac{5(3-x)}{3(1-x)^{\frac{5}{3}}}-2\sin(4x+2)$ 5. $|\sec\theta|$

6. $\frac{1}{2y-1}$ 7. $\frac{1}{2\sqrt{1-x^2}}$

23

DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

We are aware that population generally grows but in some cases decay also. There are many other areas where growth and decay are continuous in nature. Examples from the fields of Economics, Agriculture and Business can be cited, where growth and decay are continuous. Let us consider an example of bacteria growth. If there are 10,00,000 bacteria at present and say they are doubled in number after 10 hours, we are interested in knowing as to after how much time these bacteria will be 30,00,000 in number and so on.

Answers to the growth problem does not come from addition (repeated or otherwise), or multiplication by a fixed number. In fact Mathematics has a tool known as exponential function that helps us to find growth and decay in such cases. Exponential function is inverse of logarithmic function.

In this lesson, we propose to work with this tool and find the rules governing their derivatives.

OBJECTIVES

After studying this lesson, you will be able to :

- define and find the derivatives of exponential and logarithmic functions;
- find the derivatives of functions expressed as a combination of algebraic, trigonometric, exponential and logarithmic functions; and
- find second order derivative of a function.

EXPECTED BACKGROUND KNOWLEDGE

- Application of the following standard limits :

(i) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (ii) $\lim_{n\to\infty}(1+n)^{\frac{1}{n}} = e$

(iii) $\lim_{x\to\infty}\frac{e^x-1}{x} = 1$ (iv) $\lim_{x\to\infty}\frac{a^x-1}{x} = \log_e a$

(v) $\lim_{h\to 0}\frac{\left(e^h-1\right)}{h} = 1$

- Definition of derivative and rules for finding derivatives of functions.

23.1 DERIVATIVE OF EXPONENTIAL FUNCTIONS

Let $y = e^x$ be an exponential function. .....(i)

$\therefore \quad y + \delta y = e^{(x+\delta x)}$ (Corresponding small increments) .....(ii)

From (i) and (ii), we have

$$\therefore \quad \delta y = e^{x+\delta x} - e^x$$

Dividing both sides by δx and taking the limit as $\delta x \to 0$

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} e^x \frac{\left[e^{\delta x} - 1\right]}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x . 1 = e^x$$

Thus, we have $\mathbf{\frac{d}{dx}\left(e^x\right) = e^x}$.

Working rule : $\quad \frac{d}{dx}\left(e^x\right) = e^x \cdot \frac{d}{dx}(x) = e^x$

Next, let $\quad y = e^{ax+b}$.

Then $\quad y + \delta y = e^{a(x+\delta x)+b}$

[δx and δy are corresponding small increments]

$$\therefore \quad \delta y = e^{a(x+\delta x)+b} - e^{ax+b}$$

$$= e^{ax+b}\left[e^{a\delta x} - 1\right]$$

$$\therefore \quad \frac{\delta y}{\delta x} = e^{ax+b} \frac{\left[e^{a\delta x} - 1\right]}{\delta x}$$

$$= a \; e^{ax+b} \frac{e^{a\delta x} - 1}{a\delta x} \qquad \text{[Multiply and divide by a]}$$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = a \cdot e^{ax+b} \cdot \lim_{\delta x \to 0} \frac{e^{a\delta x} - 1}{a \; \delta x}$$

or $\quad \frac{dy}{dx} = a \cdot e^{ax+b} \cdot 1 \qquad \left[\lim_{x \to 0} \frac{e^x - 1}{x} = 1\right]$

$$= a e^{ax+b}$$

Working rule :

$$\frac{d}{dx}\left(e^{ax+b}\right) = e^{ax+b}\frac{d}{dx}(ax+b) = e^{ax+b}\cdot a$$

$$\therefore \quad \frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$$

Example 23.1 Find the derivative of each of the follwoing functions :

(i) e^{5x} (ii) e^{ax} (iii) $e^{-\frac{3x}{2}}$

Soution : (i) Let $y = e^{5x}$.

Then $y = e^t$ where $5x = t$

$\therefore \quad \frac{dy}{dt} = e^t$ and $5 = \frac{dt}{dx}$

We know that, $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = e^t \cdot 5 = 5e^{5x}$

Alternatively $\frac{d}{dx}\left(e^{5x}\right) = e^{5x}\cdot\frac{d}{dx}(5x) = e^{5x}\cdot 5 = 5e^{5x}$

(ii) Let $y = e^{ax}$.

Then $y = e^t$ when $t = ax$

$\therefore \quad \frac{dy}{dt} = e^t$ and $\frac{dt}{dx} = a$

We know that, $\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = e^t \cdot a$

Thus, $\frac{dy}{dx} = a\cdot e^{ax}$

(iii) Let $y = e^{\frac{-3x}{2}}$

$\therefore \quad \frac{dy}{dt} = e^{\frac{-3}{2}x}\,\frac{d}{dx}\left(\frac{-3}{2}x\right)$

Thus, $\frac{dy}{dt} = \frac{-3}{2}e^{\frac{-3x}{2}}$

Example 23.2 Find the derivative of each of the following :

(i) $y = e^x + 2\cos x$ (ii) $y = e^{x^2} + 2\sin x - \frac{5}{3}e^x + 2e$

Solution : (i) $y = e^x + 2\cos x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(e^x\right) + 2\frac{d}{dx}\left(\cos x\right) = e^x - 2\sin x$$

(ii) $$y = e^{x^2} + 2\sin x - \frac{5}{3}e^x + 2e$$

$$\therefore \quad \frac{dy}{dx} = e^{x^2}\frac{d}{dx}\left(x^2\right) + 2\cos x - \frac{5}{3}e^x + 0 \quad \text{[Since e is constant]}$$

$$= 2xe^{x^2} + 2\cos x - \frac{5}{3}e^x$$

Example 23.3 Find $\frac{dy}{dx}$, when

(i) $y = e^{x\cos x}$ (ii) $y = \frac{1}{x}e^x$ (iii) $y = e^{\frac{1-x}{1+x}}$

Solution : (i) $y = e^{x\cos x}$

$$\therefore \quad \frac{dy}{dx} = e^{x\cos x}\frac{d}{dx}\left(x\cos x\right)$$

$$\therefore \quad \frac{dy}{dx} = e^{x\cos x}\left[x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}(x)\right]$$

$$= e^{x\cos x}\left[-x\sin x + \cos x\right]$$

(ii) $$y = \frac{1}{x}e^x$$

$$\therefore \quad \frac{dy}{dx} = e^x\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\frac{d}{dx}\left(e^x\right) \qquad \text{[Using product rule]}$$

$$= \frac{-1}{x^2}e^x + \frac{1}{x}e^x$$

$$= \frac{e^x}{x^2}[-1 + x] = \frac{e^x}{x^2}[x - 1]$$

(iii) $$y = e^{\frac{1-x}{1+x}}$$

$$\frac{dy}{dx} = e^{\frac{1-x}{1+x}}\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$= e^{\frac{1-x}{1+x}}\left[\frac{-1.(1+x) - (1-x).1}{(1+x)^2}\right]$$

$$= e^{\frac{1-x}{1+x}}\left[\frac{-2}{(1+x)^2}\right] = \frac{-2}{(1+x)^2}e^{\frac{1-x}{1+x}}$$

Example 23.4 Find the derivative of each of the following functions :

(i) $e^{\sin x} \cdot \sin e^x$ (ii) $e^{ax} \cdot \cos(bx + c)$

Solution : $y = e^{\sin x} \cdot \sin e^x$

$$\therefore \quad \frac{dy}{dx} = e^{\sin x} \cdot \frac{d}{dx}\left(\sin e^x\right) + \sin e^x \frac{d}{dx} e^{\sin x}$$

$$= e^{\sin x} \cdot \cos e^x \cdot \frac{d}{dx}\left(e^x\right) + \sin e^x \cdot e^{\sin x} \frac{d}{dx}(\sin x)$$

$$= e^{\sin x} \cdot \cos e^x \cdot e^x + \sin e^x \cdot e^{\sin x} \cdot \cos x$$

$$= e^{\sin x}[e^x \cdot \cos e^x + \sin e^x \cdot \cos x]$$

(ii) $y = e^{ax} \cos(bx + c)$

$$\therefore \quad \frac{dy}{dx} = e^{ax} \cdot \frac{d}{dx}\cos(bx + c) + \cos(bx + c)\frac{d}{dx}e^{ax}$$

$$= e^{ax} \cdot [-\sin(bx + c)]\frac{d}{dx}(bx + c) + \cos(bx + c)e^{ax}\frac{d}{dx}(ax)$$

$$= -e^{ax}\sin(bx + c) \cdot b + \cos(bx + c)e^{ax}.a$$

$$= e^{ax}[-b\sin(bx + c) + a\cos(bx + c)]$$

Example 23.5 Find $\frac{dy}{dx}$, if $y = \frac{e^{ax}}{\sin(bx + c)}$

Solution :

$$\frac{dy}{dx} = \frac{\sin(bx + c)\frac{d}{dx}e^{ax} - e^{ax}\frac{d}{dx}[\sin(bx + c)]}{\sin^2(bx + c)}$$

$$= \frac{\sin(bx + c).e^{ax}.a - e^{ax}\cos(bx + c).b}{\sin^2(bx + c)}$$

$$= \frac{e^{ax}[a\sin(bx + c) - b\cos(bx + c)]}{\sin^2(bx + c)}$$

CHECK YOUR PROGRESS 23.1

1. Find the derivative of each of the following functions :

(a) e^{5x} (b) e^{7x+4} (c) $e^{\sqrt{2}x}$ (d) $e^{\frac{-7}{2}x}$ (e) e^{x^2+2x}

2. Find $\frac{dy}{dx}$, if

(a) $y=\frac{1}{3}e^{x}-5e$ (b) $y=\tan x+2\sin x+3\cos x-\frac{1}{2}e^{x}$

(c) $y=5\sin x-2e^{x}$ (d) $y=e^{x}+e^{-x}$

3. Find the derivative of each of the following functions :

(a) $f(x)=e^{\sqrt{x+1}}$ (b) $f(x)=e^{\sqrt{\cot x}}$

(c) $f(x)=e^{x\sin^2 x}$ (d) $f(x)=e^{x\sec^2 x}$

4. Find the derivative of each of the following functions :

(a) $f(x)=(x-1)e^{x}$ (b) $f(x)=e^{2x}\sin^2 x$

5. Find $\frac{dy}{dx}$, if

(a) $y=\frac{e^{2x}}{\sqrt{x^2+1}}$ (b) $y=\frac{e^{2x}\cdot\cos x}{x\sin x}$

23.2 DERIVATIVE OF LOGARITHMIC FUNCTIONS

We first consider logarithmic function

Let $y=\log x$ .....(i)

$\therefore\quad y+\delta y=\log(x+\delta x)$ .....(ii)

(δx and δy are corresponding small increments in x and y)

From (i) and (ii), we get

$$\delta y=\log(x+\delta x)-\log x$$

$$=\log\frac{x+\delta x}{x}$$

$$\therefore\quad \frac{\delta y}{\delta x}=\frac{1}{\delta x}\log\left[1+\frac{\delta x}{x}\right]$$

$$=\frac{1}{x}\cdot\frac{x}{\delta x}\log\left[1+\frac{\delta x}{x}\right]$$ [Multiply and divide by x]

$$=\frac{1}{x}\log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

Taking limits of both sides, as $\delta x\to 0$, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{1}{x}\lim_{\delta x\to 0}\log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

$$\frac{dy}{dx}=\frac{1}{x}\cdot\log\left\{\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right\}$$

$$=\frac{1}{x}\log e \qquad \left[\because \lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}=e\right]$$

$$=\frac{1}{x}$$

Thus, $$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\log x})=\frac{\mathbf{1}}{\mathbf{x}}$$

Next, we consider logarithmic function

$$y=\log(ax+b) \qquad ...(i)$$

$\therefore$ $$y+\delta y=\log[a(x+\delta x)+b] \qquad ...(ii)$$

[δx and δy are corresponding small increments]

From (i) and (ii), we get

$$\delta y=\log[a(x+\delta x)+b]-\log(ax+b)$$

$$=\log\frac{a(x+\delta x)+b}{ax+b}$$

$$=\log\frac{(ax+b)+a\delta x}{ax+b}$$

$$=\log\left[1+\frac{a\delta x}{ax+b}\right]$$

$\therefore$ $$\frac{\delta y}{\delta x}=\frac{1}{\delta x}\log\left[1+\frac{a\delta x}{ax+b}\right]$$

$$=\frac{a}{ax+b}\cdot\frac{ax+b}{a\delta x}\log\left[1+\frac{a\delta x}{ax+b}\right] \qquad \left[\text{Multiply and divide by } \frac{a}{ax+b}\right]$$

$$=\frac{a}{ax+b}\log\left[1+\frac{a\delta x}{ax+b}\right]^{\frac{ax+b}{a\delta x}}$$

Taking limits on both sides as $\delta x\to 0$

$\therefore$ $$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{a}{ax+b}\lim_{\delta x\to 0}\log\left[1+\frac{a\delta x}{ax+b}\right]^{\frac{ax+b}{a\delta x}}$$

or $$\frac{dy}{dx}=\frac{a}{ax+b}\log e \qquad \left[\because \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e\right]$$

or, $$\frac{dy}{dx}=\frac{a}{ax+b}$$

Working rule :

$$\frac{d}{dx}\log(ax+b)=\frac{1}{ax+b}\frac{d}{dx}(ax+b)$$

$$=\frac{1}{ax+b}\times a=\frac{a}{ax+b}$$

Example 23.6 Find the derivative of each of the functions given below :

(i) $y=\log x^5$ (ii) $y=\log\sqrt{x}$ (iii) $y=(\log x)^3$

Solution : (i) $y=\log x^5=5\log x$

$\therefore$ $$\frac{dy}{dx}=5\cdot\frac{1}{x}=\frac{5}{x}$$

(ii) $$y=\log\sqrt{x}=\log x^{\frac{1}{2}} \quad \text{or} \quad y=\frac{1}{2}\log x$$

$\therefore$ $$\frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{x}=\frac{1}{2x}$$

(iii) $$y=(\log x)^3$$

$\therefore$ $y=t^3$, when $t=\log x$

$\Rightarrow$ $$\frac{dy}{dt}=3t^2 \text{ and } \frac{dt}{dx}=\frac{1}{x}$$

We know that, $$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=3t^2\cdot\frac{1}{x}$$

$\therefore$ $$\frac{dy}{dx}=3(\log x)^2\cdot\frac{1}{x}$$

$\therefore$ $$\frac{dy}{dx}=\frac{3}{x}(\log x)^2$$

Example 23.7 Find, $\frac{dy}{dx}$ if

(i) $y=x^3\log x$ (ii) $y=e^x\log x$

Solution :

(i) $$y=x^3\log x$$

$\therefore$ $$\frac{dy}{dx}=\log x\frac{d}{dx}(x^3)+x^3\frac{d}{dx}(\log x)$$ [Using Product rule]

$$=3x^2\log x+x^3\cdot\frac{1}{x}$$

$$=x^2(3\log x+1)$$

(ii) $$y = e^x \log x$$

$$\therefore \quad \frac{dy}{dx} = e^x \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx} e^x$$

$$= e^x \cdot \frac{1}{x} + e^x \cdot \log x$$

$$= e^x \left[\frac{1}{x} + \log x\right]$$

Example 23.8 Find the derivative of each of the following functions :

(i) log tan x (ii) log [cos (log x)]

Solution : (i) Let $y = \log \tan x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\tan x} \frac{d}{dx}(\tan x)$$

$$= \frac{1}{\tan x} \cdot \sec^2 x$$

$$= \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}$$

$$= \operatorname{cosec} x . \sec x$$

(ii) Let $y = \log [\cos (\log x)]$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos(\log x)} \cdot \frac{d}{dx}[\cos(\log x)]$$

$$= \frac{1}{\cos(\log x)} \left[- \sin \log x \frac{d}{dx}(\log x)\right]$$

$$= \frac{-\sin(\log x)}{\cos(\log x)} \cdot \frac{1}{x}$$

$$= -\frac{1}{x} \tan(\log x)$$

Example 23.9 Find $\frac{dy}{dx}$, if $y = \log(\sec x + \tan x)$

Solution : $y = \log (\sec x + \text{tax } x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{1}{\sec x + \tan x} \cdot \left[\sec x \tan x + \sec^2 x\right]$$

$$= \frac{1}{\sec x + \tan x} \cdot \sec x \left[\sec x + \tan x\right]$$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

$$= \sec x$$

Example 23.10 Find $\frac{dy}{dx}$, if

$$y = \frac{(4x^2 - 1)(1 + x^2)^{\frac{1}{2}}}{x^3 (x - 7)^{\frac{3}{4}}}$$

Solution : Although, you can find the derivative directly using quotient rule (and product rule) but if you take logarithm on both sides, the product changes to addition and division changes to subtraction. This simplifies the process:

$$y = \frac{(4x^2 - 1)(1 + x^2)^{\frac{1}{2}}}{x^3 (x - 7)^{\frac{3}{4}}}$$

Taking logarithm on both sides, we get

$$\therefore \qquad \log y = \log\left[\frac{\left(4x^2 - 1\right)\left(1 + x^2\right)^{\frac{1}{2}}}{x^3 \left(x - 7\right)^{\frac{3}{4}}}\right]$$

or $$\log y = \log\left(4x^2 - 1\right) + \frac{1}{2}\log\left(1 + x^2\right) - 3\log x - \frac{3}{4}\log(x - 7)$$

[Using log properties]

Now, taking derivative on both sides, we get

$$\frac{d}{dx}(\log y) = \frac{1}{4x^2 - 1} \cdot 8x + \frac{1}{2(1 + x^2)} \cdot 2x - \frac{3}{x} - \frac{3}{4} \cdot \left(\frac{1}{x - 7}\right)$$

$$\Rightarrow \qquad \frac{1}{y} \cdot \frac{dy}{dx} = \frac{8x}{4x^2 - 1} + \frac{x}{1 + x^2} - \frac{3}{x} - \frac{3}{4(x - 7)}$$

$$\therefore \qquad \frac{dy}{dx} = y\left[\frac{8x}{4x^2 - 1} + \frac{x}{1 + x^2} - \frac{3}{x} - \frac{3}{4(x - 7)}\right]$$

$$= \frac{(4x^2 - 1)\sqrt{1 + x^2}}{x^3 (x - 7)^{\frac{3}{4}}}\left[\frac{8x}{4x^2 - 1} + \frac{x}{1 + x^2} \quad \frac{3}{x} \quad \frac{3}{4(x - 7)}\right]$$

CHECK YOUR PROGRESS 23.2

1. Find the derivative of each the functions given below:

   (a) $f(x) = 5 \sin x \;\square\; 2 \log x$ (b) $f(x) = \log \cos x$

2. Find $\frac{dy}{dx}$, if

   (a) $y = e^{x^2} \log x$ (b) $y = \frac{e^{x^2}}{\log x}$

3. Find the derivative of each of the following functions :

   (a) $y = \log(\sin \log x)$ (b) $y = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

   (c) $y = \log\left[\frac{a + b\tan x}{a - b\tan x}\right]$ (d) $y = \log(\log x)$

4. Find $\frac{dy}{dx}$, if

   (a) $y = (1+x)^{\frac{1}{2}}(2-x)^{\frac{2}{3}}(x^2+5)^{\frac{1}{7}}(x-9)^{-\frac{3}{2}}$ (b) $y = \frac{\sqrt{x}(1-2x)^{\frac{3}{2}}}{(3+4x)^{\frac{5}{4}}(3-7x^2)^{\frac{1}{4}}}$

23.3 DERIVATIVE OF LOGARITHMIC FUNCTION (CONTINUED)

We know that derivative of the function x^n w.r.t. x is nx^{n-1}, where n is a constant. This rule is not applicable, when exponent is a variable. In such cases we take logarithm of the function and then find its derivative.

Therefore, this process is useful, when the given function is of the type $[f(x)]^{g(x)}$. For example, a^x, x^x etc.

Note : Here f(x) may be constant.

Derivative of a^x w.r.t. x

Let $y = a^x$, $a > 0$

Taking log on both sides, we get

$$\log y = \log a^x = x \log a \qquad [\log m^n = n \log m]$$

$$\therefore \quad \frac{d}{dx}(\log y) = \frac{d}{dx}(x \log a) \quad \text{or} \quad \frac{1}{y}\cdot\frac{dy}{dx} = \log a \times \frac{d}{dx}(x)$$

$$\text{or} \quad \frac{dy}{dx} = y \log a$$

$$= a^x \log a$$

Thus, $\quad \mathbf{\frac{d}{dx}a^x = a^x \log a}, \qquad a > 0$

Example 23.11 Find the derivative of each of the following functions :

(i) $y = x^x$ (ii) $y = x^{\sin x}$

Solution : (i) $y = x^x$

Taking logrithms on both sides, we get

$$\log y = x \log x$$

Taking derivative on both sides, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log x\frac{d}{dx}(x) + x\frac{d}{dx}(\log x) \qquad \text{[Using product rule]}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1\cdot\log x + x\cdot\frac{1}{x}$$

$$= \log x + 1$$

$\therefore \quad \frac{dy}{dx} = y[\log x + 1]$

Thus, $\quad \frac{dy}{dx} = x^x(\log x + 1)$

(ii) $\quad y = x^{\sin x}$

Taking logarithm on both sides, we get

$$\log y = \sin x \log x$$

$\therefore \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(\sin x \log x)$

or $\quad \frac{1}{y}\cdot\frac{dy}{dx} = \cos x . \log x + \sin x\cdot\frac{1}{x}$

or $\quad \frac{dy}{dx} = y\left[\cos x \log x + \frac{\sin x}{x}\right]$

Thus, $\quad \frac{dy}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right]$

Example 23.12 Find the derivative, if

$$y = (\log x)^x + \left(\sin^{-1} x\right)^{\sin x}$$

Solution : Here taking logarithm on both sides will not help us as we cannot put

$(\log x)^x + (\sin^{-1} x)^{\sin x}$ in simpler form. So we put

$$u = (\log x)^x \qquad \text{and} \qquad v = (\sin^{-1} x)^{\sin x}$$

Then, $\qquad y = u + v$

$$\therefore \qquad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{......(i)}$$

Now $\qquad u = (\log x)^x$

Taking log on both sides, we have

$$\log u = \log(\log x)^x$$

$$\therefore \qquad \log u = x \log(\log x) \qquad \left[\because \log m^n = n \log m\right]$$

Now, finding the derivative on both sides, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = 1 \cdot \log(\log x) + x \frac{1}{\log x} \cdot \frac{1}{x}$$

Thus, $$\frac{du}{dx} = u\left[\log(\log x) + \frac{1}{\log x}\right]$$

Thus, $$\frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right] \qquad \text{.....(ii)}$$

Also, $\qquad v = (\sin^{-1} x)^{\sin x}$

$$\therefore \qquad \log v = \sin x \log(\sin^{-1} x)$$

Taking derivative on both sides, we have

$$\frac{d}{dx}(\log v) = \frac{d}{dx}[\sin x \log(\sin^{-1} x)]$$

$$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} + \cos x \cdot \log\left(\sin^{-1} x\right)$$

or, $$\frac{dv}{dx} = v\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \cdot \log \sin^{-1} x\right]$$

$$= \left(\sin^{-1} x\right)^{\sin x}\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \log\left(\sin^{-1} x\right)\right] \qquad \text{.....(iii)}$$

From (i), (ii) and (iii), we have

$$\frac{dy}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right] + \left(\sin^{-1} x\right)^{\sin x}\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \log \sin^{-1} x\right]$$

Example 23.13 If $x^y = e^{x-y}$, prove that

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

Solution : It is given that $x^y = e^{x-y}$ ...(i)

Taking logarithm on both sides, we get

$$y \log x = (x-y)\log e$$
$$= (x-y)$$

or $$y(1+\log x) = x \quad [\because \log e = 1]$$

or $$y = \frac{x}{1+\log x} \qquad \text{(ii)}$$

Taking derivative with respect to x on both sides of (ii), we get

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x\left(\frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

Example 23.14 Find , $\frac{dy}{dx}$ if

$$e^x \log y = \sin^{-1}x + \sin^{-1}y$$

Solution : We are given that

$$e^x \log y = \sin^{-1}x + \sin^{-1}y$$

Taking derivative with respect to x of both sides, we get

$$e^x\left(\frac{1}{y}\frac{dy}{dx}\right) + e^x \log y = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$$

or $$\left[\frac{e^x}{y} - \frac{1}{\sqrt{1-y^2}}\right]\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - e^x \log y$$

or $$\frac{dy}{dx} = \frac{y\sqrt{1-y^2}\left[1 - e^x\sqrt{1-x^2}\log y\right]}{\left[e^x\sqrt{1-y^2} - y\right]\sqrt{1-x^2}}$$

Example 23.15 Find $\frac{dy}{dx}$, if $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\ldots\infty}}$

Solution : We are given that

$$y = (\cos x)^{(\cos x)^{(\cos x)\ldots\ldots\infty}} = (\cos x)^y$$

Taking logarithm on both sides, we get

$$\log y = y \log \cos x$$

Differentiating (i) w.r.t.x, we get

$$\frac{1}{y}\frac{dy}{dx} = y\,\frac{1}{\cos x}(-\sin x) + \log(\cos x)\cdot\frac{dy}{dx}$$

or $$\left[\frac{1}{y} - \log(\cos x)\right]\frac{dy}{dx} = -y\tan x$$

or $$\left[1 - y\log(\cos x)\right]\frac{dy}{dx} = -y^2\tan x$$

or $$\frac{dy}{dx} = \frac{-y^2\tan x}{1 - y\log(\cos x)}$$

CHECK YOUR PROGRESS 23.3

1. Find the derivative with respect to x of each the following functions :

   (a) $y = 5^x$ (b) $y = 3^x + 4^x$ (c) $y = \sin(5^x)$

2. Find $\frac{dy}{dx}$, if

   (a) $y = x^{2x}$ (b) $y = (\cos x)^{\log x}$ (c) $y = (\log x)^{\sin x}$

   (d) $y = (\tan x)^x$ (e) $y = (1 + x^2)^{x^2}$ (f) $y = x^{(x^2 + \sin x)}$

3. Find the derivative of each of the functions given below :

   (a) $y = (\tan x)^{\cot x} + (\cot x)^x$ (b) $y = x^{\log x} + (\sin x)^{\sin^{-1} x}$

   (c) $y = x^{\tan x} + (\sin x)^{\cos x}$ (d) $y = (x)^{x^2} + (\log x)^{\log x}$

4. If $y = (\sin x)^{(\sin x)^{(\sin x)\ldots\ldots\infty}}$, show that

   $$\frac{dy}{dx} = \frac{y^2\cot x}{1 - y\log(\sin x)}$$

5. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\infty}}}$, show that

   $$\frac{dy}{dx} = \frac{1}{x(2x-1)}$$

23.4 SECOND ORDER DERIVATIVES

In the previous lesson we found the derivatives of second order of trigonometric and inverse trigonometric functions by using the formulae for the derivatives of trigonometric and inverse trigonometric functions, various laws of derivatives, including chain rule, and power rule discussed earlier in lesson 21. In a similar manner, we will discuss second order derivative of exponential and logarithmic functions :

Example 23.16 Find the second order derivative of each of the following :

(i) e^x (ii) $\cos(\log x)$ (iii) x^x

Solution : (i) Let $y = e^x$

Taking derivative w.r.t. x on both sides, we get $\dfrac{dy}{dx} = e^x$

Taking derivative w.r.t. x on both sides, we get $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(e^x) = e^x$

$$\therefore \qquad \frac{d^2y}{dx^2} = e^x$$

(ii) Let $y = \cos(\log x)$

Taking derivative w.r.t x on both sides, we get

$$\frac{dy}{dx} = -\sin(\log x)\cdot\frac{1}{x} = \frac{-\sin(\log x)}{x}$$

Taking derivative w.r.t. x on both sides, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\frac{\sin(\log x)}{x}\right]$$

or
$$= -\frac{x\cdot\cos(\log x)\cdot\frac{1}{x} - \sin(\log x)}{x^2}$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{\sin(\log x) - \cos(\log x)}{x^2}$$

(iii) Let $y = x^x$

Taking logarithm on both sides, we get

$$\log y = x\log x \qquad ....(i)$$

Taking derivative w.r.t. x of both sides, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x = 1 + \log x$$

or
$$\frac{dy}{dx} = y(1 + \log x) \qquad ....(ii)$$

Taking derivative w.r.t. x on both sides we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[y(1+\log x)]$$

$$= y \cdot \frac{1}{x} + (1+\log x)\frac{dy}{dx} \qquad \text{...(iii)}$$

$$= \frac{y}{x} + (1+\log x)y(1+\log x)$$

$$= \frac{y}{x} + (1+\log x)^2 y \qquad \text{(Using (ii))}$$

$$= y\left[\frac{1}{x} + (1+\log x)^2\right]$$

$$\therefore \quad \frac{d^2y}{dx^2} = x^x\left[\frac{1}{x} + (1+\log x)^2\right]$$

Example 23.17 If $y = e^{a\cos^{-1}x}$, show that

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0 \cdot$$

Solution : We have, $y = e^{a\cos^{-1}x}$ .....(i)

$$\therefore \quad \frac{dy}{dx} = e^{a\cos^{-1}x}\frac{-a}{\sqrt{1-x^2}}$$

$$= -\frac{ay}{\sqrt{1-x^2}} \qquad \text{Using (i)}$$

or $$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2\left(1-x^2\right) - a^2y^2 = 0 \qquad \text{.....(ii)}$$

Taking derivative of both sides of (ii), we get

$$\left(\frac{dy}{dx}\right)^2(-2x) + 2\left(1-x^2\right)\times\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - a^2\cdot 2y\cdot\frac{dy}{dx} = 0$$

or $$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$ [Dividing through out by $2\cdot\frac{dy}{dx}$]

CHECK YOUR PROGRESS 23.4

1. Find the second order derivative of each of the following :

(a) $x^4 e^{5x}$ (b) $\tan\left(e^{5x}\right)$ (c) $\dfrac{\log x}{x}$

2. If $y = a\cos(\log x) + b\sin(\log x)$, show that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

3. If $y = e^{\tan^{-1} x}$, prove that

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$

LET US SUM UP

- (i) $\dfrac{d}{dx}(e^x) = e^x$ (ii) $\dfrac{d}{dx}(a^x) = a^x \log a \;;\; a > 0$
- If μ is a derivable function of x, then

  (i) $\dfrac{d}{dx}(e^{\mu}) = e^{\mu} \cdot \dfrac{d\mu}{dx}$ (ii) $\dfrac{d}{dx}(a^{\mu}) = a^{\mu} \cdot \log a \cdot \dfrac{d\mu}{dx} \;; a > 0$

  (iii) $\dfrac{d}{dx}(e^{ax+b}) = e^{ax+b} \cdot a = ae^{ax+b}$
- (i) $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$ (ii) $\dfrac{d}{dx}(\log \mu) = \dfrac{1}{\mu} \cdot \dfrac{d\mu}{dx}$, if μ is a derivable function of x.
- (iii) $\dfrac{d}{dx}\log(ax+b) = \dfrac{1}{ax+b} \cdot a = \dfrac{a}{ax+b}$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. Find the derivative of each of the following functions :

(a) $(x^x)^x$ (b) $x^{\left(x^x\right)}$

2. Find $\frac{dy}{dx}$, if

(a) $y = a^{x \log \sin x}$ (b) $y = (\sin x)^{\cos^{-1} x}$

(c) $y = \left(1 + \frac{1}{x}\right)^{x^2}$ (d) $y = \log\left[e^x \left(\frac{x-4}{x+4}\right)^{\frac{3}{4}}\right]$

3. Find the derivative of each of the functions given below :

(a) $f(x) = \cos x \log(x) e^{x^2} x^x$ (b) $f(x) = (\sin^{-1} x)^2 \cdot x^{\sin x} \cdot e^{2x}$

4. Find the derivative of each of the following functions :

(a) $y = (\tan x)^{\log x} + (\cos x)^{\sin x}$ (b) $y = x^{\tan x} + (\sin x)^{\cos x}$

5. Find $\frac{dy}{dx}$, if

(a) If $y = \frac{x^4 \sqrt{x+6}}{(3x+5)^2}$ (b) If $y = \frac{e^x + e^{-x}}{(e^x - e^{-x})}$

6. Find $\frac{dy}{dx}$, if

(a) If $y = a^x \cdot x^a$ (b) $y = 7^{x^2+2x}$

7. Find the derivative of each of the following functions :

(a) $y = x^2 e^{2x} \cos 3x$ (b) $y = \frac{2^x \cot x}{\sqrt{x}}$

8. If $y = x^{x^{x^{x \cdots \infty}}}$, prove that $x\frac{dy}{dx} = \frac{y^2}{1 - y \log x}$

Notes

ANSWERS

CHECK YOUR PROGRESS 23.1

1. (a) $5e^{5x}$ (b) $7e^{7x+4}$ (c) $\sqrt{2}e^{\sqrt{2}x}$ (d) $-\frac{7}{2}e^{-\frac{7}{2}x}$ (e) $2(x+1)e^{x^2+2x}$

2. (a) $\frac{1}{3}e^x$ (b) $\sec^2 x+2\cos x-3\sin x-\frac{1}{2}e^x$

   (c) $5\cos x-2e^x$ (d) e^x-e^{-x}

3. (a) $\frac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$ (b) $e^{\sqrt{\cot x}}\left[\frac{-\operatorname{cosec}^2 x}{2\sqrt{\cot x}}\right]$

   (c) $e^{x\sin^2 x}[\sin x+2x\cos x]\sin x$

   (d) $e^{x\sec^2 x}\left[\sec^2 x+2x\sec^2 x\tan x\right]$

4. (a) xe^x (b) $2e^{2x}\sin x(\sin x+\cos x)$

5. (a) $\frac{2x^2-x+2}{(x^2+1)^{3/2}}e^{2x}$ (b) $\frac{e^{2x}[(2x-1)\cot x-x\operatorname{cosec}^2 x]}{x^2}$

CHECK YOUR PROGRESS 23.2

1. (a) $5\cos x-\frac{2}{x}$ (b) $-\tan x$

2. (a) $e^{x^2}\left[2x\log x+\frac{1}{x}\right]$ (b) $\frac{2x^2\log x-1}{x(\log x)^2}.e^{x^2}$

3. (a) $\frac{\cot(\log x)}{x}$ (b) $\sec x$

   (c) $\frac{2ab\sec^2 x}{a^2-b^2\tan^2 x}$ (d) $\frac{1}{x\log x}$

4. (a) $(1+x)^{\frac{1}{2}}(2-x)^{\frac{2}{3}}(x^2+5)^{\frac{1}{7}}(x+9)^{-\frac{3}{2}}\times\left[\frac{1}{2(1+x)}-\frac{2}{3(2-x)}+\frac{2x}{7(x^2-5)}-\frac{3}{2(x+9)}\right]$

   (b) $\frac{\sqrt{x}(1-2x)^{\frac{3}{2}}}{(3+4x)^{\frac{5}{4}}(3-7x^2)^{\frac{1}{4}}}\left[\frac{1}{2x}-\frac{3}{1-2x}-\frac{5}{3+4x}+\frac{7x}{2(3-7x^2)}\right]$

CHECK YOUR PROGRESS 23.3

1. (a) $5^x \log 5$ (b) $3^x \log 3 + 4^x \log 4$ (c) $\cos 5^x \, 5^x \log 5$

2. (a) $2x^{2x}(1+\log x)$ (b) $(\cos x)^{\log x}\left[\dfrac{\log\cos x}{x} - \tan x \log x\right]$

(c) $(\log x)^{\sin x}\left[\cos x \log(\log x) + \dfrac{\sin x}{x \log x}\right]$

(d) $(\tan x)^x\left[\log\tan x + \dfrac{x}{\sin x \cos x}\right]$

(e) $(1+x)^{x^2}\left[2x\log(1+x^2) + 2\dfrac{x^3}{1+x^2}\right]$

(f) $x^{(x^2+\sin x)}\left[\dfrac{x^2+\sin x}{x} + (2x+\cos x)\log x\right]$

3. (a) $\text{cosec}^2 x\,(1-\log\tan x)(\tan x)^{\cot x} + \left(\log\cot x - x\,\text{cosec}^2 x \tan x\right)(\cot x)^x$

(b) $2x^{(\log x - 1)}\log x + (\sin x)^{\sin^{-1}x}\left[\cot x \sin^{-1} x + \dfrac{\log\sin x}{\sqrt{1-x^2}}\right]$

(c) $x^{\tan x}\left(\dfrac{\tan x}{x} + \sec^2 x \log x\right) + (\sin x)^{\cos x}[\cos x \cot x - \sin x \log\sin x]$

(d) $(x)^{x^2}\cdot x(1+2\log x) + (\log x)^{\log x}\left[\dfrac{1+\log(\log x)}{x}\right]$

CHECK YOUR PROGRESS 23.4

1. (a) $e^{5x}\left(25x^4 + 40x^3 + 12x^2\right)$ (b) $25e^{5x}\sec^2\left(e^{5x}\right)\left\{1 + 2e^{5x}\tan e^{5x}\right\}$

(c) $\dfrac{2\log x - 3}{x^3}$

TERMINAL EXERCISE

1. (a) $(x^x)^x[x + 2x\log x]$ (b) $x^{(x)^x}[x^{x-1} + \log x(\log x + 1)x^x]$

2. (a) $a^{x\log\sin x}[\log\sin x + x\cot x]\log a$

(b) $(\sin x)^{\cos^{-1}x}\left[\cos^{-1} x \cot x - \dfrac{\log\sin x}{\sqrt{1-x^2}}\right]$

(c) $\left(1+\frac{1}{x}\right)^{x^2}\left[2x\log\left(x+\frac{1}{x}\right)-\frac{1}{1+\frac{1}{x}}\right]$ (d) $1+\frac{3}{4(x-4)}-\frac{3}{4(x+4)}$

3. (a) $\cos x\log(x)e^{x^2}\cdot x^x\left[\tan x+\frac{1}{x\log x}+2x+1+\log x\right]$

(b) $(\sin^{-1}x)^2.x^{\sin x}e^{2x}\left[\frac{2}{\sqrt{1-x^2}\sin^{-1}x}+\cos x\log x+\frac{\sin x}{x}+2\right]$

4. (a) $(\tan x)^{\log x}\left[2\operatorname{cosec}2x\log x+\frac{1}{x}\log\tan x\right]$

$+(\cos x)^{\sin x}[-\sin x\tan x+\cos x\log(\cos x)]$

(b) $x^{\tan x}\left[\frac{\tan x}{x}+\sec^2 x\log x\right]+(\sin x)^{\cos x}[\cot x\cos x-\sin x\log\sin x)]$

5. (a) $\frac{x^4\sqrt{x+6}}{(3x+5)^2}\left[\frac{4}{x}+\frac{1}{2(x+6)}-\frac{6}{(3x+5)}\right]$ (b) $\frac{-4e^{2x}}{(e^{2x}-1)^2}$

6. (a) $a^x.x^{a-1}[a+x\log_e a]$ (b) $7^{x^2+2x}(2x+2)\log_e 7$

7. (a) $x^2e^{2x}\cos 3x\left\{\frac{2}{x}+2-3\tan 3x\right\}$

(b) $\frac{2^x\cot x}{\sqrt{x}}\left[\log 2-2\operatorname{cosec}2x-\frac{1}{2x}\right]$

24

TANGENTS AND NORMALS

In the previous lesson, you have learnt that the slope of a line is the tangent of the angle which this line makes with the positive direction of x-axis. It is denoted by the letter 'm'. Thus, if θ is the angle which a line makes with the positive direction of x-axis, then m is given by tan θ.

You have also learnt that the slope m of a line, passing through two points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

In this lesson, we shall find the equations of tangents and normals to different curves, using the knowledge of differential calculus. We shall also study about Rolle's Theorem and Mean Value Theorems and their applications.

OBJECTIVES

After studying this lesson, you will be able to :

- define tangent and normal to a curve (graph of a function) at a point;
- find equations of tangents and normals to a curve under given conditions;
- state Rolle's Theorem and Lagrange's Mean Value Theorem; and
- test the validity of the above theorems and apply them to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of coordinate Geometry
- Concept of tangent and normal to a curve
- Concept of diferential coefficient of various functions
- Geometrical meaning of derivative of a function at a point

MODULE - V
Calculus

Notes

24.1 SLOPE OF TANGENT AND NORMAL

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be a point on it then the slope PT at $P(x_1, y_1)$ is given by

$$\left(\frac{dy}{dx}\right) \text{at } (x_1, y_1) \qquad \text{....(i)}$$

and (i) is equal to $\tan\theta$

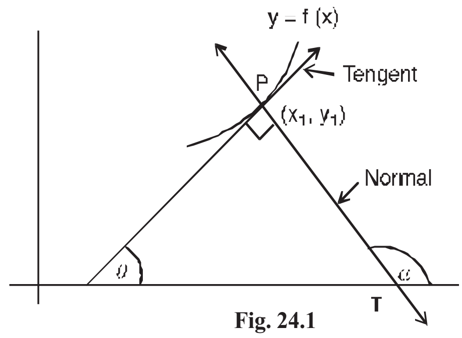


Fig. 24.1

We know that a normal to a curve is a line perpendicular to the tangent at the point of contact

We know that $\alpha = \frac{\pi}{2} + \theta$ (From Fig. 24.1)

$$\Rightarrow \qquad \tan\alpha = \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$$

$$= -\frac{1}{\tan\theta}$$

$\therefore$ Slope of normal $= -\frac{1}{m} = \frac{-1}{\left(\frac{dy}{dx}\right)}$ at (x_1, y_1) or $-\left(\frac{dx}{dy}\right)$ at (x_1, y_1)

Note

1. The tangent to a curve at any point will be parallel to x-axis if $\theta = 0$, i.e, the derivative at the point will be zero.

   i.e. $\left(\frac{dx}{dy}\right)$ at $(x_1, y_1) = 0$

2. The tangent at a point to the curve $y = f(x)$ will be parallel to y-axis if $\frac{dy}{dx} = 0$ at that point.

Let us consider some examples :

Example 24.1 Find the slope of tangent and normal to the curve

$$x^2 + x^3 + 3xy + y^2 = 5 \text{ at } (1, 1)$$

Solution : The equation of the curve is

$$x^2 + x^3 + 3xy + y^2 = 5 \qquad \text{.....(i)}$$

Differentialing (i),w.r.t. x, we get

$$2x + 3x^2 + 3\left[x\frac{dy}{dx} + y.1\right] + 2y\frac{dy}{dx} = 0 \qquad .....(ii)$$

Substituting $x = 1, \quad y = 1$, in (ii), we get

$$2\times1+3\times1+3\left[\frac{dy}{dx}+1\right]+2\frac{dy}{dx}=0$$

or $\quad 5\frac{dy}{dx} = -8 \Rightarrow \quad \frac{dy}{dx} = -\frac{8}{5}$

$\therefore$ The slope of tangent to the curve at (1, 1) is $-\frac{8}{5}$

$\therefore$ The slope of normal to the curve at (1, 1) is $\frac{5}{8}$

Example 24.2 Show that the tangents to the curve $y = \frac{1}{6}\left[3x^5 + 2x^3 - 3x\right]$ at the points $x = \pm3$ are parallel.

Solution : The equation of the curve is $y = \frac{3x^5 + 2x^3 - 3x}{6}$ .....(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\left(15x^4 + 6x^2 - 3\right)}{6}$$

$$\left(\frac{dy}{dx}\right)_{x=3} \text{at} = \frac{\left[15(3)^4 + 6(3)^2 - 3\right]}{6}$$

$$= \frac{1}{6}[15\times9\times9+54-3]$$

$$= \frac{3}{6}[405+17] = 211$$

$$\left(\frac{dy}{dx}\right) \text{at} x = -3 = \frac{1}{6}\left[15(-3)^4 + 6(-3)^2 - 3\right] = 211$$

$\therefore$ The tangents to the curve at $x = \pm3$ are parallel as the slopes at $x = \pm3$ are equal.

Example 24.3 The slope of the curve $6y^3 = px^2 + q$ at $(2, -2)$ is $\frac{1}{6}$.

Find the values of p and q.

Solution : The equation of the curve is

$$6y^3 = px^2 + q \qquad .....(i)$$

Differentiating (i) w.r.t. x, we get

$$18y^2 \frac{dy}{dx} = 2px \qquad .....(ii)$$

Putting $x = 2, y = -2,$ we get

$$18(-2)^2 \frac{dy}{dx} = 2p \; 2 = 4p$$

$$\therefore \qquad \frac{dy}{dx} = \frac{p}{18}$$

It is given equal to $\frac{1}{6}$

$$\therefore \qquad \frac{1}{6} = \frac{p}{18} \Rightarrow p = 3$$

$\therefore$ The equation of curve becomes

$$6y^3 = 3x^2 + q$$

Also, the point (2, –2) lies on the curve

$$\therefore \qquad 6(-2)^3 = 3(2)^2 + q$$

$$\Rightarrow \qquad -48 - 12 = q \quad \text{or} \quad q = -60$$

$\therefore$ The value of $p = 3,\ q = -60$

CHECK YOUR PROGRESS 24.1

1. Find the slopes of tangents and normals to each of the curves at the given points :

   (i) $y = x^3 - 2x$ at $x = 2$ (ii) $x^2 + 3y + y^2 = 5$ at (1, 1)

   (iii) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$

2. Find the values of p and q if the slope of the tangent to the curve $xy + px + qy = 2$ at (1, 1) is 2.

3. Find the points on the curve $x^2 + y^2 = 18$ at which the tangents are parallel to the line x+y = 3.

4. At what points on the curve $y = x^2 - 4x + 5$ is the tangent perpendiculat to the line $2y + x - 7 = 0$.

24.2 EQUATIONS OF TANGENT AND NORMAL TO A CURVE

We know that the equation of a line passing through a point (x_1, y_1) and with slope m is

$$y - y_1 = m(x - x_1)$$

As discussed in the section before, the slope of tangent to the curve $y = f(x)$ at (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)$ at (x_1, y_1) and that of normal is $\left(-\frac{dx}{dy}\right)$ at (x_1, y_1)

$\therefore$ Equation of tangent to the curve $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} [x - x_1]$$

And, the equation of normal to the curve $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{-1}{\frac{dy}{dx}}\right)_{(x_1, y_1)} [x - x_1]$$

Note

(i) The equation of tangent to a curve is parallel to x-axis if $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$. In that case the equation of tangent is $y = y_1$.

(ii) In case $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \rightarrow$ c, the tangent at (x_1, y_1) is parallel to $y-$axis and its equation is $x = x_1$

Let us take some examples and illustrate

Example 24.4 Find the equation of the tangent and normal to the circle $x^2 + y^2 = 25$ at the point (4, 3)

Solution : The equation of circle is

$$x^2 + y^2 = 25 \qquad \text{....(i)}$$

Differentialing (1), w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-x}{y}$$

$$\therefore \qquad \left(\frac{dy}{dx}\right)_{\text{at }(4,3)} = -\frac{4}{3}$$

$\therefore$ Equation of tangent to the circle at (4, 3) is

$$y-3=-\frac{4}{3}(x-4)$$

or $\quad 4(x-4)+3(y-3)=0 \quad$ or, $\quad 4x+3y=25$

Also, slope of the normal $= \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(4,3)}} = \dfrac{3}{4}$

$\therefore$ Equation of the normal to the circle at (4,3) is

$$y-3=\frac{3}{4}(x-4)$$

or $\quad 4y-12=3x-12$

$\Rightarrow \quad 3x=4y$

$\therefore$ Equation of the tangent to the circle at (4,3) is $4x+3y=25$

Equation of the normal to the circle at (4,3) is $3x=4y$

Example 24.5 Find the equation of the tangent and normal to the curve $16x^2+9y^2=144$ at the point (x_1, y_1) where $y_1>0$ and $x_1=2$

Solution : The equation of curve is

$$16x^2+9y^2=144 \qquad .....(i)$$

Differentiating (i), w.r.t. x we get

$$32x+18y\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=-\frac{16x}{9y}$$

As $x_1=2$ and (x_1, y_1) lies on the curve

$\therefore \quad 16(2)^2+9(y^2)=144$

$\Rightarrow \quad y^2=\dfrac{80}{9} \Rightarrow y=\pm\dfrac{4}{3}\sqrt{5}$

As $\quad y_1>0 \Rightarrow y=\dfrac{4}{3}\sqrt{5}$

$\therefore$ Equation of the tangent to the curve at $\left(2, \dfrac{4}{3}\sqrt{5}\right)$ is

$$y-\frac{4}{3}\sqrt{5}=\left(-\frac{16x}{9y}\right)_{\text{at}\left(2,\frac{4\sqrt{5}}{3}\right)}[x-2]$$

or $$y-\frac{4}{3}\sqrt{5}=-\frac{16}{9}\cdot\frac{2\times 3}{4\sqrt{5}}(x-2) \quad \text{or} \quad y-\frac{4}{3}\sqrt{5}+\frac{8}{3\sqrt{5}}(x-2)=0$$

or $$3\sqrt{5}y-\frac{4}{3}\sqrt{5}\cdot 3\sqrt{5}+8(x-2)=0$$

$$3\sqrt{5}y-20+8x-16=0 \quad \text{or} \quad 3\sqrt{5}y+8x=36$$

Also, equation of the normal to the curve at $\left(2,\frac{4}{3}\sqrt{5}\right)$ is

$$y-\frac{4}{3}\sqrt{5}=\left(\frac{9y}{16x}\right)_{\text{at}\left(2,\frac{4}{3}\sqrt{5}\right)}[x-2]$$

$$y-\frac{4}{3}\sqrt{5}=\frac{9}{16}\times\frac{2\sqrt{5}}{3}(x-2)$$

$$y-\frac{4}{3}\sqrt{5}=\frac{3\sqrt{5}}{8}(x-2)$$

$$3\times 8(y)-32\sqrt{5}=9\sqrt{5}(x-2)$$

$$24y-32\sqrt{5}=9\sqrt{5}\,x-18\sqrt{5} \quad \text{or} \quad 9\sqrt{5}x-24y+14\sqrt{5}=0$$

Example 24.6 Find the points on the curve $\frac{x^2}{9}-\frac{y^2}{16}=1$ at which the tangents are parallel to x-axis.

Solution : The equation of the curve is

$$\frac{x^2}{9}-\frac{y^2}{16}=1 \qquad \text{....(i)}$$

Differentiating (i) w.r.t. x we get

$$\frac{2x}{9}-\frac{2y}{16}\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=\frac{16x}{9y}$$

For tangent to be parallel to x-axis, $\frac{dy}{dx}=0$

$$\Rightarrow \qquad \frac{16x}{9y} = 0 \qquad \Rightarrow \qquad x = 0$$

Putting x = 0 in (i), we get $y^2 = -16 \;\; y = \pm 4i$

This implies that there are no real points at which the tangent to $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is parallel to x-axis.

Example 24.7 Find the equation of all lines having slope – 4 that are tangents to the curve $y = \frac{1}{x-1}$

Solution : $$y = \frac{1}{x-1} \qquad \text{.....(i)}$$

$$\therefore \qquad \frac{dy}{dx} = -\frac{1}{(x-1)^2}$$

It is given equal to – 4

$$\therefore \qquad \frac{-1}{(x-1)^2} = -4$$

$$\Rightarrow \qquad (x-1)^2 = \frac{1}{4}$$

$$\Rightarrow \qquad x = 1 \pm \frac{1}{2} \Rightarrow x = \frac{3}{2}, \frac{1}{2}$$

Substituting $x = \frac{1}{2}$ in (i), we get

$$y = \frac{1}{\frac{1}{2} - 1} = \frac{1}{-\frac{1}{2}} = -2$$

When $x = \frac{3}{2}, \quad y = 2$

$\therefore$ The points are $\left(\frac{3}{2}, 2\right), \left(\frac{1}{2}, -2\right)$

$\therefore$ The equations of tangents are

(a) $$y - 2 = -4\left(x - \frac{3}{2}\right)$$

$$\Rightarrow \qquad y - 2 = -4x + 6 \qquad \text{or} \qquad 4x + y = 8$$

(b) $$y + 2 = -4\left(x - \frac{1}{2}\right)$$

$$\Rightarrow \qquad y + 2 = -4x + 2 \qquad \text{or} \qquad 4x + y = 0$$

Example 24.8 Find the equation of the normal to the curve $y = x^3$ at (2 , 8)

Solution : $y = x^3 \qquad \Rightarrow \frac{dy}{dx} = 3x^2$

$$\therefore \qquad \left(\frac{dy}{dx}\right)_{at\, x=2} = 12$$

$\therefore$ Slope of the normal $= -\frac{1}{12}$

$\therefore$ Equation of the normal is

$$y - 8 = -\frac{1}{12}(x-2)$$

or $\quad 12(y-8) + (x-2) = 0 \quad$ or $\quad x + 12y = 98$

CHECK YOUR PROGRESS 24.2

1. Find the equation of the tangent and normal at the indicated points :

   (i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

   (ii) $y = x^2$ at $(1, 1)$

   (iii) $y = x^3 - 3x + 2$ at the point whose x–coordinate is 3

2. Find the equation of the targent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1)

3. Find the equation of the tangent to the hyperbola

   $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad$ at (x_0, y_0)

4. Find the equation of normals to the curve

   $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$

5. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$

24.3 ROLLE'S THEOREM

Let us now study an important theorem which reveals that between two points a and b on the graph of y = f (x) with equal ordinates f (a) and f (b), there exists at least one point c such that the tangent at $[c, f(c)]$ is parallel to x-axis. (see Fig. 24.2).

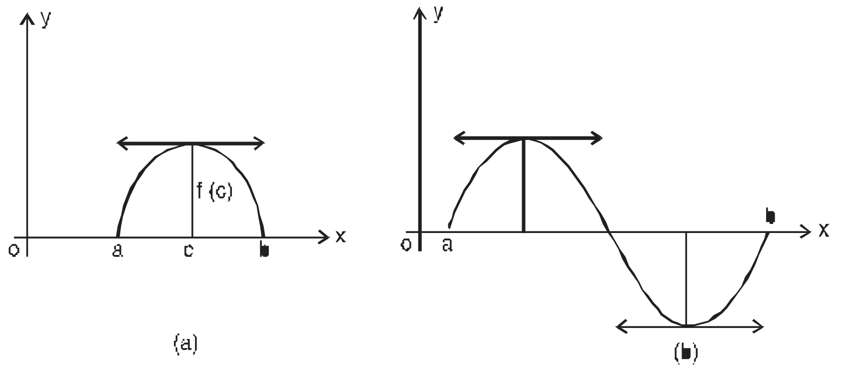


Fig. 24.2

24.3.1 Mathematical formulation of Rolle's Theorem

Let f be a real function defined in the closed interval [a, b] such that

(i) f is continuous in the closed interval [a, b]

(ii) f is differentiable in the open inteval (a, b)

(iii) f (a) = f (b)

Then there is at least one point c in the open inteval (a, b) such that $f\,'(c) = 0$

Remarks

(i) The remarks "at least one point" says that there can be more than one point $c \in (a, b)$ such that $f\,'(c) = 0$.

(ii) The condition of continuity of f on [a, b] is essential and can not be relaxed

(iii) The condition of differentiability of f on (a,b) is also essential and can not be relaxed.

For example $f(x) = |x|, x \in [-1,1]$ is continuous on $[-1,1]$ and differentiable on (–1, 1) and Rolle's Theorem is valid for this

Let us take some examples

Example 24.9 Verify Rolle's for the function

$$f(x) = x(x-1)(x-2), x \in [0,2]$$

Solution :
$$f(x) = x(x-1)(x-2)$$
$$= x^3 - 3x^2 + 2x$$

(i) f (x) is a polynomial function and hence continuous in [0, 2]

(ii) f (x) is differentiable on (0,2)

(iii) Also f (0) = 0 and f (2) = 0

$\therefore \qquad f(0) = f(2)$

$\therefore$ All the conditions of Rolle's theorem are satisfied.

Also, $\qquad f'(x) = 3x^2 - 6x + 2$

$\therefore f'(c) = 0$ gives $3c^2 - 6c + 2 = 0 \quad \Rightarrow \quad c = \dfrac{6 \pm \sqrt{36-24}}{6}$

$\Rightarrow \qquad c = 1 \pm \dfrac{1}{\sqrt{3}}$

We see that both the values of c lie in (0, 2)

Example 24.10 Discuss the applicability of Rolle's Theorem for

$$f(x) = \sin x - \sin 2x, \; x \in [0, \pi] \qquad \text{....(i)}$$

(i) is a sine function. It is continuous and differentiable on $(0, \pi)$

Again, we have, $f(0) = 0$ and $f(\pi) = 0$

$\Rightarrow \qquad f(\pi) = f(0) = 0$

$\therefore$ All the conditions of Rolle's theorem are satisfied

Now $\qquad f'(c) = 2\left[2\cos^2 c - 1\right] - \cos c = 0$

or $\qquad 4\cos^2 c - \cos c - 2 = 0$

$\therefore \qquad \cos c = \dfrac{1 \pm \sqrt{1+32}}{8}$

$$= \frac{1 \pm \sqrt{33}}{8}.$$

As $\sqrt{33} < 6$

$\therefore \qquad \cos c < \dfrac{7}{8} = 0.875$

which shows that c lies between 0 and π

CHECK YOUR PROGRESS 24.3

Verify Rolle's Theorem for each of the following functions :

(i) $f(x) = \dfrac{x^3}{3} - \dfrac{5x^2}{3} + 2x, \; x \in [0, 3]$ (ii) $f(x) = x^2 - 1$ on $[-1, 1]$

(iii) $f(x) = \sin x + \cos x - 1$ on $\left(0, \dfrac{\pi}{2}\right)$ (iv) $f(x) = (x^2 - 1)(x - 2)$ on $[-1, 2]$

24.4 LANGRANGE'S MEAN VALUE THEOREM

This theorem improves the result of Rolle's Theorem saying that it is not necessary that tangent may be parallel to x-axis. This theorem says that the tangent is parallel to the line joining the end points of the curve. In other words, this theorem says that there always exists a point on the graph, where the tangent is parallel to the line joining the end-points of the graph.

24.4.1 Mathematical Formulation of the Theorem

Let f be a real valued function defined on the closed interval [a, b] such that

(a) f is continuous on [a, b], and

(b) f is differentiable in (a, b)

(c) $f(b) \neq f(a)$

then there exists a point c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Remarks

When f (b) = f (a), f '(c) = 0 and the theorem reduces to Rolle's Theorem

Let us consider some examples

Example 24.11 Verify Langrange's Mean value theorem for

$$f(x) = (x-3)(x-6)(x-9) \text{ on } [3, 5]$$

Solution : $f(x) = (x-3)(x-6)(x-9)$

$$= (x-3)(x^2 - 15x + 54)$$

or $f(x) = x^3 - 18x^2 + 99x - 162$ ...(i)

(i) is a polynomial function and hence continuous and differentiable in the given interval

Here, $f(3) = 0,\ f(5) = (2)(-1)(-4) = 8$

$\therefore$ $f(3) \neq f(5)$

$\therefore$ All the conditions of Mean value Theorem are satisfied

$\therefore$ $f'(c) = \dfrac{f(5) - f(3)}{5 - 3} = \dfrac{8 - 0}{2} = 4$

Now $f'(x) = 3x^2 - 36x + 99$

$\therefore$ $3c^2 - 36c + 99 = 4$ or $3c^2 - 36c + 95 = 0$

$\therefore$ $c = \dfrac{36 \pm \sqrt{1296 - 1140}}{6} = \dfrac{36 \pm 12.5}{6}$

$= 8.08$ or 3.9

$c = 3.9 \in (3,5)$

$\therefore$ Langranges mean value theorem is verified

Example 24.12 Find a point on the parabola $y=(x-4)^2$ where the tangent is parallel to the chord joining $(4,0)$ and $(5,1)$

Solution : Slope of the tangent to the given curve at any point is given by $(f'(x))$ at that point.

$$f'(x)=2(x-4)$$

Slope of the chord joining $(4,0)$ and $(5,1)$ is

$$\frac{1-0}{5-4}=1 \qquad \left[\because m=\frac{y_2-y_1}{x_2-x_1}\right]$$

$\therefore$ According to mean value theorem

$$2(x-4)=1 \quad \text{or} \quad (x-4)=\frac{1}{2}$$

$$\Rightarrow \qquad x=\frac{9}{2}$$

which lies between 4 and 5

Now $\qquad y=(x-4)^2$

When $\qquad x=\frac{9}{2},\ y=\left(\frac{9}{2}-4\right)^2=\frac{1}{4}$

$\therefore$ The required point is $\left(\frac{9}{2},\frac{1}{4}\right)$

CHECK YOUR PROGRESS 24.4

1. Check the applicability of Mean Value Theorem for each of the following functions :

   (i) $f(x)=3x^2-4$ on $[2,3]$ $\qquad$ (ii) $f(x)=\log x$ on $[1,2]$

   (iii) $f(x)=x+\frac{1}{x}$ on $[1,3]$ $\qquad$ (iv) $f(x)=x^3-2x^2-x+3$ on $[0,1]$

2. Find a point on the parabola $y=(x+3)^2$, where the tangent is parallel to the chord joining $(3,0)$ and $(-4,1)$

LET US SUM UP

- The equation of tangent at (x_1, y_1) to the curve $y = f(x)$ is given by

$$y - y_1 = [f'(x)]_{at(x_1, y_1)} \{x - x_1\}$$

- The equation of normal at (x_1, y_1) to the curve $y = f(x)$ is given by

$$y - y_1 = \left[\frac{-1}{f'(x)}\right]_{(x_1, y_1)} (x - x_1)$$

- The equation of tangent to a curve $y = f(x)$ at (x_1, y_1) and parallel to x-axis is given by $y = y_1$ and parallel to y-axis is given by $x = x_1$
- **Rolle's Theorem states :** If f (x) is a function which is

  (i) continuous in the closed interval [a,b]

  (ii) differentiable in the open interval (a,b)

  (iii) $f(a) = f(b)$

  then there exists a point c in (a,b) such that $f'(c) = 0$
- Lagranges Mean Value Theorem states that if f (x) is a function which is

  (i) continuous in the closed interval [a,b]

  (ii) differentiable in the open interval (a,b)

  (iii) $f(b) \neq f(a)$

  then there exists a point c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. Find the slopes of tangents and normals to each of the following curves at the indicated points :

(i) $y=\sqrt{x}$ at x=9 (ii) $y=x^3+x$ at $x=2$

(iii) $x=a(\theta-\sin\theta), y=a(1+\cos\theta)$ at $\theta=\frac{\pi}{2}$

(iv) $y=2x^2+\cos x$ at $x=0$ (v) $xy=6$ at (1,6)

2. Find the equations of tangent and normal to the curve

$$x=a\cos^3\theta,\ y=a\sin^3\theta \text{ at } \theta=\frac{\pi}{4}$$

3. Find the point on the curve $\frac{x^2}{9}-\frac{y^2}{16}=1$ at which the tangents are parallel to y-axis.

4. Find the equation of the tangents to the curve

$y=x^2-2x+5$, (i) which is parallel to the line $2x+y+7=0$ (ii) which is perpendicular to the line $5(y-3x)=12$

5. Show that the tangents to the curve $y=7x^3+11$ at the points x = 2 and $x=-2$ are parallel

6. Find the equation of normal at the point

$\left(am^2, am^3\right)$ to the curve $ay^2=x^3$

7. Verify Rolle's Theorem for each of the following functions:

(i) $f(x)=\left(x^2-1\right)(x-2)$ on $[-1,2]$ (ii) $f(x)=\frac{x(x-2)}{x-1}$ on $[0,2]$

(iii) $f(x)=\frac{8x^2}{3}-2x,\ x\in\left[0,\frac{3}{4}\right]$

8. If Rolle's theorem holds for $f(x)=x^3+bx^2+ax,\ x\in[1,3]$ with $c=2+\frac{1}{\sqrt{3}}$, find the values of a and b.

9. Verify Mean Value Theorem for each of the following functions.

(i) $f(x)=ax^2+bx^2+cx+d$ on $[0,1]$ (ii) $f(x)=\frac{1}{4x+1}$ on $[-1,4]$

(iii) $y=(x+3)^2$ on $[-4,3]$

10. Find a point on the parabola $f(x)=(x-3)^2$, where the tangent is parallel to the chord joining the points $(3,0)$ and $(4,1)$

ANSWERS

CHECK YOUR PROGRESS 24.1

1. (i) $10, -\frac{1}{10}$ (ii) $-\frac{2}{5}, \frac{5}{2}$ (iii) $1, -1$

2. $p = 5,\ q = -4$ 3. $(3, 3), (-3, -3)$ 4. $(3, 2)$

CHECK YOUR PROGRESS 24.2

1.

	Tangent	Normal
(i)	$y + 10x = 5$	$x - 10y + 50 = 0$
(ii)	$2x - y = 1$	$x + 2y - 3 = 0$
(iii)	$24x - y = 52$	$x + 24y = 483$

2. $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ 3. $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$

4. $x + 14y - 254 = 0,\ x + 14y + 86 = 0$

CHECK YOUR PROGRESS 24.3

(i) $c = \frac{5 \pm \sqrt{7}}{3}$ (ii) $c = 0$ (iii) $c = \frac{\pi}{4}$ (iv) $c = \frac{2 \pm \sqrt{7}}{3}$

CHECK YOUR PROGRESS 24.4

1. (i) $c = 2.5$ (ii) $c = 1/\log_e^2$ (iii) $c = \sqrt{3}$ (iv) $c = \frac{1}{3}$ 2. $\left(-\frac{43}{14}, \frac{1}{196}\right)$

TERMINAL EXRCISE

1. (i) $\frac{1}{6}, -6$ (ii) $13, -\frac{1}{13}$ (iii) $1, -1$ (iv) 0, not defined (v) $-6, \frac{1}{6}$

2. $2\sqrt{2}\,(x + y) = a;\ x + y = 0$ 3. $(3, 0), (-3, 0)$

4. (i) $2x + y - 5 = 0$ (ii) $12x + 36y = 155$

6. $2x + 3my - am^2\left(2 + 3m^2\right) = 0$

7. $c = \frac{2 \pm \sqrt{7}}{3}$ (ii) At no real point (iii) $c = \frac{3}{8}$ 8. $a = 11,\ b = -6$

9. (i) $c = \frac{1}{2}$ (ii) Not applicable (iii) $c = -\frac{1}{2}$ 10. $\left(\frac{7}{2}, \frac{1}{4}\right)$

25

MAXIMA AND MINIMA

You are aware that in any transaction the total amount paid increases with the number of items purchased. Consider a function as $f(x) = 2x + 1, x > 0$. Let the function f(x) represent the amount required for purchasing 'x' number of items.

The graph of the function $y = f(x) = 2x + 1, x > 0$ for different values of x is shown in Fig. 25.1.

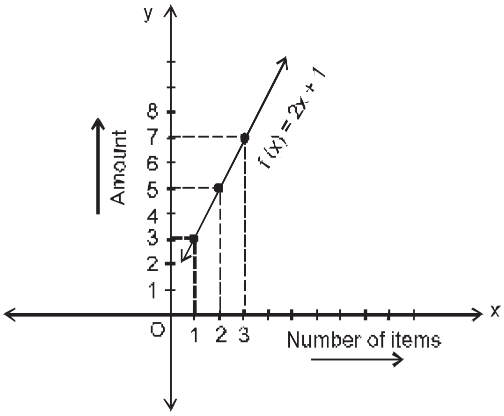


Fig. 25.1

A look at the graph of the above function prompts us to believe that the function is increasing for positive values of x i.e., for x>0 Can you think of another example in which value of the function decreases when x increases ? Any such relation would be relationship between time and manpower/person(s) involved. You have learnt that they are inversely propotional . In other words we can say that time required to complete a certain work increases when number of persons (manpower) invovled decreases and vice-versa. Consider such similar function as

$$g(x) = \frac{2}{x}, x > 0$$

The values of the function g (x) for different values of x are plotted in Fig. 25.2.

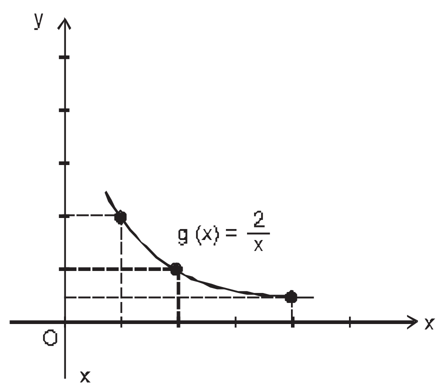


Fig. 25.2

All these examples, despite the diversity of the variables involved, have one thing in common : function is either increasing or decreasing.

In this lesson, we will discuss such functions and their characteristics. We will also find out intervals in which a given function is increasing/decreasing and its application to problems on maxima and minima.

OBJECTIVES

After studying this lesson, you will be able to :

- define monotonic (increasing and decreasing) functions;
- establish that $\frac{dy}{dx} > 0$ in an interval for an increasing function and $\frac{dy}{dx} < 0$ for a decreasing function;
- define the points of maximum and minimum values as well as local maxima and local minima of a function from the graph;
- establish the working rule for finding the maxima and minima of a function using the first and the second derivatives of the function; and
- work out simple problems on maxima and minima.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of function and types of function
- Differentiation of a function
- Solutions of equations and the inquations

25.1 INCREASING AND DECREASING FUNCTIONS

You have already seen the common trends of an increasing or a decreasing function. Here we will try to establish the condition for a function to be an increasing or a decreasing.

Let a function $f(x)$ be defined over the closed interval [a,b].

Let $x_1, x_2 \in [a,b]$, then the function f(x) is said to be an increasing function in the given interval if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$. It is said to be strictly increasing if $f(x_2) > f(x_1)$ for all $x_2 > x_1$, $x_1, x_2 \in [a,b]$.

In Fig. 25.3, sin x increases from -1 to +1 as x increases from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

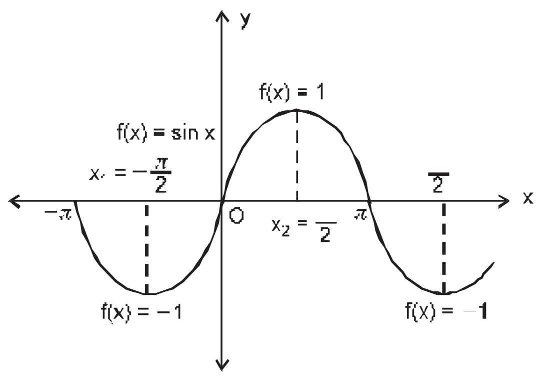


Fig. 25. 3

Note : A function is said to be an increasing function in an interval if f (x + h)> f (x) for all x belonging to the interval when h is positive.

A function f(x) defined over the closed interval [a, b] is said to be a decreasing function in the given interval, if $f(x_2) \leq f(x_1)$, whenever $x_2 > x_1$, $x_1, x_2 \in [a,b]$. It is said to be strictly decreasing if $f(x_1) > f(x_2)$ for all $x_2 > x_1$, $x_1, x_2 \in [a,b]$.

In Fig. 25.4, cos x decreases from 1 to -1 as x increases from 0 to π.

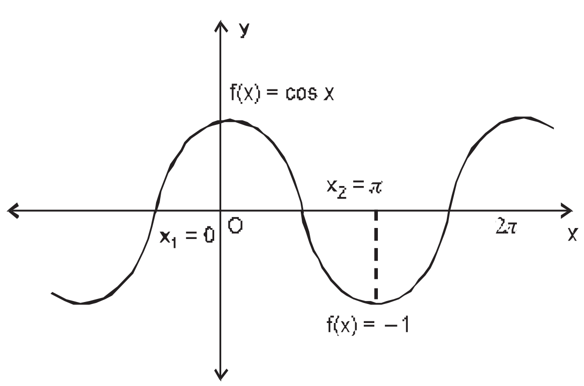


Fig. 25.4

Note : A function is said to be a decreasing in an internal if $f(x+h) < f(x)$ for all x belonging to the interval when h is positive.

25.2 MONOTONIC FUNCTIONS

Let x_1, x_2 be any two points such that $x_1 < x_2$ in the interval of definition of a function f(x). Then a function f(x) is said to be monotonic if it is either increasing or decreasing. It is said to be monotonically increasing if $f(x_2) \geq f(x_1)$ for all $x_2 > x_1$ belonging to the interval and monotonically decreasing if $f(x_1) \geq f(x_2)$.

Example 25.1 Prove that the function $f(x) = 4x + 7$ is monotonic for all values of $x \in R$.

Solution : Consider two values of x say $x_1, x_2 \in R$

such that $x_2 > x_1$ .....(1)

Multiplying both sides of (1) by 4, we have $4x_2 > 4x_1$ .....(2)

Adding 7 to both sides of (2), to get

$$4x_2 + 7 > 4x_1 + 7$$

We have $f(x_2) > f(x_1)$

Thus, we find $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

Hence the given function $f(x) = 4x + 7$ is monotonic function. (monotonically increasing).

Example 25.2 Show that

$$f(x) = x^2$$

is a strictly decreasing function for all x < 0.

Solution : Consider any two values of x say x_1, x_2 such that

$$x_2 > x_1, \qquad x_1, x_2 < 0 \qquad \text{.......(i)}$$

Order of the inequality reverses when it is multiplied by a negative number. Now multiplying (i) by x_2, we have

$$x_2 \cdot x_2 < x_1 \cdot x_2$$

or, $x_2^2 < x_1 x_2$ ......(ii)

Now multiplying (i) by x_1, we have

$$x_1 \cdot x_2 < x_1 \cdot x_1$$

or, $x_1 x_2 < x_1^2$ .....(iii)

From (ii) and (iii), we have

$$x_2^2 < x_1 x_2 < x_1^2$$

or, $\quad x_2^2 < x_1^2$

or, $\quad f(x_2) < f(x_1)$ .....(iv)

Thus, from (i) and (iv), we have for

$$x_2 > x_1,$$

$$f(x_2) < f(x_1)$$

Hence, the given function is strictly decreasing for all $x < 0$.

CHECK YOUR PROGRESS 25.1

1. (a) Prove that the function

$$f(x) = 3x + 4$$

is monotonic increasing function for all values of $x \in R$.

(b) Show that the function

$$f(x) = 7 - 2x$$

is monotonically decreasing function for all values of $x \in R$.

(c) Prove that $f(x) = ax + b$ where a, b are constants and $a > 0$ is a strictly increasing function for all real values of x.

2. (a) Show that $f(x) = x^2$ is a strictly increasing function for all real $x > 0$.

(b) Prove that the function $f(x) = x^2 - 4$ is monotonically increasing for $x > 2$ and monotonically decreasing for $-2 < x < 2$ where $x \in R$.

Theorem 1 : If f(x) is an increasing function on an open interval]a, b[, then its derivative $f'(x)$ is positive at this point for all $x \in [a,b]$.

Proof : Let (x, y) or [x, f (x)] be a point on the curve y = f (x)

For a positive δx, we have

$$x + \delta x > x$$

Now, function f(x) is an increasing function

$\therefore \quad f(x + \delta x) > f(x)$

or, $\quad f(x + \delta x) - f(x) > 0$

or, $\quad \dfrac{f(x + \delta x) - f(x)}{\delta x} > 0 \quad [\because \delta x > 0]$

Taking δx as a small positive number and proceeding to limit, when $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} > 0$$

or, $$f'(x) > 0$$

Thus, if $y = f(x)$ is an increasing function at a point, then f '(x) is positive at that point.

Theorem 2 : If f (x) is a decreasing function on an open interval]a, b[then its derivative f '(x) is negative at that point for all $x \in [a, b]$.

Proof : Let (x, y) or [x, f(x)] be a point on the curve y = f(x)

For a positive δx, we have $x + \delta x > x$

Since the function is a decreasing function

$\therefore$ $$f(x+\delta x) < f(x) \qquad \delta x > 0$$

or, $$f(x+\delta x) - f(x) < 0$$

Dividing by δx, we have $$\frac{f(x+\delta x) - f(x)}{\delta x} < 0 \quad \delta x > 0$$

or, $$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} < 0$$

or, $$f'(x) < 0$$

Thus, if $y = f(x)$ is a decreasing function at a point, then, $f'(x)$ is negative at that point.

Note : If $f(x)$ is derivable in the closed interval [a,b], then f (x) is

(i) increasing over [a,b], if f '(x)>0 in the open interval]a,b[

(ii) decreasing over [a,b], if f '(x)<0 in the open interval]a,b[.

25.3 RELATION BETWEEN THE SIGN OF THE DERIVATIVE AND MONOTONICITY OF FUNCTION

Consider a function whose curve is shown in the Fig. 25.5

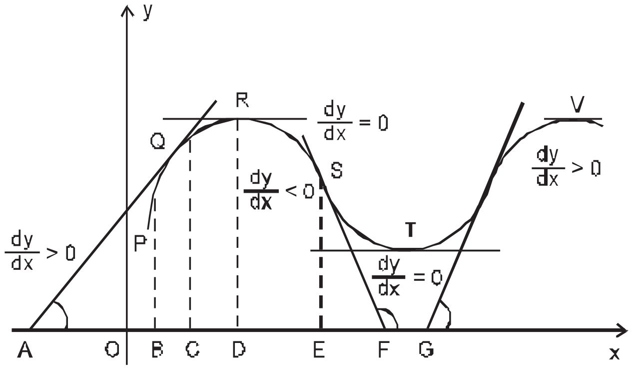


Fig. 25.5

We divide, our study of relation between sign of derivative of a function and its increasing or decreasing nature (monotonicity) into various parts as per Fig. 25.5

(i) P to R (ii) R to T (iii) T to V

(i) We observe that the ordinate (y-coordinate) for every succeeding point of the curve from P to R increases as also its x-coordinate. If (x_2, y_2) are the coordinates of a point that succeeds (x_1, y_1) then $x_2 > x_1$ yields $y_2 > y_1$ or $f(x_2) > f(x_1)$.

Also the tangent at every point of the curve between P and R makes acute angle with the positive direction of x-axis and thus the slope of the tangent at such points of the curve (except at R) is positive. At R where the ordinate is maximum the tangent is parallel to x-axis, as a result the slope of the tangent at R is zero.

We conclude for this part of the curve that

(a) The function is monotonically increasing from P to R.

(b) The tangent at every point (except at R) makes an acute angle with positive direction of x-axis.

(c) The slope of tangent is positive i.e. $\frac{dy}{dx} > 0$ for all points of the curve for which y is increasing.

(d) The slope of tangent at R is zero i.e. $\frac{dy}{dx} = 0$ where y is maximum.

(ii) The ordinate for every point between R and T of the curve decreases though its x-coordinate increases. Thus, for any point $x_2 > x_1$ yelds $y_2 < y_1$, or $f(x_2) < f(x_1)$.

Also the tangent at every point succeeding R along the curve makes obtuse angle with positive direction of x-axis. Consequently, the slope of the tangent is negative for all such points whose ordinate is decreasing. At T the ordinate attains minimum value and the tangent is parallel to x-axis and as a result the slope of the tangent at T is zero.

We now conclude :

(a) The function is monotonically decreasing from Rto T.

(b) The tangent at every point, except at T, makes obtuse angle with positive direction of x-axis.

(c) The slope of the tangent is negative i.e., $\frac{dy}{dx} < 0$ for all points of the curve for which y is decreasing.

(d) The slope of the tangent at T is zero i.e. $\frac{dy}{dx} = 0$ where the ordinate is minimum.

(iii) Again, for every point from T to V

The ordinate is constantly increasing, the tangent at every point of the curve between T and V makes acute angle with positive direction of x-axis. As a result of which the slope of the tangent at each of such points of the curve is positive.

Conclusively,

$$\frac{dy}{dx} > 0$$

at all such points of the curve except at Tand V, where $\frac{dy}{dx}=0$. The derivative $\frac{dy}{dx}<0$ on one side, $\frac{dy}{dx}>0$ on the other side of points R, T and V of the curve where $\frac{dy}{dx}=0$.

Example 25.3 Find for what values of x, the function

$$f(x)=x^2-6x+8$$

is increasing and for what values of x it is decreasing.

Solution : $f(x)=x^2-6x+8$

$$f'(x)=2x-6$$

For $f(x)$ to be increasing, $f'(x)>0$

i.e., $2x-6>0$ or, $2(x-3)>0$

or, $x-3>0$ or, $x>3$

The function increases for x>3.

For $f(x)$ to be decreasing,

$$f'(x)<0$$

i.e., $2x-6<0$ or, $x-3<0$

or, $x<3$

Thus, the function decreases for x < 3.

Example 25.4 Find the interval in which $f(x)=2x^3-3x^2-12x+6$ is increasing or decreasing.

Solution : $f(x)=2x^3-3x^2-12x+6$

$$f'(x)=6x^2-6x-12$$

$$=6\left(x^2-x-2\right)$$

$$=6(x-2)(x+1)$$

For $f(x)$ to be increasing function of x,

$$f'(x)>0$$

i.e. $6(x-2)(x+1)>0$ or, $(x-2)(x+1)>0$

Since the product of two factors is positive, this implies either both are positive or both are negative.

Either $x-2>0$ and $x+1>0$ or $x-2<0$ and $x+1<0$

i.e. $x > 2$ and $x > -1$

$x > 2$ implies $x > -1$

$\therefore$ $x > 2$

i.e. $x < 2$ and $x < -1$

$x < -1$ implies $x < 2$

$\therefore$ $x < -1$

Hence f (x) is increasing for $x > 2$ or $x < -1$.

Now, for f (x) to be decreasing,

$$f'(x) < 0$$

or, $6(x-2)(x+1) < 0$ or, $(x-2)(x+1) < 0$

Since the product of two factors is negative, only one of them can be negative, the other positive.

Therefore,

Either

$x - 2 > 0$ and $x + 1 < 0$

i.e. $x > 2$ and $x < -1$

There is no such possibility that $x > 2$ and at the same time $x < -1$

or

$x - 2 < 0$ and $x + 1 > 0$

i.e. $x < 2$ and $x > -1$

This can be put in this form

$-1 < x < 2$

$\therefore$ The function is decreasing in $-1 < x < 2$.

Example 25.5 Determine the intervals for which the function

$$f(x) = \frac{x}{x^2+1} \text{ is increasing or decreasing.}$$

Solution :
$$f'(x) = \frac{(x^2+1)\frac{dx}{dx} - x\cdot\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - x\cdot(2x)}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$$\therefore \quad f'x = \frac{(1-x)(1+x)}{(x^2+1)^2}$$

As $(x^2+1)^2$ is positive for all real x.

Therefore, if $-1 < x < 0$, $(1-x)$ is positive and $(1+x)$ is positive, so $f'(x) > 0$;

$\therefore$ If $0 < x < 1$, $(1-x)$ is positive and $(1+x)$ is positive, so f $f'(x) > 0$;

If $x < -1$, $(1-x)$ is positive and $(1+x)$ is negative, so $f'(x) < 0$;

$x > 1$, $(1-x)$ is negative and $(1+x)$ is positive, so $f'(x) < 0$;

Thus we conclude that

the function is increasing for $-1 < x < 0$ and $0 < x < 1$

or, for $-1 < x < 1$

and the function is decreasing for $x < -1$ or $x > 1$

Note : Points where $f'(x) = 0$ are critical points. Here, critical points are $x = -1$, $x = 1$.

Example 25.6 Show that

(a) $f(x) = \cos x$ is decreasing in the interval $0 \le x \le \pi$.

(b) $f(x) = x - \cos x$ is increasing for all x.

Solution :(a) $f(x) = \cos x$

$$f'(x) = -\sin x$$

$f(x)$ is decreasing

If $f'(x) < 0$

i.e., $-\sin x < 0$

i.e., $\sin x > 0$

sin x is positive in the first quadrant and in the second quadrant, therefore, sin x is positive in $0 \le x \le \pi$

$\therefore$ $f(x)$ is decreasing in $0 \le x \le \pi$

(b) $f(x) = x - \cos x$

$$f'(x) = 1 - (-\sin x)$$

$$= 1 + \sin x$$

Now, we know that the minimum value of sinx is –1 and its maximum; value is 1 i.e.,sin x lies between –1 and 1 for all x,

i.e., $-1 \le \sin x \le 1$ or $1 - 1 \le 1 + \sin x \le 1 + 1$

or $0 \le 1 + \sin x \le 2$

or $0 \le f'(x) \le 2$

or $0 \le f'(x)$

or $f'(x) \ge 0$

$\Rightarrow$ $f(x) = x - \cos x$ is increasing for all x.

CHECK YOUR PROGRESS 25.2

Find the intervals for which the followiong functions are increasing or decreasing.

1. (a) $f(x) = x^2 - 7x + 10$ (b) $f(x) = 3x^2 - 15x + 10$

2. (a) $f(x) = x^3 - 6x^2 - 36x + 7$ (b) $f(x) = x^3 - 9x^2 + 24x + 12$

3. (a) $y = -3x^2 - 12x + 8$ (b) $f(x) = 1 - 12x - 9x^2 - 2x^3$

4. (a) $y = \dfrac{x-2}{x+1}, x \neq -1$ (b) $y = \dfrac{x^2}{x-1}, x \neq 1$ (c) $y = \dfrac{x}{2} + \dfrac{2}{x}, x \neq 0$

5. (a) Prove that the function log sin x is decreasing in $\left[\dfrac{\pi}{2}, \pi\right]$

   (b) Prove that the function cos x is increasing in the interval $[\pi, 2\pi]$

   (c) Find the intervals in which the function $\cos\left(2x + \dfrac{\pi}{4}\right), 0 \leq x \leq \pi$ is decreasing or increasing.

Find also the points on the graph of the function at which the tangents are parallel to x-axis.

25.4 MAXIMUM AND MINIMUM VALUES OF A FUNCTION

We have seen the graph of a continuous function. It increases and decreases alternatively. If the value of a continious function increases upto a certain point then begins to decrease, then this point is called point of maximum and corresponding value at that point is called maximum value of the function. A stage comes when it again changes from decreasing to increasing . If the value of a continuous function decreases to a certain point and then begins to increase, then value at that point is called minimum value of the function and the point is called point of minimum.

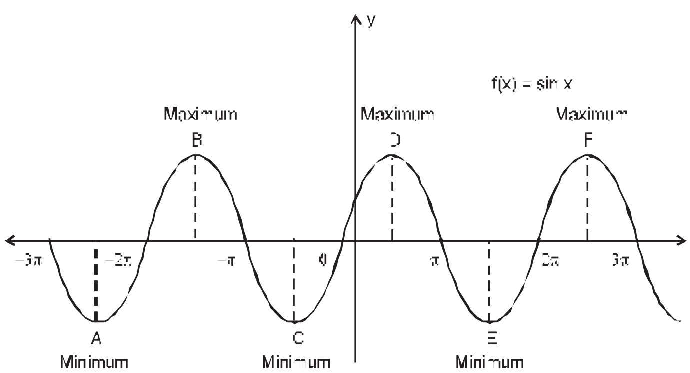


Fig. 25.6

Fig. 25.6 shows that a function may have more than one maximum or minimum values. So, for continuous function we have maximum (minimum) value in an interval and these values are not absolute maximum (minimum) of the function. For this reason, we sometimes call them as local maxima or local minima.

A function f (x) is said to have a maximum or a local maximum at the point x = a where a – b < a < a + b (See Fig. 25.7), if $f(a) \geq f(a \pm b)$ for all sufficiently small positive b.

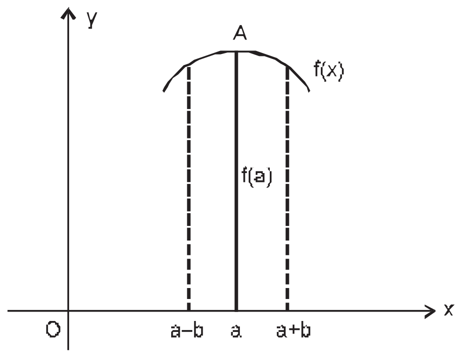


Fig. 25.7

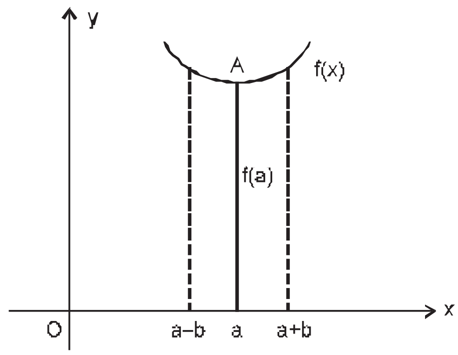


Fig. 25.8

A maximum (or local maximum) value of a function is the one which is greater than all other values on either side of the point in the immediate neighbourhood of the point.

A function f (x) is said to have a minimum (or local minimum) at the point x = a if $f(a) \leq f(a \pm b)$ where a – b< a < a + b

for all sufficiently small positive b.

In Fig. 25.8, the function f(x) has local minimum at the point x = a.

A minimum (or local miunimum) value of a function is the one which is less than all other values, on either side of the point in the immediate neighbourhood of the point.

Note : A neighbourhood of a point x $\in$ R is defined by open internal]x–$\in$ [, when $\in$ >0.

25.5 CONDITIONS FOR MAXIMUM OR MINIMUM

We know that derivative of a function is positive when the function is increasing and the derivative is negative when the function is decreasing. We shall apply this result to find the condition for maximum or a function to have a minimum. Refer to Fig. 25.6, points B,D, F are points of maxima and points A,C,E are points of minima.

Now, on the left of B, the function is increasing and so $f'(x) > 0$, but on the right of B, the function is decreasing and, therefore, $f'(x) < 0$. This can be achieved only when f '(x) becomes zero somewhere in betwen. We can rewrite this as follows :

A function f (x) has a maximum value at a point if (i) f '(x) = 0 and (ii) f '(x) changes sign from positive to negative in the neighbourhood of the point at which f '(x)=0 (points taken from left to right).

Now, on the left of C (See Fig. 25.6), function is decreasing and f '(x) therefore, is negative and on the right of C, f (x) is increasing and so f '(x) is positive. Once again f '(x) will be zero before having positive values. We rewrite this as follows :

A function f (x) has a minimum value at a point if (i) f '(x)=0, and (ii) f '(x) changes sign from negative to positive in the neighbourhood of the point at which f '(x) = 0.

We should note here that f '(x) = 0 is necessary condition and is not a sufficient condition for maxima or minima to exist. We can have a function which is increasing, then constant and then again increasing function. In this case, f '(x) does not change sign.The value for which f '(x) = 0 is not a point of maxima or minima. Such point is called point of inflexion.

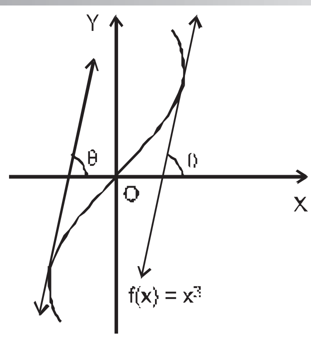


Fig. 25.9

For example, for the function $f(x) = x^3, x = 0$ is the point of inflexion as $f'(x) = 3x^2$ does not change sign as x passes through 0. f'(x) is positive on both sides of the value '0' (tangents make acute angles with x-axis) (See Fig. 25.9).

Hence $f(x) = x^3$ has a point of inflexion at x = 0.

The points where f '(x) = 0 are called stationary points as the rate ofchange of the function is zero there. Thus points of maxima and minima are stationary points.

Remarks

The stationary points at which the function attains either local maximum or local minimum values are also called extreme points and both local maximum and local minimum values are called extreme values of f (x). Thus a function attains an extreme value at x=a if f (a) is either a local maximum or a local minimum.

25.6 METHOD OF FINDING MAXIMA OR MINIMA

We have arrived at the method of finding the maxima or minima of a function. It is as follows :

(i) Find f'(x)

(ii) Put f '(x)=0 and find stationary points

(iii) Consider the sign of f '(x) in the neighbourhood of stationary points. If it changes sign from +ve to –ve, then f (x) has maximum value at that point and if f '(x) changes sign from –ve to +ve, then f (x) has minimum value at that point.

(iv) If f '(x) does not change sign in the neighbourhood of a point then it is a point of inflexion.

Example 25.7 Find the maximum (local maximum) and minimum (local minimum) points of the function $f(x) = x^3 - 3x^2 - 9x$.

Solution : Here $\quad f(x) = x^3 - 3x^2 - 9x$

$$f'(x) = 3x^2 - 6x - 9$$

Step I. Now $\quad f'(x) = 0$ gives us $3x^2 - 6x - 9 = 0$

or $\quad x^2 - 2x - 3 = 0$

or $\quad (x-3)(x+1) = 0$

or $\quad x = 3, -1$

$\therefore$ Stationary points are $\quad x = 3, x = -1$

Step II. At $x = 3$

For $x < 3$ $\quad f'(x) < 0$

and for $x > 3$ $\quad f'(x) > 0$

$\therefore$ $f'(x)$ changes sign from –ve to +ve in the neighbourhood of 3.

$\therefore$ $f(x)$ has minimum value at $x = 3$.

Step III. At $x = -1$,

For $x < -1$, $\quad f'(x) > 0$

and for $x > -1$, $\quad f'(x) < 0$

$\therefore$ $f'(x)$ changes sign from +ve to –ve in the neighbourhood of –1.

$\therefore$ $f(x)$ has maximum value at $x = -1$.

$\therefore$ $x = -1$ and $x = 3$ give us points of maxima and minima respectively. If we want to find maximum value (minimum value), then we have

$$\text{maximum value} = f(-1) = (-1)^3 - 3(-1)^2 - 9(-1)$$

$$= -1 - 3 + 9 = 5$$

and $\quad \text{minimum value} = f(3) = 3^3 - 3(3)^2 - 9(3) = -27$

$\therefore$ $(-1, 5)$ and $(3, -27)$ are points of local maxima and local minima respectively.

Example 25.8 Find the local maximum and the local minimum of the function

$$f(x) = x^2 - 4x$$

Solution : $\quad f(x) = x^2 - 4x$

$\therefore \quad f'(x) = 2x - 4 = 2(x - 2)$

Putting $f'(x) = 0$ yields $2x - 4 = 0$, i.e., $x = 2$.

We have to examine whether $x = 2$ is the point of local maximum or local minimum or neither maximum nor minimum.

Let us take $x = 1.9$ which is to the left of 2 and $x = 2.1$ which is to the right of 2 and find $f'(x)$ at these points.

$$f'(1.9) = 2(1.9 - 2) < 0$$

$$f'(2.1) = 2(2.1 - 2) > 0$$

Since $f'(x) < 0$ as we approach 2 from the left and $f'(x) > 0$ as we approach 2 from the right, therefore, there is a local minimum at $x = 2$.

We can put our findings for sign of derivatives of $f(x)$ in any tabular form including the one given below :

sign of f ' (x)

point x = 2

left of 2	right of 2
f ' (x) < 0	f ' (x) < 0

Local minimum

Example 25.9 Find all local maxima and local minima of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 8$$

Solution : $f(x) = 2x^3 - 3x^2 - 12x + 8$

$\therefore$ $f'(x) = 6x^2 - 6x - 12$

$= 6(x^2 - x - 2)$

$\therefore$ $f'(x) = 6(x+1)(x-2)$

Now solving f '(x)=0 for x, we get

$6(x+1)(x-2) = 0$

$\Rightarrow$ $x = -1, 2$

Thus $f'(x) = 0$ at $x = -1, 2.$

We examine whether these points are points of local maximum or local minimum or neither of them.

Consider the point $x = -1$

Let us take $x = -1.1$ which is to the left of -1 and $x = -0.9$ which is to the right of -1 and find f ' (x) at these points.

$f'(-1.1) = 6(-1.1+1)(-1.1-2)$, which is positive i.e. $f'(x)>0$

$f'(-0.9) = 6(-0.9+1)(-0.9-2)$, which is negative i.e. $f'(x)<0$

Thus, at $x = -1$, there is a local maximum.

Consider the point $x = 2$.

Now, let us take $x = 1.9$ which is to the left of $x = 2$ and $x = 2.1$ which is to the right of $x = 2$ and find f '(x) at these points.

$f'(1.9) = 6(1.9+1)(1.9-2)$

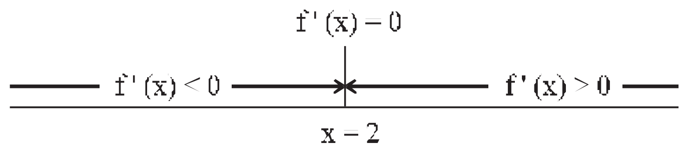
= 6×(Positive number)×(negative number)

= a negative number

i.e. $f'(1.9) < 0$

and $f'(2.1) = 6(2.1+1)(2.1-2)$, which is positive

i.e., $f'(2.1) > 0$



$\because$ $f'(x) < 0$ as we approach 2 from the left

and $f'(x) > 0$ as we approach 2 from the right.

$\therefore$ $x = 2$ is the point of local minimum

Thus f (x) has local maximum at $x = -1$, maximum value of $f(x) = -2-3+12+8 = 15$

f (x) has local minimum at $x = 2$, minimum value of $f(x) = 2(8)-3(4)-12(2)+8 = -12$

Sign of f ' (x)

Point x = – 1			Point x = 2	
Left of – 1	Right of – 1		Left of 2	Right of 2
positive	negative		negative	positive
local maximum			local minimum	

Example 25.10 Find local maximum and local minimum, if any, of the following function

$$f(x) = \frac{x}{1+x^2}$$

Solution :
$$f(x) = \frac{x}{1+x^2}$$

Then
$$f'(x) = \frac{(1+x^2)1 - (2x)x}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

For finding points of local maximum or local minimum, equate $f'(x)$ to 0.

i.e.
$$\frac{1-x^2}{(1+x^2)^2} = 0$$

$\Rightarrow \qquad 1-x^2=0$

or $\qquad (1+x)(1-x)=0 \qquad$ or $\qquad x=1,-1.$

Consider the value x = 1.

The sign of f '(x) for values of x slightly less than 1 and slightly greater than 1 changes from positive to negative. Therefore there is a local maximum at x =1, and the local maximum value $=\dfrac{1}{1+(1)^2}=\dfrac{1}{1+1}=\dfrac{1}{2}$

Now consider x = – 1.

f '(x) changes sign from negative to positive as x passes through –1, therefore, f (x) has a local minimum at x = – 1

Thus, the local minimum value $=\dfrac{-1}{2}$

Example 25.11 Find the local maximum and local minimum, if any, for the function

$$f(x)=\sin x+\cos x, 0<x<\frac{\pi}{2}$$

Solution : We have $f(x)=\sin x+\cos x$

$$f'(x)=\cos x-\sin x$$

For local maxima/minima, f '(x) = 0

$\therefore \qquad \cos x-\sin x=0$

or, $\qquad \tan x=1 \qquad$ or, $\qquad x=\dfrac{\pi}{4}$ in $0<x<\dfrac{\pi}{2}$

At $\qquad x=\dfrac{\pi}{4}$,

For $\qquad x<\dfrac{\pi}{4}$, $\cos x>\sin x$

$\therefore \qquad f'(x)=\cos x-\sin x>0$

For $\qquad x>\dfrac{\pi}{4}$, $\cos x-\sin x<0$

$\therefore \qquad f'(x)=\cos x-\sin x<0$

$\therefore$ f '(x) changes sign from positive to negative in the neighbourhood of $\dfrac{\pi}{4}$.

$\therefore \ x=\dfrac{\pi}{4}$ is a point of local maxima.

Maximum value $=f\left(\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}=\sqrt{2}$

$\therefore$ Point of local maxima is $\left(\dfrac{\pi}{4},\sqrt{2}\right)$.

CHECK YOUR PROGRESS 25.3

Find all points of local maxima and local minima of the following functions. Also , find the maxima and minima at such points.

1.	$x^2 - 8x + 12$	2.	$x^3 - 6x^2 + 9x + 15$
3.	$2x^3 - 21x^2 + 36x - 20$	4.	$x^4 - 62x^2 + 120x + 9$
5.	$(x-1)(x-2)^2$	6.	$\dfrac{x-1}{x^2+x+2}$

25.7 USE OF SECOND DERIVATIVE FOR DETERMINATION OF MAXIMUM AND MINIMUM VALUES OF A FUNCTION

We now give below another method of finding local maximum or minimum of a function whose second derivative exists. Various steps used are :

(i) Let the given function be denoted by f (x).

(ii) Find f ' (x) and equate it to zero.

(iii) Solve f ' (x)=0, let one of its real roots be x = a.

(iv) Find its second derivative, f "(x). For every real value 'a' of x obtained in step (iii), evaluate f" (a). Then if

f "(a) <0 then x =a is a point of local maximum.

f "(a)>0 then x = a is a point of local minimum.

f "(a)=0 then we use the sign of f '(x) on the left of 'a' and on the right of 'a' to arrive at the result.

Example 25.12 Find the local minimum of the following function :

$$2x^3 - 21x^2 + 36x - 20$$

Solution : Let $f(x) = 2x^3 - 21x^2 + 36x - 20$

Then $f'(x) = 6x^2 - 42x + 36$

$$= 6\left(x^2 - 7x + 6\right)$$

$$= 6(x-1)(x-6)$$

For local maximum or min imum

$$f'(x) = 0$$

or $6(x-1)(x-6) = 0 \quad \Rightarrow x = 1, 6$

Now $$f''(x) = \frac{d}{dx} f'(x)$$

$$= \frac{d}{dx}\left[6\left(x^2 - 7x + 6\right)\right]$$

$$= 12x - 42$$

$$= 6(2x - 7)$$

For $x = 1$, $f''(1) = 6(2.1 - 7) = -30 < 0$

$x = 1$ is a point of local maximum.

and $f(1) = 2(1)^3 - 21(1)^2 + 36(1) - 20 = -3$ is a local maximum.

For $x = 6$,

$f''(6) = 6(2.6 - 7) = 30 > 0$

$\therefore$ $x = 6$ is a point of local minimum

and $f(6) = 2(6)^3 - 21(6)^2 + 36(6) - 20 = -128$ is a local minimum.

Example 25.13 Find local maxima and minima (if any) for the function

$$f(x) = \cos 4x; \quad 0 < x < \frac{\pi}{2}$$

Solution : $f(x) = \cos 4x$

$\therefore$ $f'(x) = -4\sin 4x$

Now, $f'(x) = 0 \quad \Rightarrow \quad -4\sin 4x = 0$

or, $\sin 4x = 0$ or, $4x = 0, \pi, 2\pi$

or, $$x = 0, \frac{\pi}{4}, \frac{\pi}{2}$$

$\therefore$ $$x = \frac{\pi}{4} \qquad \left[\because \; 0 < x < \frac{\pi}{2}\right]$$

Now, $f''(x) = -16\cos 4x$

at $$x = \frac{\pi}{4},\; f''(x) = -16\cos\pi$$

$$= -16(-1) = 16 > 0$$

$\therefore$ f (x) is minimum at $x = \frac{\pi}{4}$

Minimum value $$f\left(\frac{\pi}{4}\right) = \cos\pi = -1$$

Example 25.14 (a) Find the maximum value of $2x^3 - 24x + 107$ in the interval [–3,–1].

(b) Find the minimum value of the above function in the interval [1,3].

Solution : Let $f(x) = 2x^3 - 24x + 107$

$$f'(x) = 6x^2 - 24$$

For local maximum or minimum,

$$f'(x) = 0$$

i.e. $\quad 6x^2 - 24 = 0 \quad \Rightarrow \quad x = -2, 2$

Out of two points obtained on solving $f'(x)=0$, only –2 belong to the interval [–3,–1]. We shall, therefore, find maximum if any at $x=-2$ only.

Now $\quad f''(x) = 12x$

$\therefore \quad f''(-2) = 12(-2) = -24$

or $\quad f''(-2) < 0$

which implies the function f (x) has a maximum at $x = -2$.

$\therefore$ Required maximum value $= 2(-2)^3 - 24(-2) + 107$

$= 139$

Thus the point of maximum belonging to the given interval [–3,–1] is –2 and, the maximum value of the function is 139.

(b) Now $\quad f''(x) = 12x$

$\therefore \quad f''(2) = 24 > 0, \qquad [\because$ 2 lies in [1, 3]]

which implies, the function f (x) shall have a minimum at $x = 2$.

$\therefore$ Required minimum $= 2(2)^3 - 24(2) + 107$

$= 75$

Example 25.15 Find the maximum and minimum value of the function

$$f(x) = \sin x(1 + \cos x) \text{ in } (0, \pi).$$

Solution : We have, $f(x) = \sin x(1 + \cos x)$

$$f'(x) = \cos x(1 + \cos x) + \sin x(-\sin x)$$

$$= \cos x + \cos^2 x - \sin^2 x$$

$$= \cos x + \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x + \cos x - 1$$

For stationary points, $f'(x) = 0$

$$\therefore \quad 2\cos^2 x + \cos x - 1 = 0$$

$$\therefore \quad \cos x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = -1, \frac{1}{2}$$

$\therefore \qquad x = \pi, \frac{\pi}{3}$

Now, $\qquad f(0) = 0$

$$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3}\left(1 + \cos\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$$

and $\qquad f(\pi) = 0$

$\therefore$ f (x) has maximum value $\frac{3\sqrt{3}}{4}$ at $x = \frac{\pi}{3}$

and miminum value 0 at x = 0 and $x = \pi$.

CHECK YOUR PROGRESS 25.4

Find local maximum and local minimum for each of the following functions using second order derivatives.

1. $2x^3 + 3x^2 - 36x + 10$
2. $-x^3 + 12x^2 - 5$
3. $(x - 1)(x + 2)^2$
4. $x^5 - 5x^4 + 5x^3 - 1$
5. $\sin x(1 + \cos x), 0 < x < \frac{\pi}{2}$
6. $\sin x + \cos x, 0 < x < \frac{\pi}{2}$
7. $\sin 2x - x, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

25.8 APPLICATIONS OF MAXIMA AND MINIMA TO PRACTICAL PROBLEMS

The application of derivative is a powerful tool for solving problems that call for minimising or maximising a function. In order to solve such problems, we follow the steps in the following order :

(i) Frame the function in terms of variables discussed in the data.

(ii) With the help of the given conditions, express the function in terms of a single variable.

(iii) Lastly, apply conditions of maxima or minima as discussed earlier.

Example 25.16 Find two positive real numbers whose sum is 70 and their product is maximum.

Solution : Let one number be x. As their sum is 70, the other number is 70–x. As the two numbers are positive, we have, $x > 0, 70 - x > 0$

$$70 - x > 0 \qquad \Rightarrow \qquad x < 70$$

$$\therefore \qquad 0 < x < 70$$

Let their product be f (x)

Then $$f(x) = x(70 - x) = 70x - x^2$$

We have to maximize the prouct f (x).

We, therefore, find f '(x) and put that equal to zero.

$$f'(x) = 70 - 2x$$

For maximum product, f ' (x) = 0

or $$70 - 2x = 0$$

or $$x = 35$$

Now f''(x) = –2 which is negative. Hence f (x) is maximum at x = 35

The other number is $70 - x = 35$

Hence the required numbers are 35, 35.

Example 25.17 Show that among rectangles of given area, the square has the least perimeter.

Solution : Let x, y be the length and breadth of the rectangle respectively.

$\therefore$ Its area = xy

Since its area is given, represent it by A, so that we have

$$A = xy$$

or $$y = \frac{A}{x} \qquad \text{... (i)}$$

Now, perimeter say P of the rectangle = 2 (x + y)

or $$P = 2\left(x + \frac{A}{x}\right)$$

$\therefore$ $$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right) \qquad \text{...(ii)}$$

For a minimum $P, \frac{dP}{dx} = 0.$

i.e. $$2\left(1 - \frac{A}{x^2}\right) = 0$$

or $$A = x^2 \quad \text{or} \quad \sqrt{A} = x$$

Now, $$\frac{d^2P}{dx^2} = \frac{4A}{x^3}, \text{ which is positive.}$$

Hence perimeter is minimum when $x = \sqrt{A}$

$\therefore$ $$y = \frac{A}{x}$$

$$= \frac{x^2}{x} = x \qquad (\because A = x^2)$$

Thus, the perimeter is minimum when rectangle is a square.

Example 25.18 An open box with a square base is to be made out of a given quantity of sheet of area a^2. Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.

Solution : Let x be the side of the square base of the box and y its height.

Total surface area of othe box $= x^2 + 4xy$

$$\therefore \qquad x^2 + 4xy = a^2 \quad \text{or} \quad y = \frac{a^2 - x^2}{4x}$$

Volume of the box, $V = \text{base area} \times \text{height}$

$$= x^2 y = x^2\left(\frac{a^2 - x^2}{4x}\right)$$

or
$$V = \frac{1}{4}\left(a^2 x - x^3\right) \qquad \text{...(i)}$$

$$\therefore \qquad \frac{dV}{dx} = \frac{1}{4}\left(a^2 - 3x^2\right)$$

For maxima/minima $\frac{dV}{dx} = 0$

$$\therefore \qquad \frac{1}{4}\left(a^2 - 3x^2\right) = 0$$

$$x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\text{Volume} = \frac{1}{4}\left(\frac{(a^3)}{\sqrt{3}} - \frac{a^3}{3\sqrt{3}}\right) = \frac{a^3}{6\sqrt{3}} \qquad \text{..(iii)}$$

Again
$$\frac{d^2V}{dx^2} = \frac{d}{dx}\frac{1}{4}\left(a^2 - 3x^2\right) = -\frac{3}{2}x$$

x being the length of the side, is positive.

$$\therefore \qquad \frac{d^2V}{dx^2} < 0$$

$\therefore$ The volume is maximum.

Hence maximum volume of the box $= \frac{a^3}{6\sqrt{3}}$.

Example 25.19 Show that of all rectangles inscribed in a given circle, the square has the maximum area.

Solution : Let ABCD be a rectangle inscribed in a circle of radius r. Then diameter AC= 2r

Let $\qquad AB = x$ and $BC = y$

Then $\qquad AB^2 + BC^2 = AC^2 \qquad$ or $\qquad x^2 + y^2 = (2r)^2 = 4r^2$

Now area A of the rectangle = xy

$$\therefore \qquad A = x\sqrt{4r^2 - x^2}$$

$$\therefore \qquad \frac{dA}{dx} = \frac{x(-2x)}{2\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2} \cdot 1$$

$$= \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

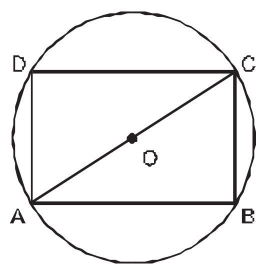


Fig. 25.10

For maxima/minima, $\frac{dA}{dx} = 0$

$$\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0 \Rightarrow x = \sqrt{2}r$$

Now $$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}\,(-4x) - \left(4r^2 - 2x^2\right)\frac{(-2x)}{2\sqrt{4r^2 - x^2}}}{\left(4r^2 - x^2\right)}$$

$$= \frac{-4x\left(4r^2 - x^2\right) + x\left(4r^2 - 2x^2\right)}{\left(4r^2 - x^2\right)^{\frac{3}{2}}}$$

$$= \frac{-4\sqrt{2}\left(2r^2\right) + 0}{\left(2r^2\right)^{\frac{3}{2}}} \qquad \text{... (Putting } x = \sqrt{2}r\text{)}$$

$$= \frac{-8\sqrt{2}r^3}{2\sqrt{2}r^3} = -4 < 0$$

Thus, A is maximum when $x = \sqrt{2}r$

Now, from (i), $\qquad y = \sqrt{4r^2 - 2r^2} = \sqrt{2}\, r$

x = y. Hence, rectangle ABCD is a square.

Example 25.20 Show that the height of a closed right circular cylinder of a given volume and least surface is equal to its diameter.

Solution : Let V be the volume, r the radius and h the height of the cylinder.

Then, $$V = \pi r^2 h$$

or $$h = \frac{V}{\pi r^2} \qquad \text{... (i)}$$

Now surface area $$S = 2\pi r h + 2\pi r^2$$

$$= 2\pi r \cdot \frac{V}{\pi r^2} + 2\pi r^2 = \frac{2V}{r} + 2\pi r^2$$

Now $$\frac{dS}{dr} = \frac{-2V}{r^2} + 4\pi r$$

For minimum surface area, $\frac{dS}{dr} = 0$

$$\therefore \quad \frac{-2V}{r^2} + 4\pi r = 0$$

or $$V = 2\pi r^3$$

From (i) and (ii), we get $$h = \frac{2\pi r^3}{\pi r^2} = 2r \qquad \text{..(ii)}$$

Again, $$\frac{d^2S}{dr^2} = \frac{4V}{r^3} + 4\pi = 8\pi + 4\pi \qquad \text{... [Using (ii)]}$$

$$= 12\pi > 0$$

$\therefore$ S is least when h = 2r

Thus, height of the cylidner = diameter of the cylinder.

Example 25.21 Show that a closed right circular cylinder of given surface has maximum volume if its height equals the diameter of its base.

Solution : Let S and V denote the surface area and the volume of the closed right circular cylinder of height h and base radius r.

Then $$S = 2\pi r h + 2\pi r^2 \qquad \text{.....(i)}$$

(Here surface is a constant quantity, being given)

$$V = \pi r^2 h$$

$$\therefore \quad V = \pi r^2 \left[\frac{S - 2\pi r^2}{2\pi r}\right]$$

$$= \frac{r}{2}\left[S - 2\pi r^2\right]$$

$$V = \frac{Sr}{2} - \pi r^3$$

$$\frac{dV}{dr} = \frac{S}{2} - \pi\left(3r^2\right)$$

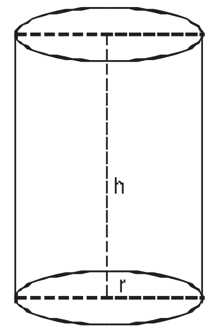


Fig. 25.11

For maximum or minimum, $\frac{dV}{dr} = 0$

i.e., $\frac{S}{2} - \pi\left(3r^2\right) = 0$

or $S = 6\pi r^2$

From (i), we have $6\pi r^2 = 2\pi rh + 2\pi r^2$

$\Rightarrow$ $4\pi r^2 = 2\pi rh$

$\Rightarrow$ $2r = h$ ...(ii)

Also, $$\frac{d^2V}{dr^2} = \frac{d}{dr}\left[\frac{S}{2} - 3\pi r^2\right]$$

$$= -6\pi r, \qquad \because \frac{d}{dr}\left(\frac{S}{2}\right) = 0$$

= a negative quantity

Hence the volume of the right circular cylinder is maximum when its height is equal to twice its radius i.e. when $h = 2r$.

Example 25.22 A square metal sheet of side 48 cm. has four equal squares removed from the corners and the sides are then turned up so as to form an open box. Determine the size of the square cut so that volume of the box is maximum.

Solution : Let the side of each of the small squares cut be x cm, so that each side of the box to be made is (48–2x) cm. and height x cm.

Now $x > 0$, $48-2x > 0$, i.e. $x < 24$

$\therefore$ x lies between 0 and 24 or $0 < x < 24$

Now, Volume V of the box

$$= (48-2x)(48-2x)\,x$$

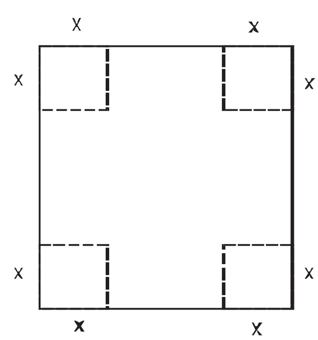


Fig. 25.12

i.e. $$V = (48-2x)^2 \cdot x$$

$\therefore$ $$\frac{dV}{dx} = (48-2x)^2 + 2(48-2x)(-2)\,x$$

$$= (48-2x)(48-6x)$$

Condition for maximum or minimum is $\frac{dV}{dx} = 0$

i.e., $$(48-2x)(48-6x) = 0$$

We have either $x = 24$, or $x = 8$

$\because$ $$0 < x < 24$$

$\therefore$ Rejecting x = 24, we have, x = 8 cm.

Now, $$\frac{d^2V}{dx^2} = 24x - 384$$

$$\left(\frac{d^2V}{dx^2}\right)_{x=8} = 192 - 384 = -192 < 0,$$

Hence for x = 8, the volume is maximum.

Hence the square of side 8 cm. should be cut from each corner.

Example 25.23 The profit function P (x) of a firm, selling x items per day is given by

$$P(x) = (150-x)\,x - 1625.$$

Find the number of items the firm should manufacture to get maximum profit. Find the maximum profit.

Solution : It is given that 'x' is the number of items produced and sold out by the firm every day. In order to maximize profit,

$$P'(x) = 0 \text{ i.e. } \frac{dP}{dx} = 0$$

or $$\frac{d}{dx}\left[(150-x)\,x - 1625\right] = 0$$

or $$150 - 2x = 0$$

or $$x = 75$$

Now, $\frac{d}{dx}P'(x) = P''(x) = -2$, a negative quantity. Hence P (x) is maximum for x = 75.

Thus, the firm should manufacture only 75 items a day to make maximum profit.

Now, Maximum Profit $= P(75) = (150-75)\,75 - 1625$

= Rs. (75×75–1625)

= Rs. (5625–1625)

= Rs. 4000

Example 25.24 Find the volume of the largest cylinder that can be inscribed in a sphere of radius 'r' cm.

Solution : Let h be the height and R the radius of the base of the inscribed cylinder. Let V be the volume of the cylinder.

Then $$V = \pi R^2 h \qquad \text{...(i)}$$

From Δ OCB, we have

$$r^2 = \left(\frac{h}{2}\right)^2 + R^2 \qquad \text{...}\left(\because OB^2 = OC^2 + BC^2\right)$$

$$\therefore \quad R^2 = r^2 - \frac{h^2}{4} \qquad \text{...(ii)}$$

Now $$V = \pi\left(r^2 - \frac{h^2}{4}\right)h = \pi r^2 h - \pi\frac{h^3}{4}$$

$$\therefore \quad \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4}$$

For maxima/minima, $\dfrac{dV}{dh} = 0$

$$\therefore \quad \pi r^2 - \frac{3\pi h^2}{4} = 0$$

$$\Rightarrow \quad h^2 = \frac{4r^2}{3} \qquad \Rightarrow \quad h = \frac{2r}{\sqrt{3}}$$

Now $$\frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$$

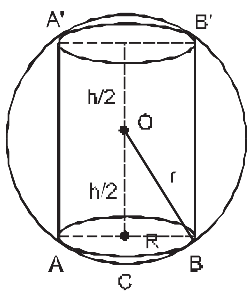


Fig. 25.13

$$\therefore \quad \frac{d^2V}{dh^2}\left(at\ h = \frac{2r}{\sqrt{3}}\right) = -\frac{3\pi\times 2r}{2\times\sqrt{3}}$$

$$= -\sqrt{3}\,\pi r < 0$$

$\therefore$ V is maximum at $h = \dfrac{2r}{\sqrt{3}}$

Putting $h = \dfrac{2r}{\sqrt{3}}$ in (ii), we get

$$R^2 = r^2 - \frac{4r^2}{4\times 3} = \frac{2r^2}{3}$$

$$\therefore \quad R = \sqrt{\frac{2}{3}}\,r$$

Maximum volume of the cylinder $= \pi R^2 h$

$$= \pi\left(\frac{2}{3}r^2\right)\frac{2r}{\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}\,cm^3.$$

CHECK YOUR PROGRESS 25.5

1. Find two numbers whose sum is 15 and the square of one multiplied by the cube of the other is maximum.
2. Divide 15 into two parts such that the sum of their squares is minimum.
3. Show that among the rectangles of given perimeter, the square has the greatest area.
4. Prove that the perimeter of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles.
5. A window is in the form of a rectangle surmounted by a semi-circle. If the perimeter be 30 m, find the dimensions so that the greatest possible amount of light may be admitted.
6. Find the radius of a closed right circular cylinder of volume 100 c.c. which has the minimum total surface area.
7. A right circular cylinder is to be made so that the sum of its radius and its height is 6 m. Find the maximum volume of the cylinder.
8. Show that the height of a right circular cylinder of greatest volume that can be inscribed in a right circular cone is one-third that of the cone.
9. A conical tent of the given capacity (volume) has to be constructed. Find the ratio of the height to the radius of the base so as to minimise the canvas requried for the tent.
10. A manufacturer needs a container that is right circular cylinder with a volume 16π cubic meters. Determine the dimensions of the container that uses the least amount of surface (sheet) material.
11. A movie theatre's management is considering reducing the price of tickets from Rs.55 in order to get more customers. After checking out various facts they decide that the average number of customers per day 'q' is given by the function where x is the amount of ticket price reduced. Find the ticket price othat result in maximum revenue.

$$q = 500+100\,x$$

where x is the amount of ticket price reduced. Find the ticket price that result is maximum revenue.

LET US SUM UP

- **Increasing function :** A function f (x) is said to be increasing in the closed interval [a,b]

  if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$

- **Decreasing function :** A function f (x) is said to be decreasing in the closed interval [a,b]

  if $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$

- f (x) is increasing in an open interval]a,b[

  if $f'(x) > 0$ for all $x \in [a,b]$

- f (x) is decreasing in an open interval]a,b[

if $f'(x) < 0$ for all $x \in [a,b]$

- **Monotonic function :**

(i) A function is said to be monotonic (increasing) if it increases in the given interval.

(ii) A function f (x) is said to be monotonic (decreasing) if it decreases in the given interval.

A function f (x) which increases and decreases in a given interval, is not monotonic.

- In an interval around the point x = a of the function f (x),

(i) if $f'(x) > 0$ on the left of the point 'a' and $f'(x) < 0$ on the right of the point x = a, then f (x) has a local maximum.

(ii) if $f'(x) < 0$ on the left of the point 'a' and $f'(x) > 0$ on the right of the point x = a, then f (x) has a local minimum.

- If f (x) has a local maximum or local minimum at x = a and f (x) is derivable at x = a, then $f'(a) = 0$

- (i) If f '(x) changes sign from positive to negative as x passes through 'a', then f (x) has a local maximum at x = a.

(ii) If f ' (x) changes sign from negative to positive as x passes othrough 'a', then f (x) has a local minimum at x = a.

- **Second order derivative Test :**

(i) If $f'(a) = 0$, and $f''(a) < 0$; then f (x) has a local maximum at x = a.

(ii) If $f'(a) = 0$, and $f''(a) > 0$; then f (x) has a local minimum at x = a.

(iii) In case $f'(a) = 0$, and $f''(a) = 0$; then to determine maximum or minimum at x = a, we use the method of change of sign of f '(x) as x passes through 'a' to.

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. Show that $f(x) = x^2$ is neither increasing nor decreasing for all $x \in R$.

Find the intervals for which the following functions are increasing or decreasing :

2. $2x^3 - 3x^2 - 12x + 6$

3. $\frac{x}{4} + \frac{4}{x}, x \neq 0$

4. $x^4 - 2x^2$

5. $\sin x - \cos x, 0 \leq x \leq 2\pi$

Find the local maxima or minima of the following functions :

6. (a) $x^3 - 6x^2 + 9x + 7$ (b) $2x^3 - 24x + 107$

(c) $x^3 + 4x^2 - 3x + 2$ (d) $x^4 - 62x^2 + 120x + 9$

7. (a) $\dfrac{1}{x^2 + 2}$ (b) $\dfrac{x}{(x-1)(x-4)}, 1 < x < 4$

(c) $x\sqrt{1-x}, \quad x < 1$

8. (a) $\sin x + \dfrac{1}{2}\cos 2x, \quad 0 \le x \le \dfrac{\pi}{2}$ (b) $\sin 2x \quad , 0 \le x \le 2\pi$

(c) $-x + 2\sin x, 0 \le x \le 2\pi$

9. For what value of x lying in the closed interval [0,5], the slope of the tangent to

$$x^3 - 6x^2 + 9x + 4$$

is maximum. Also, find the point.

10. Find the vlaue of the greatest slope of a tangent to

$-x^3 + 3x^2 + 2x - 27$ at a point of othe curve. Find also the point.

11. A container is to be made in the shape of a right circular cylinder with total surface area of 24π sq. m. Determine the dimensions of the container if the volume is to be as large as possible.

12. A hotel complex consisting of 400 two bedroom apartments has 300 of them rented and the rent is Rs. 360 per day. Management's research indicates that if the rent is reduced by x rupees then the number of apartments rented q will be $q = \dfrac{5}{4}x + 300, 0 \le x \le 80$.

Determine the rent that results in maximum revenue. Also find the maximum revenue.

ANSWERS

CHECK YOUR PROGRESS 25.2

1. (a) Increasing for $x > \frac{7}{2}$, Decreasing for $x < \frac{7}{2}$

   (b) Increasing for $x > \frac{5}{2}$, Decreasing for $x < \frac{5}{2}$

2. (a) Increasing for $x > 6$ or $x < -2$, Decreasing for $-2 < x < 6$

   (b) Increasing for $x > 4$ or $x < 2$, Decreasing for x in the interval]2,4[

3. (a) Increasing for $x < -2$; decreasing for $x > -2$

   (b) Increasing in the interval $-1 < x < 2$, Decreasing for $x > -1$ or $x < -2$

4. (a) Increasing always.

   (b) Increasing for $x > 2$, Decreasing in the interval $0 < x < 2$

   (c) Increasing for $x > 2$ or $x < -2$ Decreasing in the interval $-2 < x < 2$

5. (c) Increasing in the interval $\frac{3\pi}{8} \le x \le \frac{7\pi}{8}$

   Decreasing in the interval $0 \le x \le \frac{3\pi}{8}$

   Points at which the tangents are parallel to x-axis are $x = \frac{3\pi}{8}$ and $x = \frac{7\pi}{8}$

CHECK YOUR PROGRESS 25.3

1. Local minimum is – 4 at x = 4
2. Local minimum is 15 at x = 3, Local maximum is 19 at x = 1.
3. Local minimum is – 128 at x = 6, Local maximum is – 3 at x = 1.
4. Local minimum is – 1647 at x = – 6, Local minimum is – 316 at x = 5,

   Local maximum is 68 at x = 1.
5. Local minimum at x = 0 is – 4, Local maximum at x = – 2 is 0.
6. Local minimum at x = – 1, value $= -1$ Local maximum at x = 3, value $= \frac{1}{7}$

CHECK YOUR PROGRESS 25.4

1. Local minimum is – 34 at x = 2, Local maximum is 91 at x = – 3.
2. Local minimum is – 5 at x = 0, Local maximum is 251 at x = 8.
3. Local minimum – 4 at x = 0, Local maximum 0 at x = – 2.
4. Local minimum = – 28; x = 3, Local maximum = 0 ; x = 1.

   Neither maximum nor minimum at x = 0.

5. Local maximum $= \frac{3\sqrt{3}}{4}$; $x = \frac{\pi}{3}$

6. Local maximum $= \sqrt{2}$; $x = \frac{\pi}{4}$

7. Local minimum $= \frac{-\sqrt{3}}{2} + \frac{\pi}{6}$; $x = \frac{\pi}{6}$, Local maximum $= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$; $x = \frac{\pi}{6}$

CHECK YOUR PROGRESS 25.5

1. Numbers are 6,9.

2. Parts are 7.5, 7.5

5. Dimensions are : $= \frac{30}{\pi+4}, \frac{30}{\pi+4}$ meters each.

6. radius $= \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm ; height $= 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm

7. Maximum Volume $= 32\,\pi$ cubic meters.

9. $h = \sqrt{2}r$

10. r = 2 meters, h = 4 meters.

11. Rs. 30,00

TERMINAL EXERCISE

2. Increasing for $x > 2$ or < -1, Decreasing in the interval $-1 < x < 2$
3. Increasing for $x > 4$ or $x < -4$, Decreasing in the interval $]-4,4[$
4. Increasing for $x > 1$ or $-1 < x < 0$, Decreasing for $x < -1$ or $0 < x < 1$
5. Increasing for $0 \le x \le \frac{3\pi}{4}$ or $\frac{7\pi}{4} \le x \le 2\pi$, Decreasing for $\frac{3\pi}{4} \le x \le \frac{7\pi}{4}$.
6. (a) Local maximum is 11 at $x = 1$; local minimum is 7 at $x = 3$.

   (b) Local maximum is 139 at $x = -2$; local minimum is 75 at $x = 2$.

   (c) Local maximum is 20 at $x = -3$; local minimum is $\frac{40}{27}$ at $x = \frac{1}{3}$.

   (d) Local maximum is 68 at $x = 1$; local minimum is -316 at $x = 5$ and -1647 at $x = -6$.
7. (a) Local minimum is $\frac{1}{2}$ at $x = 0$. (b) Local maximum is -1 at $x = 2$.

   (c) Local maximum is $\frac{2}{3\sqrt{3}}$ at $x = \frac{2}{3}$.
8. (a) Local maximum is $\frac{3}{4}$ at $x = \frac{\pi}{6}$; Local minimum is $\frac{1}{2}$ at $x = \frac{\pi}{2}$;

Notes

(b) Local maximum is 1 at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$; Local minimum is –1 at $x = \frac{3\pi}{4}$

(c) Local maximum is $\frac{-\pi}{4} + \sqrt{3}$ at $x = \frac{\pi}{3}$; Local minimum is $\frac{-5\pi}{3} - \sqrt{3}$ at $x = \frac{5\pi}{3}$.

9. Greatest slope is 24 at x = 5; Coordinates of the point: (5, 24).

10. Greatest slope of a tangent is 5 at x = 1, The point is (1,–23)

11. Radius of base = 2 m, Height of cylinder = 4 m.

12. Rent reduced to Rs. 300, The maximum revenue = Rs. 1,12,500.

Partial Differentiation

Objectives

After studying this lesson, you will be able to know.

1. Definition of partial derivative corresponds closed to that of derivative of a function of a single variable.
2. The existence of first order partial derivatives does not guarantee the contінity of the function.
3. Discuss about the functions with two independent variables.

25.9 Partial Differentiation

So far we have studied the concepts of limits, continuity, differentiability of functions of a single variable. In this chapter we consider these concepts for function of two or more real variables. Most of the definitions and theorems apply to function of n independent variables as well, but we restrict our discussion to functions of two independent variables only.

Recall that the cartesian product $A \times B$ of sets A and B is the collection of all ordered pairs (a, b) with $a \in$ A and $b \in$ B. If $A \subseteq R$ and $B \subseteq R$, $A \times B$ is a subset of $R \times R$ i.e., R^2. The elements of R^2 can be considered to be points in the two dimensional plane. If fact, if $(a, b) \in R^2$, it can be identified with a unique point with cartesian coordinates (a, b) in the plane.

Suppose $A \subseteq R, B \subseteq R$ and $E = A \times B$ so that $E \subseteq R^2$. Then a function $f : E \to R$ is called a real valued function of two real variables. Also for $(x, y) \in$ E its image under f is denoted by

$f(x, y)$ is a real number, say Z, for any $(x, y) \in$ E, So that we can write $z = f(x, y)$. Some times it is convenient to write $z = z(x, y)$, where the function it self is denoted by z. Throughout this chapter by a function of two variables we mean a real valued function of two real variables.

25.9.1 Partial Derivatives

In in the function $f(x, y)$, y is held constant and the resulting function of x is differentiated with respect to x, then the resulting derivative is called the partial derivative of f with respect to x, That is denoted by $\frac{\partial f}{\partial x}$ or f_x

That is, we define $\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$ provided the limit exists.

Similarly, the partial dervative of f with respect to y denoted by $\frac{\partial f}{\partial y}$ or f_y is defined as

$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x+y+h) - f(x, y)}{h}$ provided the limit exists. The partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are called first order partial derivative of f with respect to x and y respectively.

Example 1

If $f(x, y) = x^2y^2 - y \sin x$, then

$$\frac{\partial f}{\partial x} = y^2 (2x) - y \cos x$$

$$= 2xy^2 - y \cos x$$

$$\frac{\partial f}{\partial y} = x^2 (2y) - \sin x = 2x^2 y - \sin x.$$

One can notice that the definition of a partial derivative corresponds closely to that of an ordinary derivative of a function of a single variable. Consequently many of the rules of differentiation of function of a single variable remain valid in the case of partial differentiation. However, the existance of all the first order partial derivatives at a point does not guarantee the continuity of the function there. The following example illustrates if.

Example 2.

Let $f(x, y) = \dfrac{xy}{x^2+y^2}$, if $(x, y) \neq (0, 0)$. If $(x, y) = (0, 0)$ i.e., $f(0, 0) = 0$ For $h \neq 0$, we have

$$\frac{f(h,0) - f(0,0)}{h} = 0$$

Hence $\lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h}$ exists and is equal to zero.

Hence $f_x|_{(0,\,0)} = 0$

Similarly $f_y|_{(0,\,0)} = 0$

We have already noted that $f(x, y)$ is not continuous at (0, 0_

Thus the existance of partial derivatives may not guarantee the conitnuity of f.

25.9.2 The Second order partial derivatives

As the partial derivatives of a function $f(x, y)$ are themselves function of x and y, they inteurn may process partial derivatives. We define these second order partial derivatives as follows :

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x)$$

$$= \lim_{h \to 0} \frac{f_x(x+h,\ y) - f_x(x,\ y)}{h}$$

and $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y)$$

$$= \lim_{k \to 0} \frac{f_y(x,\ y+k) - f_y(x,\ y)}{k}$$

provided these limits exists.

In addition to these, There are two other mixed second order partial derivatives, namely

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x)$$

$$= \lim_{k \to 0} \frac{f_x(x,\ y+k) - f_x(x,\ y)}{k}$$

and $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(f_y\right)$$

$$= \lim_{h \to 0} \frac{f_y(x+h,\ y) - f_y(x,\ y)}{h}$$

Note that $\dfrac{\partial^2 f}{\partial y \partial x} = f_{xy}$ for

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(f_x\right) = [f_x]_y = f_{xy}$$

similarly, $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$

Note also that $\dfrac{\partial^2 f}{\partial x^2} = f_{xx}$, $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$ we shall confine to those functions for which $f_{xy} = f_{yx}$.

Example 3

Find all the first and second order partial derivatives for $f(x, y) = e^{2x-y}$.

Sol: For the given function

$$f_x = 2e^{2x-y} \quad , \qquad f_{xx} = 4\ e^{2x-y}$$

$$f_y = -\ e^{2x-y} \quad , \qquad f_{yy} = e^{2x-y}$$

$$f_{xy} = -2\ e^{2x-y} \quad , \qquad f_{xx} = -\ 2\ e^{2x-y}$$

Example 4

Find all the first and second order partial derivatives for $f(x, y) = \cos(ax + by)$ where a and b are real constants.

Sol : For the given function

$$f_x = -\ a \sin(ax + by) \quad , \qquad f_y = -\ b \sin(ax + by)$$

$$f_{xx} = -\ a^2 \cos(ax + by), \qquad f_{yy} = -\ b^2 \cos(ax + by)$$

$$f_{xy} = [f_x]_y = -ab \cos(ax + by)$$

$$f_{yx} = [f_y]_x = -ab \cos(ax + by)$$

Example 5

If $z = e^{ax} \sin by$ where a and b are real constants find Z_x. Z_y, Z_{xx}, Z_{yy}, Z_{yx}.

Sol :

Given $Z = e^{ax} \sin by$,

$Z_x = a\, e^{ax} \sin by$, $Z_y = b\, e^{ax} \cos by$

$Z_{xx} = (Z_x)_x = a^2 e^{ax} \sin by$, $Z_{yy} = (Z_y)_y = -b^2 e^{ax} \sin by$

$Z_{xy} = (Z_x)_y = ab\, e^{ax} \cos by$ and

$Z_{yx} = (Z_y)_x = ab\, e^{ax} \cos by$

Example 6

If $f = x^2 - y^2$, show that $f_{xx} + f_{yy} = 0$.

Sol: Given that $f_x = 2x \Rightarrow f_{xx} = 2$

$$f_y = -2y \Rightarrow f_{yy} = -2$$

$$f_{xx} + f_{yy} = 2 - 2 = 0$$

$$\therefore \quad f_{xx} + f_{yy} = 0$$

25.10 Homogeneous Functions

We shall discuss an important class of functions, namely the class of homogeneous functions and an important theorem due to Euler, related to such functions. First let us give a formal definition of a homogeneous function.

25.10.1 Defintion

A non-zero function $f = f(x, y)$ is said to be homogeneous if there is a real number α such that $f(\lambda x, \lambda y) = \lambda^{\alpha} f(x, y)$ for all those values of k for which both sides of the above equation are meaningful. The number α is called the degree of the homogeneous function f.

Example 7 :

$f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3, why becuase

$$f(\lambda x, \lambda y) = \lambda^3[x^3 + y^3]$$

$$= \lambda^3 f(x, y)$$

Example 8 :

$f(x, y) = (x^2 + 4y^2)^{-2/7}$, is a homogeneous function of degree $\frac{-4}{7}$. Since

$$f(\lambda x, \lambda y) = (\lambda^2 x^2 + 4\lambda^2 y^2)^{-2/7}$$

$$f(\lambda x, \lambda y) = \lambda^{-4/7} f(x, y)$$

Check Your Progress 1

1. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for given functions.

   (i) $z = \frac{\cos x}{\operatorname{cosec} y}$ (ii) $z = 3y + x e^y$

   (iii) $z = \operatorname{Tan}^{-1}\left(\frac{x}{y}\right)$ (iv) $z = \log\left(x + \frac{y}{x^2}\right)$

   (v) $z = y e^y + x e^x$ (vi) $z = \sin(y^2 - x)$

2. If $z = e^x \cos y$ then find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}$

3. If $f = \frac{y}{x^2 + y^2}$ show that $f_{xx} + f_{yy} = 0$.

4. If $z = e^{x+y} + f(x) + g(y)$ then show that $z_{xy} = e^{x+y}$.

5. If $z = \log(\operatorname{Tan} x + \operatorname{Tan} y)$. Then show that $(\sin 2x) z_x + (\sin 2y) z_y = 2$.

6. If $u(x, y) = \dfrac{x^2 y}{x^3 + y^3}$ Then show that $x\,u_x + y\,u_y = 0$.

Let Us Sum Up

1. $$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h,\ y) - f(x,\ y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x,\ y+k) - f(x,\ y)}{k}$$

2. $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = \lim_{h \to 0} \frac{f_x(x+h,\ y) - f_x(x,\ y)}{h}$$

$$= f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y) = \lim_{k \to 0} \frac{f_y(x,\ y+k) - f_y(x,\ y)}{k}$$

$$= f_{yy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = \lim_{k \to 0} \frac{f_x(x,\ y+k) - f_x(x,\ y)}{k}$$

$$= f_{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = \lim_{h \to 0} \frac{f_y(x+h,\ y) - f_y(x,\ y)}{h}$$

$$= f_{yx}$$

Supportive Websites

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

Answers

Check Your Progress 1

(1) (i) $z_x = -\sin x \sin y$, $zy = \cos$

(ii) $z_x = e^y$, $z_y = 3 + x\,e^y$

(iii) $z_x = \dfrac{y}{x^2 + y^2}$, $z_y = \dfrac{-x}{x^2 + y^2}$

(iv) $z_x = \dfrac{x^3 - zy}{x(x^3 + y)}$, $z_y = \dfrac{1}{y + x^3}$

(v) $z_x = x\,e^x + e^x$, $z_y = y\,e^y + e^y$

(vi) $z_x = -\cos(y^2 - x)$, $z_y = 2y\cos(y^2 - x)$

(2) $\dfrac{\partial^2 z}{\partial x^2} = e^x \cos y$

$\dfrac{\partial^2 z}{\partial y^2} = -\cos y\; e^x = -z$

Successive Differentiation

Second and higher order derivatives have applications in curve sketching and in the representation of functions as infinite series. Further, they are useful in differential equations, complex analysis etc.,

Objectives

From this chapter we can get the following points.

- curve sketching
- Representation of functions as infinite series
- n^{th} order derivatives of some standard functions.

25.11 Successive Differentiation - n^{th} derivatives

In this section we define first and higher order derivatives of a real valued function f and find the n^{th} order derivatives of some standard functions.

Definition

Let f be a real valued function defined on a subset E of R. Suppose that E contains a neighbour hood (left or right of both) of each of its points. We known that f is the limit of $\dfrac{f(c+h)-f(c)}{h}$ as h tends to 0 exists. This limit is denoted by $f'(c)$ and is called the derivative of f at 'c' or the differential coefficient of f at 'c'. It f is differentiable at every point x of its domain E, then we can define a function f' on E as $f'(x)$ is the limit of $\dfrac{f(x+h)-f(x)}{h}$ as h tends to 0. The function f' thus defined is called the first derivative or first order derivative of f.

Again of the function f' is differentiable at every point x of E, then we can define function f'' on E as

$$f''(x) = \underset{x\to 0}{Lt}\ \frac{f'(x+h)-f'(x)}{h}$$

as h tends to 0.

The function thus defined is called the second derivative or second orderivative of f. This process can be continued as long as the successive derivatives are differentiable at every point of E. In general, the nth derivative or n^{th} order derivative of f is the derivative of $f^{(n-1)}$ and is denoted by $f^{(n)}$ (i.e., $f''^{\,.....(n \text{ dasehes})}$). Thus

$$f^{n}(x) = \underset{h\to 0}{Lt}\ \frac{f^{(n-1)}(x+h)-f^{n-1}(x)}{h}$$

as h tends to 0.

If $y = f(x)$ then the notations for the n^{th} derivative f are

$$\frac{d^n f}{dx^n},\ D^n f,\ \frac{d^n y}{dx^n},\ y_n$$

which are used according to our convenience. Note that

$$f'(x) = \frac{d}{dx}[f(x)] \text{ and } f^{(n)}(x) = \frac{d}{dx}[f^{(n-1)}(x)]$$

$$f^{n}(x) = \frac{d^n y}{dx^n} = y_n = D^n y$$

$$= \frac{d^{n-1}}{dx^{n-1}}[f'(x)] \text{ for } n > 1.$$

If $f^{(n)}(x)$ exists for all positive integers n and for all x in E, then f is said to be an infinitely differntiable function on E.

The above porcess of finding higher order derivative of f is called successive differentiation.

25.11.1 n^{th} order derivatives of some standard functions

(i) Let a, b and m be real numbers. Suppose that $a \neq 0$ and $f(x) = (ax+b)^m$ Then

$f^n(x) = m(m-1) \ldots (m-n+1)\, a^n\, (ax+b)^{m-n}$ $(n \in \mathrm{N})$

whenever $ax + b > 0$ and n is a positive integer.

Note: In the above theorem if m is a positive integer, then $m = n$ impllies that $f^{(n)}(x) = n!\, a^n$ and hence $f^{(n)}(x) = 0$ for $n > m$.

If m is a positive integer such that $m > n$, then

$$f^{(n)}(x) = m(m-1) \ldots\ldots (m-n+1)\, a^n\, (ax+b)^{m-n}$$

$$= \frac{m!\, a^n}{(m-n)!}(ax+b)^{m-n}$$

(ii) On taking $m = -1$ in theorem 7.1.4, we get

$$f(x) = (ax+b)^{-1} = \frac{1}{ax+b},$$

and $f^{(n)}(x) = (-1)\,(-2) \ldots\ldots (-n)\, a^n\, (ax+b)^{-1-n}$

that is, $\dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}.$

(iii) Let $f(x) = e^{ax+b}$ for all x in R. Then $f^n(x) = a^n e^{ax+b}$ for all x in R and for all positive integers n.

Example 1.

Find the 4^{th} derivative of $\dfrac{1}{2x+3}$.

Sol: $f(x) = \dfrac{1}{ax+b} \Rightarrow f^n(x) = \dfrac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

by above theorem, taking $a = 2$, $b = 3$ and $m = -1$ we get

$$f^4(x) = (-1)\,(-2)(-3)(-4)\, 2^4\, (2x+3)^{-5}$$

$$= \frac{4!\, 2^4}{(2x+3)^5}$$

Example 2

Find the 5th derivative of sin 6x.

Sol: $\frac{d^n}{dx^n}[\sin(ax+b)] = a^n \sin\left(ax+b+\frac{n\pi}{2}\right) \quad \forall n \in N$

The above formula taking $a = 6, b = 0, n = 5$

$$D^5(\sin 6x) = 6^5 \sin\left(6x + \frac{5\pi}{2}\right)$$

$$= 6^5 \sin\left(6x + \frac{\pi}{2}\right)$$

$$= 6^5 \cos 6x.$$

Example 3

Show that $f(x) = x + \cot x$ satisfies $(\sin^2 x) f''(x) + 2x = 2 f(x)$.

Sol : Now $f'(x) = 1 - \operatorname{cosec}^2 x$ .... (1)

so that $(\sin^2 x) f'(x) = \sin^2 x - 1$ .... (2)

Differentiating both sides of (2) w.r.t x,

$(\sin^2 x) f''(x) + 2 \sin x \cos x f'(x) = 2 \sin x \cos x$

Hence

$$\sin^2 x f''(x) = 2 \sin x \cos x(1 - f'(x))$$

$$= 2 \sin x \cos x \operatorname{cosec}^2 x \ (1) \text{ వల్ల}$$

$$= 2 \cot x = 2 [f(x) - x]$$

Check Your Progress 1

I 1. If $y = \frac{a+bx}{c+dx}$, Then show that $2 y_1 y_3 = 3y_2^2$

2. If $y = a + b e^{-4x}$, Then show that $y_2 + 4y_1 = 0$.

3. If $y = ax^{n+1} + b x^{-n}$, then show that $x^2 y_2 = n(n + 1)y$.

4. If $y = x + \operatorname{Tan} x$, then show that $y_2 \cos^2 x + 2x = 2y$.

5. If $a\,y^4 = (x + b)^5$, then show that $5y\,y_2 = y_1{}^2$.

6. If $ax^2 + 2h\,xy + by^2 = 1$. Then show that $\dfrac{d^2y}{dx^2} = \dfrac{h^2-ab}{(hx+by)^3}$.

II. Find the n^{th} derivative of following functions.

1. Find the n^{th} derivative of $\cos^2 x$.

2. Find the n^{th} derivative of $\dfrac{1}{3x+4}$.

3. If $y = \dfrac{2}{(x-1)(x-2)}$ Then find y_n.

25.12 Leibnitz Theorem and its Applications

Let f and g be two functions having n^{th} order derivatives over a domain $E \subseteq R$. We have seen in the previous section that $f+g$ and $f-g, kf$ (k being a constant) have n^{th} order derivatives over E and that

$$(f + g)^{(n)} = f^{(n)} + g^{(n)}$$

$$(f - g)^{(n)} = f^{(n)} - g^{(n)}$$

$$(k\,f)^{(n)} = k\,f^{(n)}$$

In this section we obtain a formula for the n^{th} derivative of the product fg of two functions f and g.

25.12.1 Theorem (Leibnitz Theorem)

Suppose that n is a positive integer. Let f and g be two functions defined on an interval I or R having n^{th} order derivative on I. The fg has n^{th} other derivative on I and

$$(f\,g^{(n)}(x) = f^{\,n}(x)\,.\,g(x) + n\,c_1\,f^{(n-1)}(x)\,g'(x) + \ldots.. + n\,c_r\,f^{(n-r)}(x)\,g^{(r)}(x) + \ldots. + f(x)\,g^{(n)}(x)$$

$$= \sum_{r=0}^{n} n_{c_r}\,f^{(n-r)}(x)\,g^{(r)}(x) \;\; \forall x \in \mathrm{I}$$

Note: Where u and v are used for $f(x)$ and $g(x)$, n^{th} derivative of u denoted by u_n and $u_0 = u$.

$$(uv)_n = n_{c_0}\,u_n\,v_0 + n_{c_1}\,u_{n-1}v_1 + \ldots. + n_{c_r}\,u_{n-r}v_r + \ldots. + n_{c_n}\,u_0 v_n$$

Example 4

If u and v are function of x having 4^{th} order derivatives, write the formula for $D^4(uv)$.

Sol: By Leibnitz theorem

$$D^4(uv) = D^4(u)\, v + 4c_1\, D^3(u).\, D(v) + 4c_2\, D^2(u)\, D^2(v) + 4c_3\, D(u)\, D^3v + u\, D^4(v)$$

$$= u_4\, v + 4\, u_3\, v_1 + 6\, u_2\, v_2 + 4\, u_1\, v_3 + u\, v_4$$

Example 5

Find $D^4\,(x \log x)$ at $x = e,\ x > 0$

Sol: By Leibnitz Theorem

$$D^4\,(x \log x) = D^4\,(\log x\, .\, x)$$

$$= D^4\,(\log x)\, x + 4\, c_1\, D^3\,(\log x)\, D(x)$$

(Since higher derivatives of x vanish

$$= \frac{-6}{x^4}.\, x + 4.\frac{2}{x^3}\, .1$$

$$= \frac{-6}{x^3} + \frac{8}{x^3} = \frac{2}{x^3}$$

$\therefore$ at $x = e$

$$D^4(x \log x) = \frac{2}{e^3}$$

Example 6

Find the n^{th} derivative of $x\, e^x$.

Sol: By Leibnitz Theorem

$$D^n\,(x\, e^x) = D^n(e^x\, x)$$

$$= D^n(e^x)\, x + nc_1\, D^{n-1}(e^x)\, D(x)$$

$$= e^x\, .\, x + n\, e^x$$

$$= (x + n)\, e^x$$

Check Your Progress 2

I. 1. Find $D^3(\sin x \log x)$ when $x = \frac{\pi}{2}$.

2. Find the sum of the middle terms in the Leibnitz expansion of $D^3(e^x \cos x)$.

II(1) Find the n^{th} derivative of the following using Leibnitz theorem.

(i) $h(x) = e^{3x+5}\, x^2$

(ii) $x^3\, e^{ax}$

(iii) $x^2\, \cos 2x$

(2) If $y = x^n \log x$, then show that $x\, y_{n+1} = n!$.

(3) If $y = \cos(m \log x)$ $(x > 0)$ then show that $x^2 y_2 + xy_1 + m^2 y = 0$ and hence deduce that

$x^2 y_{n+2} + (2n + 1)\, x\, y_{n+1} + (m^2 + n^2)\, y_n = 0.$

(4) If $y = e^{\sin^{-1} x}$, then show that $(1 - x^2)\, y_2 - x\, y_1 - y = 0$ and hence deduce that

$(1 - x^2)\, y_{n+2} - (2n + 1)\, x\, y_{n+1} - (n^2 + 1)\, y_n = 0$

Let Us Sum Up

	Function $f(x)$	n^{th} derivative $f^{(n)}(x)$
1.	(ax + b)m	$m(m-1) \ldots. (m-n+1)\, a^n\, (ax + b)^{m-n}$
2.	$\frac{1}{ax+b}$	$\frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$
3.	e^{ax+b}	$a^n\, e^{ax+b}$
4.	$\sin(ax + b)$	$a^n \sin(ax + b + \frac{n\pi}{2})$
5.	$\cos(ax + b)$	$a^n \cos(ax + b + \frac{n\pi}{2})$

6. $h(x) + g(x)$ — $h^{(n)}(x) \pm g^{(n)}(x)$

7. $k\,g$ — $k\,g^{(n)}(x)$, k ఒక స్థిర సంఖ్య

8. Leibnitz Theorem $(hg)(x)$ — $\sum_{r=0}^{n} n_{c_r}\, h^{(n-r)}(x)\, g^{(r)}(x)$

Supportive Website

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

Answers

Check Your Progress 1

II 1. $2^{n-1}\cos\left(2x + \frac{2\pi}{2}\right)$

2. $\dfrac{(-1)^n n!\, 3^n}{(3x+4)^{n+1}}$

3. $\dfrac{(-2)(-1)^n n!}{(x-1)^{n+1}} + \dfrac{2(-1)^n\, n!}{(x-2)^{n+1}}$

Check Your Progress 2

I. 1. $\dfrac{16}{\pi^3} - \dfrac{6}{\pi}$

2. $-3\, e^x (\sin x + \cos x)$

II. 1. (i) $e^{3x+5}\, 3^{n-2}\, [9x^2 + 6nx + n(n-1)]$

(ii) $e^{ax}\, [a^n x^3 + 3n\, x^2\, a^{n-1} + 3n\,(n-1)\, a^{n-2}\, x + n(n-1)(n-2)\, a^{n-3}]$

(iii) $2^{n-1}[2x^2 \cos(2x + \frac{n\pi}{2}) + 2nx \cos(2x + (n-1)\pi/2) + \frac{n(n-1)}{2} \cos(2x + (n-2)\pi/2)]$

26

INTEGRATION

In the previous lesson, you have learnt the concept of derivative of a function. You have also learnt the application of derivative in various situations.

Consider the reverse problem of finding the original function, when its derivative (in the form of a function) is given. This reverse process is given the name of integration. In this lesson, we shall study this concept and various methods and techniques of integration.

OBJECTIVES

After studying this lesson, you will be able to :

- explain integration as inverse process (anti-derivative) of differentiation;
- find the integral of simple functions like $x^n, \sin x, \cos x,$

  $\sec^2 x, \operatorname{cosec}^2 x, \sec x \tan x, \operatorname{cosec} x \cot x, \frac{1}{x}, e^x$ etc.;
- state the following results :

  (i) $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

  (ii) $\int [\pm k f(x)]\, dx = \pm k \int f(x)\, dx$
- find the integrals of algebraic, trigonometric, inverse trigonometric and exponential functions;
- find the integrals of functions by substitution method.
- evaluate integrals of the type

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{a^2 - x^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c},$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \frac{(px + q)\,dx}{ax^2 + bx + c}, \int \frac{(px + q)\,dx}{\sqrt{ax^2 + bx + c}}$$

MODULE - V
Calculus

Notes

- derive and use the result

$$\int \frac{f'(x)}{f(x)} = \ell n|f(x)| + C$$

- state and use the method of integration by parts;
- evaluate integrals of the type :

$$\int \sqrt{x^2 \pm a^2}dx, \int \sqrt{a^2 - x^2}dx, \int e^{ax} \sin bx\, dx, \int e^{ax} \cos bx\, dx,$$

$$\int (px + q)\sqrt{ax^2 + bx + c}\, dx, \int \sin^{-1} x\, dx, \int \cos^{-1} x\, dx,$$

$$\int \sin^n x \cos^m x\, dx, \int \frac{dx}{a + b\sin x}, \int \frac{dx}{a + b\cos x}$$

- derive and use the result

$$\int e^x [f(x) + f'(x)]dx = e^x f(x) + c \text{; and}$$

- integrate rational expressions using partial fractions.

EXPECTED BACKGROUND KNOWLEDGE

- Differentiation of various functions
- Basic knowledge of plane geometry
- Factorization of algebraic expression
- Knowledge of inverse trigonometric functions

26.1 INTEGRATION

Integration literally means summation. Consider, the problem of finding area of region ALMB as shown in Fig. 26.1.

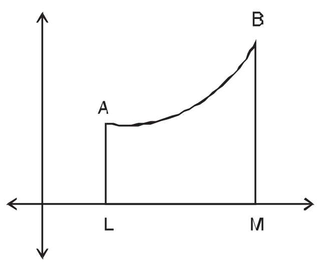


Fig. 26.1

We will try to find this area by some practical method. But that may not help every time. To solve such a problem, we take the help of integration (summation) of area. For that, we divide the figure into small rectangles (See Fig.26.2).

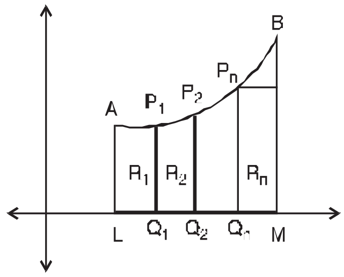


Fig. 26.2

Unless these rectangles are having their width smaller than the smallest possible, we cannot find the area.

This is the technique which Archimedes used two thousand years ago for finding areas, volumes, etc. The names of Newton (1642-1727) and Leibnitz (1646-1716) are often mentioned as the creators of present day of Calculus.

The integral calculus is the study of integration of functions. This finds extensive applications in Geometry, Mechanics, Natural sciences and other disciplines.

In this lesson, we shall learn about methods of integrating polynomial, trigonometric, exponential and logarithmic and rational functions using different techniques of integration.

26.2. INTEGRATION AS INVERSE OF DIFFERENTIATION

Consider the following examples :

(i) $\frac{d}{dx}\left(x^2\right) = 2x$ (ii) $\frac{d}{dx}(\sin x) = \cos x$ (iii) $\frac{d}{dx}\left(e^x\right) = e^x$

Let us consider the above examples in a different perspective

(i) 2x is a function obtained by differentiation of x^2.

$\Rightarrow$ x^2 is called the antiderivative of $2x$

(ii) cos x is a function obtained by differentiation of sin x

$\Rightarrow$ sin x is called the antiderivative of cos x

(iii) Similarly, e^x is called the antiderivative of e^x

Generally we express the notion of antiderivative in terms of an operation. This operation is called the operation of integration. We write

1. Integration of 2 x is x^2
2. Integration of cos x is sin x
3. Integration of e^x is e^x

The operation of integration is denoted by the symbol $\int$.

Thus

1. $\int 2x\,dx = x^2$ 2. $\int \cos x\,dx = \sin x$ 3. $\int e^x\,dx = e^x$

Remember that dx is symbol which together with symbol $\int$ denotes the operation of integration.

The function to be integrated is enclosed between $\int$ and dx.

Definition : If $\frac{d}{dx}[f(x)] = f'(x)$, then f (x) is said to be an integral of $f'(x)$ and is written as $\int f'(x)dx = f(x)$

The function f '(x) which is integrated is called the integrand.

Constant of integration

$$\text{If } y = x^2, \text{then } \frac{dy}{dx} = 2x$$

$$\therefore \qquad \int 2x\,dx = x^2$$

Now consider $\frac{d}{dx}\left(x^2 + 2\right)$ or $\frac{d}{dx}\left(x^2 + c\right)$ where c is any real constant . Thus, we see that integral of 2x is not unique. The different values of $\int 2x\,dx$ differ by some constant. Therefore, $\int 2x\,dx = x^2 + C$, where c is called the constant of integration.

Thus $\int e^x\,dx = e^x + C$, $\int \cos x\,dx = \sin x + c$

In general $\int f'(x)dx = f(x) + C$. The constant c can take any value.

We observe that the derivative of an integral is equal to the integrand.

Note : $\int f(x)dx$, $\int f(y)dy$, $\int f(z)dz$ but not like $\int f(z)dx$

Example 26.1 Find the integral of the following :

(i) x^3 (ii) x^{30} (iii) x^n

Solution :

(i) $\int x^3 dx = \frac{x^4}{4} + C$, since $\frac{d}{dx}\left(\frac{x^4}{4}\right) = \frac{4x^3}{4} = x^3$

(ii) $\int x^{30} dx = \frac{x^{31}}{31} + C$, since $\frac{d}{dx}\left(\frac{x^{31}}{31}\right) = \frac{31x^{30}}{31} = x^{30}$

(iii) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$$\text{since } \frac{d}{dx}\frac{x^{n+1}}{n+1} = \frac{1}{n+1}\frac{d}{dx}x^{n+1} = \frac{1}{n+1}(n+1)x^n = x^n$$

Example 26.2 (i) If $\frac{dy}{dx} = \cos x$, find y. (ii) If $\frac{dy}{dx} = \sin x$, find y.

Solution : (i) $\int \frac{dy}{dx}dx = \int \cos x\,dx \qquad \Rightarrow \qquad y = \sin x + C$

(ii) $\int \frac{dy}{dx}dx = \int \sin x\,dx \qquad \Rightarrow \qquad y = -\cos x + C$

26.3 INTEGRATION OF SIMPLE FUNCTIONS

We have already seen that if f (x) is any integral of f '(x), then functions of the form f (x) + C provide integral of f '(x). We repeat that C can take any value including 0 and thus

$$\int f'(x)dx = f(x) + C$$

which is an indefinite integral and it becomes a definite integral with a defined value of C.

Example 26.3 Write any 4 different values of $\int 4x^3dx$.

Solution : $\int 4x^3dx = x^4 + C$, where C is a constant.

The four different values of $\int 4x^3dx$ may be $x^4 + 1, x^4 + 2, x^4 + 3$ and $x^4 + 4$ etc.

Integrals of some simple functions given below. The validity of the integrals is checked by showing that the derivative of the integral is equal to the integrand.

	Integral	**Verification**
1.	$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ where n is a constant and $n \neq -1$.	$\because \dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1} + C\right) = x^n$
2.	$\int \sin x \, dx = -\cos x + C$	$\because \dfrac{d}{dx}(-\cos x + C) = \sin x$
3.	$\int \cos x \, dx = \sin x + C$	$\because \dfrac{d}{dx}(\sin x + C) = \cos x$
4.	$\int \sec^2 x \, dx = \tan x + C$	$\because \dfrac{d}{dx}(\tan x + C) = \sec^2 x$
5.	$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$	$\because \dfrac{d}{dx}(-\cot x + C) = \operatorname{cosec}^2 x$
6.	$\int \sec x \tan x \, dx = \sec x + C$	$\because \dfrac{d}{dx}(\sec x + C) = \sec x \tan x$
7.	$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$	$\because \dfrac{d}{dx}(-\operatorname{cosec} x + C) = \operatorname{cosec} x \cot x$
8.	$\int \dfrac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$	$\because \dfrac{d}{dx}\left(\sin^{-1} x + C\right) = \dfrac{1}{\sqrt{1-x^2}}$
9.	$\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + C$	$\because \dfrac{d}{dx}\left(\tan^{-1} x + C\right) = \dfrac{1}{1+x^2}$
10.	$\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$	$\because \dfrac{d}{dx}\left(\sec^{-1} x + C\right) = \dfrac{1}{x\sqrt{x^2-1}}$

11. $\int e^x dx = e^x + C$ $\qquad \because \frac{d}{dx}\left(e^x + C\right) = e^x$

12. $\int a^x dx = \frac{a^x}{\log a} + C$ $\qquad \because \frac{d}{dx}\left(\frac{a^x}{\log a} + C\right) = a^x = \frac{1}{x}$ if $x > 0$

13. $\int \frac{1}{x} dx = \log|x| + C$ $\qquad \because \frac{d}{dx}\left(\log|x| + C\right)$

WORKING RULE

1. To find the integral of x^n, increase the index of x by 1, divide the result by new index and add constant C to it.

2. $\int \frac{1}{f(x)} dx$ will be very often written as $\int \frac{dx}{f(x)}$.

CHECK YOUR PROGRESS 26.1

1. Write any five different values of $\int x^{\frac{5}{2}} dx$

2. Write indefinite integral of the following :

   (a) x^5 (b) $\cos x$ (c) 0

3. Evaluate :

   (a) $\int x^6 dx$ (b) $\int x^{-7} dx$ (c) $\int \frac{1}{x} dx$ (d) $\int 3^x 5^{-x} dx$

   (e) $\int \sqrt[3]{x}\, dx$ (f) $\int x^{-9}\, dx$ (g) $\int \frac{1}{\sqrt{x}} dx$ (h) $\int \sqrt[9]{x^{-8}}\, dx$

4. Evaluate :

   (a) $\int \frac{\cos\theta}{\sin^2\theta} d\theta$ (b) $\int \frac{\sin\theta}{\cos^2\theta} d\theta$

   (c) $\int \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} d\theta$ (d) $\int \frac{1}{\sin^2\theta} d\theta$

26.4 PROPERTIES OF INTEGRALS

If a function can be expressed as a sum of two or more functions then we can write the integral of such a function as the sum of the integral of the component functions, e.g. if $f(x) = x^7 + x^3$, then

$$\int f(x)\,dx = \int [x^7 + x^3]\,dx$$

$$= \int x^7 dx + \int x^3 dx$$

$$= \frac{x^8}{8} + \frac{x^4}{4} + C$$

So, in general the integral of the sum of two functions is equal to the sum of their integrals.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Similarly, if the given function

$$f(x) = x^7 - x^2$$

we can write it as $\int f(x)\,dx = \int \left(x^7 - x^2\right) dx$

$$= \int x^7 dx - \int x^2 dx$$

$$= \frac{x^8}{8} - \frac{x^3}{3} + C$$

The integral of the difference of two functions is equal to the difference of their integrals.

i.e. $$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

If we have a function f(x) as a product of a constant (k) and another function $[g(x)]$

i.e. $f(x) = kg(x)$, then we can integrate f(x) as

$$\int f(x)\,dx = \int kg(x)\,dx$$

$$= k\int g(x)\,dx$$

Integral of product of a constant and a function is product of that constant and integral of the function.

i.e. $$\int kf(x)\,dx = k\int f(x)\,dx$$

Example 26.4 Evaluate :

(ii) $\int 4^x\,dx$ (ii) $\int (2^x)(3^{-x})\,dx$

Solution :(i) $\int 4^x\,dx = \frac{4^x}{\log 4} + C$

(ii) $\int (2^x)(3^{-x})\,dx = \int \frac{2^x}{3^x}\,dx$

$$= \int \left(\frac{2}{3}\right)^x dx = \frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)} + C$$

Remember in (ii) it would not be correct to say that

$$\int 2^x 3^{-x}\, dx = \int 2^x\, dx \int 3^{-x}\, dx$$

Because $\quad \int 2^x\, dx \int 3^{-x} dx = \frac{2^x}{\log 2}\left(\frac{3^{-x}}{\log 3}\right) + C \neq \frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)} + C$

Therefore, **integral of a product of two functions is not always equal to the product of the integrals.** We shall deal with the integral of a product in a subsequent lesson.

Example 26.5 Evaluate :

(i) $\displaystyle\int \frac{dx}{\cos^n x}$, when n = 0 and n = 2 (ii) $\displaystyle\int -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\, d\theta$

Solution :

(i) When n = 0, $\displaystyle\int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^o x}$

$$= \int \frac{dx}{1} = \int dx$$

Now $\int dx$ can be written as $\int x^o dx$.

$\therefore \qquad \int dx = \int x^o dx$

$$= \frac{x^{o+1}}{0+1} + C = x + C$$

When n = 2,

$$\int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^2 x}$$

$$= \int \sec^2 x\, dx$$

$$= \tan x + C$$

(ii) $\displaystyle\int -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\, d\theta = \int \frac{-1}{\sin^2\theta}\, d\theta$

$$= -\int \operatorname{cosec}^2\theta\, d\theta$$

$$= \cot\theta + C$$

Example 26.6 Evaluate :

(i) $\int (\sin x + \cos x)\, dx$ (ii) $\int \frac{x^2+1}{x^3}\, dx$

(iii) $\int \frac{1-x}{\sqrt{x}}\, dx$ (iv) $\int \left(\frac{1}{1+x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx$

Solution : (i) $\int (\sin x + \cos x)\, dx = \int \sin x\, dx + \int \cos x\, dx = -\cos x + \sin x + C$

(ii) $$\int \frac{x^2+1}{x^3}\, dx = \int \left(\frac{x^2}{x^3} + \frac{1}{x^3} \right) dx$$

$$= \int \frac{1}{x}\, dx + \int \frac{1}{x^3}\, dx$$

$$= \log|x| + \frac{x^{-3+1}}{-3+1} + C$$

$$= \log|x| - \frac{1}{2x^2} + C$$

(iii) $$\int \frac{1-x}{\sqrt{x}}\, dx = \int \left(\frac{1}{\sqrt{x}} - \frac{x}{\sqrt{x}} \right) dx$$

$$= \int \left(x^{-\frac{1}{2}} - x^{\frac{1}{2}} \right) dx$$

$$= 2\sqrt{x} - \frac{2}{3} x^{3/2} + C$$

(iv) $$\int \left(\frac{1}{1+x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx = \int \frac{dx}{1+x^2} - \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \tan^{-1} x - \sin^{-1} x + C$$

Example 26.7 Evaluate :

(i) $\int \sqrt{1-\sin 2\theta}\, d\theta$ (ii) $\int \left(4e^x - \frac{3}{x\sqrt{x^2-1}} \right) dx$

(iii) $\int (\tan x + \cot x)^2\, dx$ (iv) $\int \left(\frac{x^6-1}{x^2-1} \right) dx$

Solution : (i) $\sqrt{1-\sin 2\theta} = \sqrt{\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta}$

$$\left[\because\ \sin^2\theta + \cos^2\theta = 1 \right]$$

$$= \sqrt{(\cos\theta - \sin\theta)^2}$$

$$= \pm(\cos\theta - \sin\theta)$$

(sign is selected depending upon the value of θ)

(a) If $\sqrt{1-\sin 2\theta} = \cos\theta - \sin\theta$

then $\int \sqrt{1-\sin 2\theta}\, d\theta = \int (\cos\theta - \sin\theta)\, d\theta$

$$= \int \cos\theta\, d\theta - \int \sin\theta\, d\theta$$

$$= \sin\theta + \cos\theta + C$$

(b) If $\int \sqrt{1-\sin 2\theta}\, d\theta = \int (-\cos\theta + \sin\theta)\, d\theta$

$$= -\int \cos\theta\, d\theta + \int \sin\theta\, d\theta$$

$$= -\sin\theta - \cos\theta + C$$

(ii) $\displaystyle\int \left(4e^x - \frac{3}{x\sqrt{x^2-1}}\right) dx = \int 4e^x dx - \int \frac{3}{x\sqrt{x^2-1}}\, dx$

$$= 4\int e^x dx - 3\int \frac{dx}{x\sqrt{x^2-1}}$$

$$= 4e^x - 3\sec^{-1} x + C$$

(iii) $\displaystyle\int (\tan x + \cot x)^2\, dx = \int \left(\tan^2 x + \cot^2 x + 2\tan x \cot x\right) dx$

$$= \int \left(\tan^2 x + \cot^2 x + 2\right) dx$$

$$= \int \left(\tan^2 x + 1 + \cot^2 x + 1\right) dx$$

$$= \int \left(\sec^2 x + \operatorname{cosec}^2 x\right) dx$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$$

$$= \tan x - \cot x + C$$

(iv) $\displaystyle\int \left(\frac{x^6 - 1}{x^2+1}\right) dx = \int \left(x^4 - x^2 + 1 - \frac{2}{x^2+1}\right) dx$ (dividing $x^6 - 1$ by $x^2 + 1$)

$$= \int x^4 dx - \int x^2 dx + \int dx - 2\int \frac{dx}{x^2+1}$$

$$= \frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1} x + C$$

Example 26.8 Evaluate :

(i) $\displaystyle\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3 dx$ (ii) $\displaystyle\int \left(\frac{4e^{5x} - 9e^{4x} - 3}{e^{3x}}\right) dx$ (iii) $\displaystyle\int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}}$

Solution:

(i) $$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3 dx = \int\left(x^{3/2}+3x\frac{1}{\sqrt{x}}+3\sqrt{x}\frac{1}{x}+\frac{1}{x^{3/2}}\right)dx$$

$$= \int x^{3/2}dx+3\int\sqrt{x}\,dx+3\int\frac{1}{\sqrt{x}}dx+\int\frac{dx}{x^{3/2}}$$

$$= \frac{x^{5/2}}{\frac{5}{2}}+3\frac{x^{3/2}}{\frac{3}{2}}+3\frac{x^{1/2}}{\frac{1}{2}}-\frac{2}{\sqrt{x}}+C$$

$$= \frac{2}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+6x^{\frac{1}{2}}-2x^{\frac{1}{2}}+C$$

(ii) $$\int\left(\frac{4e^{5x}-9e^{4x}-3}{e^{3x}}\right)dx = \int\frac{4e^{5x}}{e^{3x}}dx-\int\frac{9e^{4x}}{e^{3x}}dx-\int\frac{3dx}{e^{3x}}$$

$$= 4\int e^{2x}dx-9\int e^{x}dx-3\int e^{-3x}dx$$

$$= 2e^{2x}-9e^{x}+e^{-3x}+C$$

(iii) $$\int\frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$$

Let us first rationalise the denominator of the integrand.

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}} = \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}$$

$$\therefore \int\frac{dx}{\sqrt{x+a}+\sqrt{x+b}} = \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}$$

$$= \int\frac{\sqrt{x+a}}{a-b}dx-\int\frac{\sqrt{x+b}}{a-b}dx$$

$$= \frac{1}{a-b}\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{1}{a-b}\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}+C$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+C$$

CHECK YOUR PROGRESS 26.2

1. Evaluate :

(a) $\int\left(x+\frac{1}{2}\right)dx$ (b) $\int\frac{-x^2}{1+x^2}dx$

(c) $\int\left(10x^9-\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$ (d) $\int\left(\frac{5+3x-6x^2-7x^4-8x^6}{x^6}\right)dx$

(e) $\int\frac{x^4}{1+x^2}dx$ (f) $\int\left(\sqrt{x}+\frac{2}{\sqrt{x}}\right)^2dx$

2. Evaluate :

(a) $\int\frac{dx}{1+\cos 2x}$ (b) $\int\tan^2 x\,dx$ (c) $\int\frac{2\cos x}{\sin^2 x}dx$

(d) $\int\frac{dx}{1-\cos 2x}$ (e) $\int\frac{\sin x}{\cos^2 x}dx$ (f) $\int(\operatorname{cosec} x-\cot x)\operatorname{cosec} x\,dx$

3. Evaluate :

(a) $\int\sqrt{1+\cos 2x}\,dx$ (b) $\int\sqrt{1-\cos 2x}\,dx$ (c) $\int\frac{1}{1-\cos 2x}dx$

4. Evaluate :

(a) $\int\sqrt{x+2}\,dx$ (b) $\frac{17}{(x+1)^{17}}dx$

We know that

$$\frac{d}{dx}(ax+b)^{n+1}=(n+1)a(ax+b)^n$$

$$\Rightarrow \quad (ax+b)^n=\frac{1}{(n+1)a}\frac{d}{dx}(ax+b)^{n+1},\ n\neq -1$$

$$\int(ax+b)^n dx=\frac{(ax+b)^{n+1}}{(n+1)a}+C$$

To find the integral of $(ax+b)^n$, increase the index by one and divide the result by the increased index and the coefficient of x and add a constant of integration.

Example 26.9 Integrate the following :

(i) $(x+9)^{11}$ (ii) e^{x+7} (iii) $\cos(x+5\pi)$

(vi) $\operatorname{cosec}^2(2x+3)$

Solution :

(i) $\int (x+9)^{11} dx = \dfrac{(x+9)^{11+1}}{11+1} + C$

$$= \frac{(x+9)^{12}}{12} + C$$

(ii) $\int e^{x+7}\, dx = \int e^x .\, e^7 dx$

$= e^7 \int e^x\, dx$ (e^7 is a constant quantity)

$= e^7 .\, e^x + C = e^{x+7} + C$

(iii) $\int \cos(x+5\pi)dx = \sin(x+5\pi) + C$

$$\left[\because \frac{d}{dx}(\sin(x+5\pi)+C) = \cos(x+5\pi).\frac{d}{dx}(x+5\pi) = \cos(x+5\pi)\right]$$

(iv) $\int \operatorname{cosec}^2(2x+3)dx = \dfrac{-\cot(2x+3)}{2} + C$

$$\left[\because \frac{d}{dx}\left(\frac{-\cot(2x+3)}{2} + C\right) = -2\left(\frac{-\operatorname{cosec}^2(2x+3)}{2}\right) = \operatorname{cosec}^2(2x+3)\right]$$

Example 26.10 Evaluate : $\int (2x+5)(x-1)x^{\frac{-2}{3}} dx$

Solution : $\int (2x+5)(x-1)x^{\frac{-2}{3}} dx \quad = \int (2x^2+3x-5)x^{\frac{-2}{3}} dx$

$$= \int \left(2x^{2-\frac{2}{3}} + 3x^{1-\frac{-2}{3}} - 5x^{\frac{-2}{3}}\right) dx$$

$$= 2\int x^{4/3} dx + 3\int x^{1/3} dx - 5\int x^{-2/3} dx$$

$$= 2\frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} + 3\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} - 5\frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + C$$

$$= \frac{6}{7}x^{\frac{7}{3}} + \frac{9}{4}x^{\frac{4}{3}} - 15x^{\frac{1}{3}} + C$$

Example 26.11 Evaluate : $\int \dfrac{x^2 e^x + (x+1)^2}{x^2} dx$

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Solution : $$\int \frac{x^2e^x+(x+1)^2}{x^2}dx = \int \frac{x^2e^x+x^2+2x+1}{x^2}dx$$

$$= \int\left(e^x+1+\frac{2}{x}+\frac{1}{x^2}\right)dx$$

$$= \int e^x dx + \int dx + 2\int \frac{dx}{x} + \int \frac{dx}{x^2}$$

$$= e^x + x + 2\log|x| - \frac{1}{x} + C$$

Example 26.12 Integrate $\sec^2 x \ \operatorname{cosec}^2 x$ w.r.t. x

Solution : $$\int \sec^2 x \operatorname{cosec}^2 x\, dx = \int (1+\tan^2 x)(1+\cot^2 x)\, dx$$

$$\left[\because \sec^2 x = 1+\tan^2 x \text{ and } \operatorname{cosec}^2 x = 1+\cot^2 x\right]$$

$$= \int (1+\tan^2 x+\cot^2 x+\tan^2 x\cot^2 x)\,dx$$

$$= \int (1+\tan^2 x) + (1+\cot^2 x)dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x)dx$$

$$= \tan x - \cot x + C$$

Example 26.13 Evaluate : $\int\left(\frac{-7}{1+x^2} - \frac{6}{\sqrt{1-x^2}}\right)dx$

Solution : $$\int\left(\frac{-7}{1+x^2} - \frac{6}{\sqrt{1-x^2}}\right)dx = -7\int \frac{dx}{1+x^2} - 6\int \frac{dx}{\sqrt{1-x^2}}$$

Example 26.14 Evaluate : $\int (x+\cos x)dx$

Solution : $$\int (x+\cos x)\,dx = \int x\ dx + \int \cos x\ dx$$

$$= \frac{x^2}{2} + \sin x + C$$

26.5 TECHNIQUES OF INTEGRATION

26.5.1 Integration By Substitution

This method consists of expressing $\int f(x)\ dx$ in terms of another variable so that the resultant function can be integrated using one of the standard results discussed in the previous lesson. First, we will consider the functions of the type $f(ax+b)$, $a \neq 0$ where f (x) is a standard function.

Example 26.15 Evaluate :

(i) $\int \sin(ax+b)\,dx$ (ii) $\int \cos\left(7x+\frac{\pi}{4}\right)dx$ (iii) $\int \sin\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$

Solution : (i) $\int \sin(ax+b)\,dx$

Put $ax+b=t$.

Then $a=\frac{dt}{dx}$ or $dx=\frac{dt}{a}$

$\therefore$ $\int \sin(ax+b)\,dx = \int \sin t\,\frac{dt}{a}$ (Here the integration factor will be replaced by dt.)

$$=\frac{1}{a}\int \sin t\,dt$$

$$=\frac{1}{a}(-\cos t)+C$$

$$=-\frac{\cos(ax+b)}{a}+C$$

(ii) $\int \cos\left(7x+\frac{\pi}{4}\right)dx$

Put $7x+\frac{\pi}{4}=t \Rightarrow 7dx=dt$

$\therefore$ $\int \cos\left(7x+\frac{\pi}{4}\right)dx = \int \cos t\,\frac{dt}{7}$

$$=\frac{1}{7}\int \cos t\,dt$$

$$=\frac{1}{7}\sin t+C$$

$$=\frac{1}{7}\sin\left(7x+\frac{\pi}{4}\right)+C$$

(iii) $\int \sin\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$

Put $\frac{\pi}{4}-\frac{x}{2}=t$

Then $\quad -\frac{1}{2} = \frac{dt}{dx} \quad$ or $\quad dx = -2dt$

$$\therefore \quad \int \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) dx = -2\int \sin t \, dt$$

$$= -2(-\cos t) + C$$

$$= 2\cos t + C$$

$$= 2\cos\left(\frac{\pi}{4} - \frac{x}{2}\right) + C$$

Similarly, the integrals of the following functions will be—

$$\int \sin 2x \, dx = -\frac{1}{2}\cos 2x + C$$

$$\int \sin\left(3x + \frac{\pi}{3}\right) dx = -\frac{1}{3}\cos\left(3x + \frac{\pi}{3}\right) + C$$

$$\int \sin\left(\frac{\pi}{4} - \frac{x}{4}\right) dx = 4\cos\left(\frac{\pi}{4} - \frac{x}{4}\right) + C$$

$$\int \cos(ax + b) dx = \frac{1}{a}\sin(ax + b) + C$$

$$\int \cos 2x \, dx = \frac{1}{2}\sin 2x + C$$

Example 26.16 Evaluate :

(i) $\int (ax + b)^n \, dx$, where $n \neq -1$ $\qquad$ (ii) $\int \frac{1}{(ax + b)} dx$

Solution : (i) $\int (ax + b)^n \, dx$, where $n \neq -1$

Put $\quad ax + b = t \quad \Rightarrow \quad a = \frac{dt}{dx}$ or $dx = \frac{dt}{a}$

$$\therefore \quad \int (ax + b)^n \, dx = \frac{1}{a}\int t^n dt$$

$$= \frac{1}{a} \cdot \frac{t^{n+1}}{(n+1)} + C$$

$$= \frac{1}{a} \cdot \frac{(ax + b)^{n+1}}{n + 1} + C \qquad \text{where } n \neq -1$$

(ii) $\int \frac{1}{(ax+b)} dx$

Put $\quad ax + b = t \quad \Rightarrow \quad dx = \frac{1}{a} dt$

$$\therefore \quad \int \frac{1}{(ax+b)} dx = \int \frac{1}{a} \cdot \frac{dt}{t}$$

$$= \frac{1}{a} \log|t| + C$$

$$= \frac{1}{a} \log|ax + b| + C$$

Example 26.17 Evaluate :

(i) $\int e^{5x+7} dx$ $\qquad$ (ii) $\int e^{-3x-3} dx$

Solution : (i) $\int e^{5x+7} dx$

Put $\quad 5x + 7 = t \quad \Rightarrow \quad dx = \frac{dt}{5}$

$$\therefore \quad \int e^{5x+7} dx = \frac{1}{5} \int e^t dt$$

$$= \frac{1}{5} e^t + C$$

$$= \frac{1}{5} e^{5x+7} + C$$

(ii) $\int e^{-3x-3} dx$

Put $\quad -3x - 3 = t \quad \Rightarrow \quad dx = \frac{1}{-3} dt$

$$\therefore \quad \int e^{-3x-3} dx = -\frac{1}{3} \int e^t dt$$

$$= -\frac{1}{3} e^t + C$$

$$= -\frac{1}{3} e^{-3x-3} + C$$

Likewise $\quad \int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$

Similarly, using the substitution ax + b = t, the integrals of the following functions will be :

$$\int (ax+b)^n \, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{(ax+b)} dx = \frac{1}{a}\log|ax+b| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

$$\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + C$$

$$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$$

$$\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$$

Example 26.18 Evaluate :

(i) $\int \sin^2 x \, dx$ (ii) $\int \sin^3 x \, dx$ (iii) $\int \cos^3 x \, dx$ (iv) $\int \sin 3x \sin 2x \, dx$

Solution : We use trigonometrical identities and express the functions in terms of sines and cosines of multiples of x

(i) $$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\int 1dx - \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

(ii) $$\int \sin^3 x \, dx = \int \frac{3\sin x - \sin 3x}{4} dx \qquad \left[\because \sin 3x = 3\sin x - 4\sin^3 x\right]$$

$$= \frac{1}{4}\int (3\sin x - \sin 3x)\,dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3} \right] + C$$

(iii) $\int \cos^3 x \, dx = \int \frac{\cos 3x + 3\cos x}{4} dx \quad \left[\because \cos 3x = 4\cos^3 x - 3\cos x \right]$

$$= \frac{1}{4}\int (\cos 3x + 3\cos x)\, dx$$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x \right] + C$$

(iv) $\int \sin 3x \sin 2x \, dx = \frac{1}{2}\int 2\sin 3x \sin 2x \, dx$

$[\because 2\sin A \sin B = \cos(A - B) - \cos(A + B)]$

$$= \frac{1}{2}\int (\cos x - \cos 5x)\, dx$$

$$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5} \right] + C$$

CHECK YOUR PROGRESS 26.3

1. Evaluate :

(a) $\int \sin(4 - 5x)\, dx$ (b) $\int \sec^2 (2 + 3x)\, dx$

(c) $\int \sec\left(x + \frac{\pi}{4} \right) dx$ (d) $\int \cos(4x + 5)\, dx$

(e) $\int \sec(3x + 5) \tan(3x + 5)\, dx$

(f) $\int \operatorname{cosec}(2 + 5x) \cot(2 + 5x)\, dx$

2. Evaluate :

(a) $\int \frac{dx}{(3 - 4x)^4}$ (b) $\int (x + 1)^4 dx$ (c) $\int (4 - 7x)^{10} dx$

(d) $\int (4x - 5)^3 dx$ (e) $\int \frac{1}{3x - 5} dx$ (f) $\int \frac{1}{\sqrt{5 - 9x}} dx$

(g) $\int (2x + 1)^2 dx$ (h) $\int \frac{1}{x + 1} dx$

3. Evaluate :

(a) $\int e^{2x+1} dx$ (b) $\int e^{3-8x} dx$ (c) $\int \frac{1}{e^{(7+4x)}} dx$

4. Evaluate :

(a) $\int \cos^2 x dx$ (b) $\int \sin^3 x \cos^3 x \, dx$

(c) $\int \sin 4x \cos 3x \, dx$ (d) $\int \cos 4x \cos 2x \, dx$

26.5.2 Integration of Function of The Type $\frac{f'(x)}{f(x)}$

To evaluate $\int \frac{f'(x)}{f(x)} dx$, we put f (x) = t. Then f ' (x) dx = dt.

$$\therefore \qquad \int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log|f(x)| + C$$

Integral of a function, whose numerator is derivative of the denominator, is equal to the logarithm of the denominator.

Example 26.19 Evaluate :

(i) $\int \frac{2x}{x^2+1} dx$ (ii) $\int \frac{4x^3}{5+x^4} dx$ (iii) $\int \frac{dx}{2\sqrt{x}\left(3+\sqrt{x}\right)}$

Solution :

(i) Now 2x is the derivative of $x^2 + 1$.

$\therefore$ By applying the above result, we have

$$\int \frac{2x}{x^2+1} dx = \log\left|x^2+1\right| + C$$

(ii) $4x^3$ is the derivative of $5 + x^4$.

$$\therefore \qquad \int \frac{4x^3}{5+x^4} dx = \log\left|5+x^4\right| + C$$

(iii) $\frac{1}{2\sqrt{x}}$ is the derivative of $3+\sqrt{x}$

$$\int \frac{dx}{2\sqrt{x}\left(3+\sqrt{x}\right)} = \log\left|3+\sqrt{x}\right| + C$$

Example 26.20 Evaluate :

(i) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$ (ii) $\int \frac{e^{2x}-1}{e^{2x}+1} dx$

Solution :

(i) $e^x + e^{-x}$ is the derivative of $e^x - e^{-x}$

$$\therefore \qquad \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \log\left|e^x - e^{-x}\right| + C$$

Alternatively,

For $\qquad \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$,

Put $\qquad e^x - e^{-x} = t.$

Then $\qquad \left(e^x + e^{-x}\right) dx = dt$

$$\therefore \qquad \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log\left|e^x - e^{-x}\right| + C$$

(ii) $\displaystyle\int \frac{e^{2x} - 1}{e^{2x} + 1} dx$

Here $e^{2x} - 1$ is not the derivative of $e^{2x} + 1$. But if we multiply the numerator and denominator by e^{-x}, the given function will reduce to

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \log\left|e^x + e^{-x}\right| + C$$

$$\therefore \qquad \int \frac{e^{2x} - 1}{e^{2x} + 1} dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} = \log\left|e^x + e^{-x}\right| + C$$

$$\left[\because \left(e^x - e^{-x}\right) \text{ is the derivative of } \left(e^x + e^{-x}\right)\right]$$

CHECK YOUR PROGRESS 26.4

1. Evaluate :

(a) $\displaystyle\int \frac{x}{3x^2 - 2} dx$ (b) $\displaystyle\int \frac{2x + 1}{x^2 + x + 1} dx$ (c) $\displaystyle\int \frac{2x + 9}{x^2 + 9x + 30} dx$

(d) $\displaystyle\int \frac{x^2 + 1}{x^3 + 3x + 3} dx$ (e) $\displaystyle\int \frac{2x + 1}{x^2 + x - 5} dx$ (f) $\displaystyle\int \frac{dx}{\sqrt{x}\left(5 + \sqrt{x}\right)}$

(g) $\displaystyle\int \frac{dx}{x\left(8 + \log x\right)}$

2. Evaluate :

(a) $\int \frac{e^x}{2 + be^x} dx$ (b) $\int \frac{dx}{e^x - e^{-x}}$

26.5.3 Integration by Substitution

Example 26.21

(i) $\int \tan x dx$ (ii) $\int \sec x dx$ (iii) $\int \frac{1 - \tan x}{1 + \tan x} dx$

(iv) $\int \frac{(1 - \sin x)}{(1 + \cos x)} dx$ (v) $\int \operatorname{cosec}^5 x \cot x dx$ (vi) $\int \frac{\sin x}{\sin(x - a)} dx$

Solution :

(i) $$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

$$= -\int \frac{-\sin x}{\cos x} dx$$

$$= -\log|\cos x| + C \quad (\because -\sin x \text{ is derivative of } \cos x)$$

$$= \log\left|\frac{1}{\cos x}\right| + C \quad \text{or} \quad = \log|\sec x| + C$$

$\therefore$ $$\int \tan x dx = \log|\sec x| + C$$

Alternatively,

$$\int \tan x dx = \int \frac{\sin x \, dx}{\cos x}$$

$$= -\int \frac{-\sin x \, dx}{\cos x}$$

Put $\cos x = t.$

Then $-\sin x \, dx = dt$

$\therefore$ $$\int \tan x dx = -\int \frac{dt}{t}$$

$$= -\log|t| + C$$

$$= -\log|\cos x| + C$$

$$= \log\left|\frac{1}{\cos x}\right| + C$$

$$= \log|\sec x| + C$$

(ii) $\int \sec x \, dx$

sec x can not be integrated as such because sec x by itself is not derivative of any function. But this is not the case with $\sec^2 x$ and sec x tan x. Now $\int \sec x \, dx$ can be written as

$$\int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} dx$$

Put $\sec x + \tan x = t.$

Then $(\sec x \tan x + \sec^2 x) dx = dt$

$$\therefore \quad \int \sec x \, dx = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log|\sec x + \tan x| + C$$

(iii) $$\int \frac{1 - \tan x}{1 + \tan x} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

Put $\cos x + \sin x = t$

So that $(-\sin x + \cos x) dx = dt$

$$\therefore \quad \int \frac{1 - \tan x}{1 + \tan x} dx = \int \frac{1}{t} dt$$

$$= \log|t| + C$$

$$= \log|\cos x + \sin x| + C$$

(iv) $$\int \frac{1 + \sin x}{1 + \cos x} dt = \int \frac{1}{1 + \cos x} dx + \int \frac{\sin x}{1 + \cos x} dx$$

$$= \int \frac{1}{2\cos^2\left(\frac{x}{2}\right)} dx + \int \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}$$

$$= \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right) dx + \int \tan\frac{x}{2} dx$$

Put $\frac{x}{2} = t \Rightarrow \frac{1}{2} dx = dt$

$$\int \frac{1 + \sin x}{1 + \cos x} dx = \int \sec^2 t \, dt + 2\int \tan t \, dt$$

$$= \tan t - 2\log|\cos t| + C$$

$$= \tan\frac{x}{2} - 2\log\left|\cos\left(\frac{x}{2}\right)\right| + C$$

(v) $\int \operatorname{cosec}^5 x \cot x \, dx = -\int -\operatorname{cosec}^4 x \operatorname{cosec} x \cot x \, dx$

Put $\operatorname{cosec} x = t$.

Then $-\operatorname{cosec} x \cot x \, dx = dt$

$\therefore$ $\int \operatorname{cosec}^5 x \cot x \, dx = -\int t^4 dt$

$$= -\frac{t^5}{5} + C$$

$$= \frac{-(\operatorname{cosec} x)^5}{5} + C$$

(vi) $\int \frac{\sin x}{\sin(x-a)} dx$

Put $x - a = t$

Then $dx = dt$ and $x = t + a$

$\therefore$ $\int \frac{\sin x}{\sin(x-a)} dx = \int \frac{\sin(t+a)}{\sin t} dt$

$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t} dt$$

$(\because \sin(A+B) = \sin A \cos B + \cos A \sin B)$

$$= \cos a \int dt + \sin a \int \cot t \, dt$$

(cos a and sin a are constants.)

$$= \cos a \cdot t + \sin a \log|\sin t| + C$$

$$= (x-a)\cos a + \sin a \log|\sin(x-a)| + C$$

Example 26.22 Evaluate $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

Solution : Put $x = a \sin\theta$ $\Rightarrow$ $dx = a\cos\theta \, d\theta$

$\therefore$ $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{a\cos\theta}{\sqrt{a^2 - a^2\sin^2\theta}} d\theta$

$$= \frac{1}{a}\int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}} d\theta$$

$$= \frac{1}{a}\int \frac{1}{\cos\theta} d\theta$$

$$= \frac{1}{a}\int \sec\theta\, d\theta$$

$$= \frac{1}{a}\log\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{a}\log\left|\frac{1+\sin\theta}{\cos\theta}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{1+\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{a+x}{\sqrt{a^2-x^2}}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{\sqrt{a+x}}{\sqrt{a-x}}\right| + C$$

$$= \frac{1}{a}\log\left|\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}\right| + C$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

Example 26.23 Evaluate : $\int \frac{1}{x^2-a^2}dx$

Solution : Put $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \quad \int \frac{1}{x^2-a^2}dx = \int \frac{a\sec\theta\tan\theta\, d\theta}{a^2\sec^2\theta - a^2}$$

$$= \frac{1}{a}\int \frac{\sec\theta\tan\theta}{\tan^2\theta}d\theta \qquad \left(\tan^2\theta = \sec^2\theta - 1\right)$$

$$= \frac{1}{a}\int \frac{\sec\theta}{\tan\theta}d\theta$$

$$= \frac{1}{a}\int \frac{1}{\sin\theta}d\theta = \frac{1}{a}\int \operatorname{cosec}\theta\, d\theta$$

$$= \frac{1}{a}\log\left|\operatorname{cosec}\theta - \cot\theta\right| + C$$

$$= \frac{1}{a} \log \left| \frac{1 - \cos\theta}{\sin\theta} \right| + C$$

$$= \frac{1}{a} \log \left| \frac{1 - \frac{a}{x}}{\sqrt{1 - \frac{a^2}{x^2}}} \right| + C$$

$$= \frac{1}{a} \log \left| \frac{x - a}{\sqrt{x^2 - a^2}} \right| + C$$

$$= \frac{1}{a} \log \left| \frac{\sqrt{x - a}}{\sqrt{x + a}} \right| + C$$

$$= \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + C$$

Example 26.24 $\int \frac{1}{a^2 + x^2} dx$

Solution : Put $x = a \tan\theta \Rightarrow dx = a \sec^2\theta d\theta$

$$\therefore \qquad \int \frac{1}{a^2 + x^2} dx = \int \frac{a \sec^2\theta}{a^2\left(1 + \tan^2\theta\right)} d\epsilon$$

$$= \frac{1}{a} \int d\epsilon$$

$$= \frac{1}{a}\theta + C \qquad \left(\frac{x}{a} = \tan\epsilon \Rightarrow \tan^{-1}\frac{x}{a} = \theta\right)$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Example 26.25 $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

Put $\quad x = a \sin\theta \quad \Rightarrow \quad dx = a \cos\theta d\epsilon$

$$\therefore \qquad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{a\cos\theta}{\sqrt{a^2 - a^2\sin^2\theta}} d\epsilon$$

$$= \int \frac{a\cos\theta}{a\cos\theta} d\theta$$

$$= \int d\epsilon$$

$$= \theta + C$$

$$= \sin^{-1}\frac{x}{a} + C$$

Example 26.26 $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$

Solution : Let $\quad x = a\sec\theta \;\Rightarrow\; dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \quad \int \frac{1}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta}{a\sqrt{\sec^2\theta - 1}}$$

$$= \int \sec\theta d\theta$$

$$= \log|\sec\theta + \tan\theta| + C$$

$$= \log\left|\frac{x}{a} + \frac{1}{a}\sqrt{x^2 - a^2}\right| + C$$

$$= \log\left| x + \sqrt{x^2 - a^2}\right| + C$$

Example 26.27 $\int \frac{1}{\sqrt{a^2 + x^2}}\, dx$

Solution : Put $x = a\tan\theta \;\; dx = a\sec^2\theta\, d\theta$

$$= \int \sec\theta\, d\theta \qquad \text{[As example 26.25]}$$

$$= \log|\sec\theta + \tan\theta| + C$$

$$= \log\left|\frac{1}{a}\sqrt{a^2 + x^2} + \frac{x}{a}\right| + C$$

$$= \log\left|\sqrt{a^2 + x^2} + x\right| + C$$

Example 26.28 $\int \frac{x^2 + 1}{x^4 + 1}\, dx$

Solution : Since x^2 is not the derivative of $x^4 + 1$, therfore, we write the given integral as

$$\int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\, dx$$

Let $\qquad x - \frac{1}{x} = t$

$$\therefore \quad \left(1 + \frac{1}{x^2}\right)dx = dt$$

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Also $\qquad x^2-2+\frac{1}{x^2}=t^2 \quad \Rightarrow \quad x^2+\frac{1}{x^2}=t^2+2$

$$\therefore \qquad \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{dt}{t^2+2}$$

$$= \int \frac{dt}{(t)^2+(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)+C$$

Example 26.29 $\int \frac{x^2-1}{x^4+1}dx$

Solution : $\int \frac{x^2-1}{x^4+1}dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

Put $\qquad x+\frac{1}{x}=t.$

Then $\qquad \left(1-\frac{1}{x^2}\right)dx = dt$

Also $\qquad x^2+2+\frac{1}{x^2}=t^2$

$\Rightarrow \qquad x^2+\frac{1}{x^2}=t^2-2$

$$\therefore \qquad \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{dt}{t^2-2}$$

$$= \int \frac{dt}{(t)^2-(\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+C$$

$$= \frac{1}{2\sqrt{2}} \log \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| + C$$

Example 26.30 $\int \frac{x^2}{x^4 + 1} dx$

Solution : In order to solve it, we will reduce the given integral to the integrals given in Examples 26.28 and 26.29.

i.e., $$\int \frac{x^2}{x^4 + 1} dx = \frac{1}{2} \int \left[\frac{x^2 + 1}{x^4 + 1} + \frac{x^2 - 1}{x^4 + 1} \right] dx$$

$$= \frac{1}{2} \int \frac{x^2 + 1}{x^4 + 1} dx + \frac{1}{2} \int \frac{x^2 - 1}{x^4 + 1} dx$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} \log \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| \right] + C$$

Example 26.31 $\int \frac{1}{x^4 + 1} dx$

Solution : We can reduce the given integral to the following form

$$\frac{1}{2} \int \frac{(x^2 + 1) - (x^2 - 1)}{x^4 + 1} dx$$

$$= \frac{1}{2} \int \frac{x^2 + 1}{x^4 + 1} dx - \frac{1}{2} \int \frac{x^2 - 1}{x^4 + 1} dx$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{2}} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| \right] + C$$

Example 26.32 (a) $\int \frac{1}{x^2 - x + 1} dx$ (b) $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$

Solution : (a) $$\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{x^2 - x + \frac{1}{4} - \frac{1}{4} + 1} dx$$

$$= \int \frac{1}{\left(x - \frac{1}{2} \right)^2 + \frac{3}{4}} dx$$

$$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

(b) $$\int \frac{x^2-1}{x^4+x^2+1}\,dx = \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx$$

Put $x+\frac{1}{x} = t. \quad \Rightarrow \quad \left(1-\frac{1}{x^2}\right)dx = dt$

Also $x^2+2+\frac{1}{x^2} = t^2$

$\Rightarrow$ $x^2+1+\frac{1}{x^2} = t^2 - 1$

$\therefore$ $$\int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx = \int \frac{dt}{t^2-1}$$

$$= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$$

Example 26.33 $\int \sqrt{\tan x}\,dx$

Solution : Let $\tan x = t^2 \quad \Rightarrow \quad \sec^2 x\,dx = 2t\,dt$

$\Rightarrow$ $$dx = \frac{2t}{\sec^2 x}\,dt$$

$$= \frac{2t}{1+t^4}\,dt$$

$\therefore \qquad \int \sqrt{\tan x}\, dx = \int t \left(\frac{2t}{1+t^4} \right) dt$

$$= \int \frac{2t^2}{1+t^4} dt$$

$$= \int \left(\frac{t^2+1}{t^4+1} + \frac{t^2-1}{t^4+1} \right) dt$$

$$= \int \frac{t^2+1}{t^4+1} dt + \int \frac{t^2-1}{t^4+1} dt$$

Proceed according to example 26.28 and Example 26.29 solved before.

Example 26.34 $\int \sqrt{\cot x}\, dx$

Solution : Let $\cot x = t^2 \Rightarrow -\operatorname{cosec}^2 x\, dx = 2t\, dt$

$\Rightarrow \qquad dx = \frac{-2t}{\operatorname{cosec}^2 x} dt$

$$= -\frac{2t}{t^4+1} dt$$

$\therefore \qquad \int \sqrt{\cot x}\, dx = -\int t \left(\frac{2t}{t^4+1} \right) dt$

$$= -\int \frac{2t^2}{t^4+1} dt$$

$$= -\int \left(\frac{t^2+1}{t^4+1} + \frac{t^2-1}{t^4+1} \right) dt$$

Proceed according to Examples 26.28 and 26.29 solved before.

Example 26.35 $\int \left(\sqrt{\tan x} + \sqrt{\cot x} \right) dx$

Let $\quad \sin x - \cos x = t \Rightarrow (\cos x + \sin x)\, dx = dt$

Also $\quad 1 - 2\sin x \cos x = t^2$

$\Rightarrow \quad 1 - t^2 = 2\sin x \cos x$

$\Rightarrow \quad \frac{1-t^2}{2} = \sin x \cos x$

$$\therefore \quad \int \frac{\sin x - \cos x}{\sqrt{\cos x \sin x}}\,dx = \int \frac{dt}{\sqrt{\frac{1-t^2}{2}}}$$

$$= \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\sin^{-1}[\sin x - \cos x] + C$$

(Using the result of Example 26.25)

Example 26.36 Evaluate :

(a) $\displaystyle\int \frac{dx}{\sqrt{8+3x-x^2}}$ (b) $\displaystyle\int \frac{dx}{x(1-2x)}$

Solution :

(a)
$$\int \frac{dx}{\sqrt{8+3x-x^2}} = \int \frac{dx}{\sqrt{8-\left(x^2-3x\right)}}$$

$$= \int \frac{dx}{\sqrt{8-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}$$

$$= \sin^{-1}\left[\frac{\left(x-\frac{3}{2}\right)}{\frac{\sqrt{41}}{2}}\right] + C$$

$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right) + C$$

(b)
$$\int \frac{dx}{x(1-2x)} = \int \frac{dx}{\sqrt{x-2x^2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{x}{2}-x^2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{1}{16}-\left[x^2-\frac{x}{2}+\frac{1}{16}\right]}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left\{ \frac{\left(x - \frac{1}{4}\right)}{\left(\frac{1}{4}\right)} \right\} + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} (4x - 1) + C$$

CHECK YOUR PROGRESS 26.5

1. Evaluate :

(a) $\int \frac{x^2}{x^2 - 9} dx$ (b) $\int \frac{e^x}{e^{2x} + 1} dx$ (c) $\int \frac{x}{1 + x^4} dx$

(d) $\int \frac{dx}{\sqrt{16 - 9x^2}}$ (e) $\int \frac{dx}{1 + 3\sin^2 x}$ (f) $\int \frac{dx}{\sqrt{3 - 2x - x^2}}$

(g) $\int \frac{dx}{3x^2 + 6x + 21}$ (h) $\int \frac{dx}{\sqrt{5 - 4x - x^2}}$ (i) $\int \frac{dx}{x\sqrt{3x^2 - 12}}$

(j) $\int \frac{d\theta}{\sin^4 \theta + \cos^4 \theta}$ (k) $\int \frac{e^x dx}{\sqrt{1 + e^{2x}}}$ (l) $\int \sqrt{\frac{1 + x}{1 - x}} dx$

(m) $\int \frac{dx}{\sqrt{2ax - x^2}}$ (n) $\int \frac{3x^2}{\sqrt{9 - 16x^6}} dx$ (o) $\int \frac{(x + 1)}{\sqrt{x^2 + 1}} dx$

(p) $\int \frac{dx}{\sqrt{9 + 4x^2}}$ (q) $\int \frac{\sin\theta}{\sqrt{4\cos^2 \theta - 1}} d\theta$ (r) $\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 4}} dx$

(s) $\int \frac{1}{(x + 2)^2 + 1} dx$ (t) $\int \frac{1}{\sqrt{16x^2 + 25}} dx$

26.6 INTEGRATION BY PARTS

In differentiation you have learnt that

$$\frac{d}{dx}(fg) = f\frac{d}{dx}(g) + g\frac{d}{dx}(f)$$

or $$f\frac{d}{dx}(g) = \frac{d}{dx}(fg) - g\frac{d}{dx}(f) \qquad (1)$$

Also you know that $\int \frac{d}{dx}(fg)\,dx = fg$

Integrating (1), we have

$$\int f\frac{d}{dx}(g)\,dx = \int \frac{d}{dx}(fg)dx - \int g\frac{d}{dx}(f)dx$$

$$= fg - \int g\frac{d}{dx}(f)dx \qquad (2)$$

if we take $f = u(x);\ \frac{d}{dx}(g) = v(x),$

(2) becomes $\int u(x)v(x)dx$

$$= u(x)\,.\int v(x)dx - \int\left[\frac{d}{dx}(u(x))\int v(x)dx\right]dx$$

$= \underbrace{\text{I function} \times \text{integral of II function}}_{A} - \int \underbrace{[\text{differential coefficient of I function} \times \text{integral of II function}]dx}_{B}$

Here the important factor is the choice of I and II function in the product of two functions because either can be I or II function. For that the indicator will be part 'B' of the result above.

The first function is to be chosen such that it reduces to a next lower term or to a constant term after subsequent differentiations.

Inequations of integration like

$$x\ \sin x,\ x\cos^2 x,\ x^2 e^x$$

(i) algebraic function should be taken as the first function.

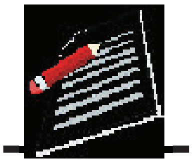
(ii) If there is no algebraic function then look for a function which simplifies the production in 'B' as above; the choice can be in order of preference like choosing first function.

(i) an inverse function (ii) a logarithmic function

(iii) a trigonometric function (iv) an exponential function

The following example will give a practice to the concept of choosing first function.

	I function	**II function**
1. $\int x\cos x\ dx$	x (being algebraic)	cos x
2. $\int x^2 e^x dx$	x^2 (being algebraic)	e^x
3. $\int x^2 \log x\,dx$	log x	x^2



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	I function	II function
4. $\int \frac{\log x}{(1+x^2)}$	logx	$\frac{1}{(1+x)^2}$
5. $\int x \sin^{-1} x\ dx$	$\sin^{-1}x$	x
6. $\int \log x\ dx$	log x	1
		(In single function of logarithm and inverse trigonometric we take unity as II function)
7. $\sin^{-1} x\ dx$	$\sin^{-1}x$	1

Example 26.37 Evaluate :

$$\int x \cos x\ dx$$

Solution : Taking the polynomial (algebraic function) x as the first function and trigonometric function cos x as the second function, we get

$$\underset{\mathrm{I}}{\int x}\ \underset{\mathrm{II}}{\cos x}\ dx = x\int \cos x\ dx - \int\left[\frac{d}{dx}(x)\ .\ \int \cos x\, dx\right]dx$$

$$\left(\text{I function} \times \text{Integral of II function} - \int\left[\frac{d}{dx}(\text{I function}).\int(\text{II fuction dx})\right]dx\right)$$

$$= x \sin x - \int 1\ .\ \sin x\ dx$$

$$= x \sin x - [-\cos x] + C$$

$$= x \sin x + \cos x + C$$

Example 26.38 Evaluate :

$$\int x^2 \sin x\ dx$$

Solution : Taking algebraic function x^2 as I function and sin x as II function, we have,

$$\int \underset{\mathrm{I}}{x^2}\ \underset{\mathrm{II}}{\sin x}\ dx = x^2\int \sin x - \int\left[\frac{d}{dx}(x^2)\int \sin x\ dx\right]dx$$

$$= -x^2 \cos x - 2\int x(-\cos x)dx$$

$$= -x^2 \cos x + 2\int x \cos x\, dx \qquad (1)$$

Again, $\int x \cos x\, dx = x \sin x + \cos x + C \qquad (2)$

Substituting (2) in (1) we have

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$$\int x^2 \sin x\,dx = -x^2 \cos x + 2[x \sin x + \cos x] + C$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Example 26.39 Evaluate :

$$\int x^2 \log x \; dx$$

Solution : In order of preference log x is to be taken as I function.

$$\therefore \quad \int \underset{\text{I}}{\log x}\; \underset{\text{II}}{x^2}\, dx = \frac{x^3}{3}\log x - \int \frac{1}{x}.\frac{x^3}{3}dx$$

$$= \frac{x^3}{3}\log x - \int \frac{x^2}{3}dx$$

$$= \frac{x^3}{3}\log x - \frac{1}{3}\left(\frac{x^2}{3}\right) + C$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C$$

Example 26.40 Evaluate :

$$\int \frac{\log x}{(1+x)^2}\, dx$$

Solution : $$\int \frac{\log x}{(1+x)^2}\, dx = \int \underset{\text{I}}{\log x}.\underset{\text{II}}{\frac{1}{(1+x)^2}}\, dx$$

$$= \log x\left(-\frac{1}{1+x}\right) - \int \frac{1}{x} \times \left(\frac{-1}{1+x}\right)dx$$

$$= \frac{-\log x}{1+x} + \int \frac{1}{x(1+x)}dx$$

$$= \frac{-\log x}{1+x} + \int \left[\frac{1}{x} - \frac{1}{1+x}\right]dx$$

$$= \frac{-\log x}{1+x} + \int \frac{1}{x}dx - \int \frac{1}{1+x}dx$$

$$= \frac{-\log x}{1+x} + \log|x| - \log|1+x| + C$$

$$= \frac{-\log x}{1+x} + \log\left|\frac{x}{1+x}\right| + C$$

Example 26.41 Evaluate :

$$\int x\, e^{2x}\, dx$$

Solution :

$$\int x\, e^{2x}\, dx = x\frac{e^{2x}}{2} - \int 1.\frac{e^{2x}}{2}dx$$

$$= x\frac{e^{2x}}{2} - \frac{1}{2}\left(\frac{e^{2x}}{2}\right) + C$$

$$= x\frac{e^{2x}}{2} - \frac{1}{4}e^{2x} + C.$$

Example 26.42 Evaluate :

$$\int \sin^{-1} x\, dx$$

Solution :

$$\int \sin^{-1} x\, dx = \int \sin^{-1} x.1.dx$$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

Let $1 - x^2 = t$

$\Rightarrow$ $-2x - dx = dt$

$\Rightarrow$ $x\, dx = \frac{-1}{2}dt$

$\therefore$ $$\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= -\sqrt{t} + C$$

$$= -\sqrt{1-x^2} + C$$

$\therefore$ $$\int \sin^{-1} x\, dx = x\sin^{-1} x + \sqrt{1-x^2} + C$$

CHECK YOUR PROGRESS 26.6

Evaluate :

1. (a) $\int x\sin x\, dx$ (b) $\int (1+x^2)\cos 2x\, dx$ (c) $\int x\sin 2x\, dx$

2. (a) $\int x\tan^2 x\, dx$ (b) $\int x^2 \sin^2 x\, dx$

3. (a) $\int x^3 \log 2x\, dx$ (b) $(1-x^2)\log x\, dx$ (c) $\int (\log x)^2\, dx$

4. (a) $\int \frac{\log x}{x^n}dx$ (b) $\int \frac{\log(\log x)}{x}dx$

5. (a) $\int x^2 e^{3x}\,dx$ (b) $\int x\, e^{3x}\,dx$

6. $\int x\,(\log x)^2\,dx$

7. (a) $\int \sec^{-1} x\,dx$ (b) $\int x \cot^{-1} x\,dx$

26.7 INTEGRAL OF THE FORM $\int e^x[f(x)+f'(x)]dx$

$$\int e^x[f(x)+f'(x)]dx$$

Where f '(x) is the differentiation of f(x). In such type of integration while integrating by parts the solution will be $e^x(f(x)) + C$.

For example, consider

$$\int e^x[\tan x + \log \sec x]dx \qquad (1)$$

Let $\quad f(x) = \log \sec x,$

then $\quad f'(x) = \dfrac{\sec x \tan x}{\sec x} = \tan x$

So (1) can be rewritten as

$$\int e^x[f'(x)+f(x)]dx = e^x(f(x)) + C = e^x \log \sec x + C$$

Alternatively, you can evaluate it as under :

$$\int e^x[\tan x + \log \sec x]dx = \int \underset{\mathrm{I}}{e^x}\ \underset{\mathrm{II}}{\tan x}\,dx + \int e^x \log \sec x\,dx$$

$$= e^x \log \sec x - \int e^x \log \sec x\,dx + \int e^x \log \sec x\,dx$$

$$= e^x \log \sec x + C$$

Example 26.43 Evaluate the following :

(a) $\int e^x\left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)dx$ (b) $\int e^x\left(\dfrac{1+x\log x}{x}\right)dx$

(c) $\int \dfrac{x\,e^x}{(x+1)^2}dx$ (d) $\int e^x\left[\dfrac{1+\sin x}{1+\cos x}\right]dx$

Solution :

(a) $\int e^x\left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)dx = \int e^x\left[\dfrac{1}{x} + \dfrac{d}{dx}\left(\dfrac{1}{x}\right)\right]dx = e^x\left(\dfrac{1}{x}\right)$

(b) $\int e^x\left(\dfrac{1+x\log x}{x}\right)dx = \int e^x\left(\dfrac{1}{x} + \log x\right)dx$

$$= \int e^x \left(\log x + \frac{d}{dx}(\log x) \right) dx$$

$$= e^x \log x + C$$

(c) $$\int \frac{x\ e^x}{(x+1)^2} dx = \int \frac{x+1-1}{(x+1)^2} e^x\ dx$$

$$= \int e^x \left(\frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx$$

$$= \int e^x \left(\frac{1}{x+1} + \frac{d}{dx}\left(\frac{1}{(x+1)} \right) \right) dx$$

$$= e^x \left(\frac{1}{x+1} \right) + C$$

(d) $$\int e^x \left[\frac{1+\sin x}{1+\cos} \right] dx = \int e^x \left[\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} \right] dx$$

$$= \int e^x \left[\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2} \right] dx$$

$$= \int e^x \left[\tan\frac{x}{2} + \frac{d}{x}\left(\tan\frac{x}{2} \right) \right] dx$$

$$= e^x\ \tan\frac{x}{2} + C$$

Example 26.44 Evaluate the following :

(a) $\int \sec^3 x\ dx$ (b) $\int e^x \sin x\ dx$

Solution :

(a) $\int \sec^3 x\ dx$

Let $I = \int \sec x - \sec^2 x\ dx$

$$= \sec x\ .\ \tan x - \int \sec x \tan\ x\ .\ \tan x\ dx$$

$\therefore$ $I = \sec x \tan x - \int (\sec^3 x - \sec\ x)\ dx$ $\left(\because \tan^2 x = \sec^2 x - 1\right)$

or $I = \sec x \tan x - \int \sec^3 x\ dx + \int \sec\ x\ dx$

or $2I = \sec x \tan x + \int \sec x\ dx$

or $\quad I = \sec x \tan x + \log|\sec x + \tan x| + C_1$

or $\quad I = \frac{1}{2}[\sec x \tan x + \log|\sec x + \tan x|] + C$

(b) $\int e^x \sin x \, dx$

Let $\quad I = \int e^x \sin x \, dx$

$$= e^x(-\cos x) - \int e^x(-\cos x)\, dx$$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

$$= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x \, dx\right)$$

$\therefore \quad I = -e^x \cos x + e^x \sin x - I$

or $\quad 2I = -e^x \cos x + e^x \sin x$

or $\quad I = \frac{e^x}{2}(\sin x - \cos x) + C$

Example 26.45 Evaluate :

$$\int \sqrt{a^2 - x^2}\, dx$$

Solution :

Let $\quad I = \int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - x^2} \,.\, 1 \, dx$

Integrating by parts only and taking 1 as the second function, we have

$$I = \left(\sqrt{a^2 - x^2}\right)x - \int \frac{1}{2\sqrt{a^2 - x^2}}(-2x) \,.\, x \, dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{2\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}\, dx - \int \sqrt{a^2 - x^2}\, dx$$

$\therefore \quad I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) - I$

or $\quad 2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)$

or $$I=\frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]+C$$

Similarly, $$\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

$\therefore$ $$\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2+x^2}}{2}-\frac{a^2}{2}\log\left|x+\sqrt{a^2-x^2}\right|+C$$

Example 26.46 Evaluate :

(a) $\int\sqrt{16x^2+25}\,dx$ (b) $\int\sqrt{16-x^2}\,dx$ (c) $\int\sqrt{1+x-2x^2}\,dx$

Solution :

(a) $$\int\sqrt{16x^2+25}\,dx=4\int\sqrt{x^2+\frac{25}{16}}\,dx=4\int\sqrt{x^2+\left(\frac{5}{4}\right)^2}\,dx$$

Using the formula for $\int\sqrt{(x^2+a^2)}\,dx$ we get,

$$\int\sqrt{16x^2+25}\,dx=\left[\frac{x}{2}\sqrt{x^2+\frac{25}{16}}+\frac{25}{32}\log\left|x+\sqrt{x^2+\frac{25}{16}}\right|\right]+C$$

$$=\frac{x}{8}\sqrt{16x^2+25}+\frac{25}{8}\log\left|4x+\sqrt{16x^2+25}\right|+C$$

(b) Using the formula for $\int\sqrt{(a^2-x^2)}\,dx$ we get,

$$\int\sqrt{16-x^2}\,dx=\int\sqrt{(4)^2-x^2}\,dx$$

$$=\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\frac{x}{4}+C$$

(c) $$\int\sqrt{1+x-2x^2dx}=\sqrt{2}\int\sqrt{\frac{1}{2}+\frac{x}{2}-x^2}\,dx$$

$$=\sqrt{2}\int\sqrt{\left[\frac{1}{2}-\left(x^2-\frac{x}{2}+\frac{1}{16}\right)+\frac{1}{16}\right]}dx$$

$$=\sqrt{2}\int\sqrt{\left(\frac{3}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}\,dx$$

$$=\sqrt{2}\left[\frac{x-\frac{1}{4}}{2}\sqrt{\frac{9}{16}-\left(x-\frac{1}{4}\right)^2}+\frac{9}{16\times 2}\sin^{-1}\frac{x-\frac{1}{4}}{\frac{3}{4}}\right]+C$$

$$= \sqrt{2}\left[\frac{4x-1}{8}\cdot\frac{1}{\sqrt{2}}\sqrt{1+x-2x^2}+\frac{9}{32}\sin^{-1}\frac{4x-1}{3}\right]+C$$

$$= \frac{4x-1}{8}\sqrt{1+x-2x^2}+\frac{9\sqrt{2}}{32}\sin^{-1}\frac{4x-1}{3}+C$$

CHECK YOUR PROGRESS 26.6

Evaluate :

1. (a) $\int e^x \sec x[1+\tan x]dx$ (b) $\int e^x[\sec x+\log|\sec x+\tan x|]dx$

2. (a) $\int \frac{x-1}{x^2}e^x\,dx$ (b) $\int e^x\left(\sin^{-1}x-\frac{1}{\sqrt{1-x^2}}\right)dx$

3. $\int e^x\frac{(x-1)}{(x+1)^3}dx$

4. $\int \frac{xe^x}{(x+1)^2}dx$

5. $\int \frac{x+\sin x}{1+\cos x}dx$

6. $\int e^x \sin 2x\,dx$

26.8 INTEGRATION BY USING PARTIAL FRACTIONS

By now we are equipped with the various techniques of integration.

But there still may be a case like $\frac{4x+5}{x^2+x-6}$, where the substitution or the integration by parts may not be of much help. In this case, we take the help of another technique called **technique of integration using partial function**.

Any proper rational fraction $\frac{p(x)}{q(x)}$ can be expressed as the sum of rational functions, each having a single factor of q(x). Each such fraction is known as **partial fraction** and the process of obtaining them is called decomposition or resolving of the given fraction into partial fractions.

For example, $\frac{3}{x+2}+\frac{5}{x-1}=\frac{8x+7}{(x+2)(x-1)}=\frac{8x+7}{x^2+x-2}$

Here $\frac{3}{x+2}, \frac{5}{x-1}$ are called partial fraction of $\frac{8x+7}{x^2+x-2}$.

If $\frac{f(x)}{g(x)}$ is a proper fraction and g(x) can be resolved into real factors then,

(a) correspnding to each non repeated linear factor ax + b, there is a partial fraction of the form

(b) for $(ax+b)^2$ we take the sum of two partial fractions as

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

For $(ax+b)^3$ we take the sum of three partial fraction as

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$$

and so on.

(c) For a non - fractorisable quadratic polynomial ax^2+bx+c there is a partial fraction

$$\frac{Ax+B}{ax^2+bx+c}$$

Therefore, if $g(x)$ is a proper fraction $\frac{f(x)}{g(x)}$ and can be resolved into real factors, then $\frac{f(x)}{g(x)}$ can be written in the following form :

Factor in the denominator	**Corresponding parital fraction**
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$
$(ax+b)^3$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^2$	$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}$

where A, B, C, D are arbitraty constants.

The rational functions which we shall consider for integration will be those whose denominators can be fracted into linear and quadratic factors.

Example 26.47 Evaluate :

$$\int\frac{2x+5}{x^2-x-2}\,dx$$

Solution : $$\frac{2x+5}{x^2-x-2}=\frac{2x+5}{(x-2)(x+1)}$$

Let $$\frac{2x+5}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1} \qquad (1)$$

Multiplying both sides by (x - 2) (x + 1), we have

$$2x + 5 = A(x + 1) + B(x - 2)$$

Putting $x = 2$, we $9 = 3A$ or $A = 3$

Putting $x = -1$, we get $3 = -3B$ or $B = -1$

Substituting these values in (1), we have

$$\frac{2x+5}{(x-2)(x+1)}=\frac{3}{x-2}-\frac{1}{x+1}$$

$$\Rightarrow \qquad \int\frac{2x+5}{(x-2)(x+1)}dx=\int\frac{3}{x-2}dx-\int\frac{1}{x+1}dx$$

$$= 3\log|x-2| - \log|x+1| + C$$

Example 26.48 Evaluate :

(a) $\displaystyle\int\frac{x^2-x-1}{x^3-x^2-6x}dx$ (b) $\displaystyle\int\frac{1}{(x^2-1)(x+1)}dx$

Solution :

(a) $$\frac{x^2-x-1}{x^3-x^2-6x}=\frac{x^2-x-1}{x(x-1)(x+2)}$$

Let $$\frac{x^2-x-1}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+2} \qquad (1)$$

Multiplying both sides by $x(x-3)(x+2)$, we have

$x^2 - x - 1 = A(x-3)(x+2) + Bx(x+2) + Cx(x-3)$

Putting $x = 3$, we get $15B = 5$ or $B = \frac{1}{3}$

Putting $x = 0$, we get $-6A = -1$ or $A = \frac{1}{6}$

Putting $x = -2$, we get $10C = 5$ or $C = \frac{1}{2}$

Substituting these values in (1), we have

$$\frac{x^2-x-1}{x(x-3)(x+2)}=\frac{1}{6x}+\frac{1}{3(x+3)}+\frac{1}{2(x+2)}$$

$$\Rightarrow \quad \int \frac{x^2 - x - 1}{x^3 - x^2 - 6x}\,dx = \int \frac{1}{6x}\,dx + \int \frac{1}{3(x-3)}\,dx + \int \frac{1}{2(x+2)}\,dx$$

$$= \frac{1}{6}\log|x| + \frac{1}{3}\log|x-3| + \frac{1}{2}\log|x+2| + C$$

(b) $$\frac{1}{(x^2-1)(x+1)} = \frac{1}{(x+1)(x-1)(x+1)} = \frac{1}{(x-1)(x+1)^2}$$

Let $$\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Multiplying both sides by $(x^2 - 1)(x + 1)$, we have

$$1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

Putting $x = 1$, we get $\qquad A = \frac{1}{4}$

Putting $x = -1$, we get $\qquad C = -\frac{1}{2}$

$$0 = A + B$$

$$\Rightarrow \qquad B = -\frac{1}{4}$$

$$\therefore \quad \int \frac{1}{(x^2-1)(x+1)}\,dx = \int \frac{1}{4(x-1)}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{(x+1)^2}\,dx$$

$$= \frac{1}{4}\log|x-1| - \frac{1}{4}\log|x+1| - \frac{1}{2}\left(-\frac{1}{x+1}\right) + C$$

$$= \frac{1}{4}\log|x-1| - \frac{1}{4}\log|x+1| + \frac{1}{2(x+1)} + C$$

Example 26.49 Evaluate :

$$\int \frac{x^3 + x + 1}{x^2 - 1}\,dx$$

Solution : Let $\qquad I = \int \frac{x^3 + x + 1}{x^2 - 1}\,dx$

Now $$\frac{x^3 + x + 1}{x^2 - 1} = x + \frac{2x+1}{x^2 - 1} = x + \frac{2x+1}{(x+1)(x-1)}$$

$$\therefore \quad I = \int \left(x + \frac{2x+1}{(x+1)(x-1)} \right) dx$$

Let $$\frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \qquad (2)$$

$\Rightarrow$ $$2x + 1 = A(x - 1) + B(x + 1)$$

Putting $x = 1$, we get $\quad B = \frac{3}{2}$

Putting $x = -1$, we get $\quad A = \frac{1}{2}$

Substituting the values of A and B in (2) and integrating, we have

$$\int \frac{2x+1}{(x^2-1)} dx = \frac{1}{2}\int \frac{1}{(x+1)} dx + \frac{3}{2}\int \frac{1}{x-1} dx$$

$$= \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| \qquad (3)$$

$\therefore$ From (1) and (3), we have

$$I = \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$$

Example 26.50 Evaluate :

(a) $\displaystyle\int \frac{8}{(x+2)(x^2+4)} dx$ (b) $\displaystyle\int \frac{1}{x(x^3+1)} dx$

Solution :

(a) $$\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

(As x^2+4 is not factorisable into linear factors)

Multiplying both sides by $(x + 2)(x^2 + 4)$, we have

$$8 = A(x^2 + 4) + (Bx + C)(x + 2)$$

On comparing the corresponding coefficients of powers of x on both sides, we get

$$\left.\begin{aligned} 0 &= A + B \\ 0 &= 2B + C \\ 8 &= 4A + 2C \end{aligned}\right\} \Rightarrow A = 1,\ B = -1,\ C = 2$$

$\therefore$ $$\int \frac{8}{(x+2)(x^2+4)} dx = \int \frac{1}{x+2} dx - \int \frac{x-2}{x^2-4} dx$$

$$= \int \frac{1}{x+2} dx - \frac{1}{2}\int \frac{2x}{x^2+4} dx + 2\int \frac{dx}{x^4+4}$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + 2.\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + \tan^{-1}\frac{x}{2} + C$$

(b) $$\frac{1}{x(x^3+1)} = \frac{1}{x(x+1)(x^2-x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1}$$

Multiplying both sides by $x(x^3+1)$ or $x(x+1)(x^2-x+1)$, we have

$I = A(x+1)(x^2-x+1) + Bx(x^2-x+1) + (Cx+D)(x)(x+1)$

On comparing the corresponding coefficients of powers of x on both sides, we get

$$\left.\begin{array}{l} 0 = A+B+C \\ 0 = -B+C+D \\ 0 = B+D \\ 1 = A \end{array}\right\} \Rightarrow A = 1,\ B = -\frac{1}{3},\ C = -\frac{2}{3},\ D = \frac{1}{3}$$

$$\frac{1}{x(x^3+1)} = \frac{1}{x} - \frac{1}{3(x+1)} - \frac{2x-1}{3(x^2-x+1)}$$

$$\int \frac{1}{x(x^3+1)}dx = \int \frac{1}{x}dx - \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{2x-1}{x^2-x+1}dx$$

$$= \log|x| - \frac{1}{3}\log|x+1| - \frac{1}{3}\log|x^2-x+1| + C$$

Example 26.51 Evaluate :

$$\int \frac{dx}{\sin x - \sin 2x}$$

Solution :

Let $$I = \int \frac{dx}{\sin x - \sin 2x} = \int \frac{dx}{\sin x - 2\sin x\ \cos x}$$

$$= \int \frac{dx}{\sin x(1-2\cos x)}$$

Multiplying numerator and denominator by sin x, we have

$$I = \int \frac{\sin x\, dx}{\sin^2 x(1-2\cos x)} = \int \frac{\sin x\, dx}{(1-\cos^2 x)(1-2\cos x)}$$

$$= \int \frac{\sin x\, dx}{(1+\cos x)(1-\cos x)(1-2\cos x)}$$

Put $\cos x = t$, then $-\sin x \; dx - dt$

$$\therefore \qquad I = \int \frac{-dt}{(1+t)(1-t)(1-2t)}$$

Let $\dfrac{-1}{(1+t)(1-t)(1-2t)} = \dfrac{A}{1-t} + \dfrac{B}{1+t} + \dfrac{C}{1-2t}$

Then, $-1 = A(1 + t)(1 - 2t) + B(1 - t)(1 - 2t) + C(1 - t)(1 + t)$

Putting $t = -1$, we get $B = -\dfrac{1}{6}$

Putting $t = 1$, we get $A = \dfrac{1}{2}$

Putting $t = \dfrac{1}{2}$, we get $C = -\dfrac{4}{3}$

$$\therefore \qquad I = \frac{1}{2}\int \frac{1}{(1-t)}dt - \frac{1}{6}\int \frac{1}{1+t}dt - \frac{4}{3}\int \frac{dt}{1-2t}$$

$$= -\frac{1}{2}\log|1-t| - \frac{1}{6}\log|1+t| + \frac{4}{6}\log|1-2t| + C$$

$$= -\frac{1}{2}\log|1-\cos x| - \frac{1}{6}\log|1+\cos x| + \frac{2}{3}\log|1-2\cos x| + C$$

Example 26.52 Evaluate :

$$\int \frac{2\sin 2\theta - \cos\theta}{4-\cos^2\theta - 4\sin\theta}\, d\theta$$

Solution :

Let $$I = \int \frac{2\sin 2\theta - \cos\theta}{4-\cos^2\theta - 4\sin\theta}\, d\theta$$

$$= \int \frac{(4\sin\theta - 1)\cos\theta d\theta}{2+\sin^2\theta - 4\sin\theta}$$

Let $\sin\theta = t$, then $\cos\theta \, d\theta = dt$

$$\therefore \qquad I = \int \frac{4t-1}{3+t^2-4t}\, dt$$

Let $\dfrac{4t-1}{3+t^2-4t} = \dfrac{A}{t-3} + \dfrac{B}{t-1}$

Thus $4t - 1 = A(t - 1) + B(t - 3)$

Put $t = 1$ then $B = -\dfrac{3}{2}$

Put $\quad t = 1 \quad$ then $A = \dfrac{11}{2}$

$$\therefore \quad I = \frac{11}{2}\int\left(\frac{1}{t-3}\right)dt - \frac{3}{2}\int\frac{dt}{t-1}$$

$$= \frac{11}{2}\log|t-3| - \frac{3}{2}\log|t-1| + C$$

$$= \frac{11}{2}\log|\sin\theta - 3| - \frac{3}{2}\log|\text{sain}\,\theta - 1| + C$$

Example 26.53 Evaluate :

$$\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}$$

Solution :

Let
$$I = \int\frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}d\theta = \int\frac{\tan\theta(1+\tan^2\theta)}{1+\tan^3\theta}d\theta$$

$$= \int\frac{\tan\theta + \sec^2\theta}{1+\tan^3\theta}d\theta$$

Let $\quad \tan\theta = t$, then $\sec^2\theta\, d\theta = dt$

$$\therefore \quad I = \int\frac{t\,dt}{1+t^3} = \int\frac{t\,dt}{(1+t)(1-t+t^2)} \qquad (1)$$

Let
$$\frac{t\,dt}{(1+t)(1-t+t^2)} = \frac{A}{1+t} + \frac{Bt+C}{1-t+t^2}$$

Then $\quad t = A(1-t+t^2) + (Bt+C)(1+t)$

Comparing the coefficients of t, we get

$$A + B = 0, \quad -A + B + C = 1, \quad A + C = 0$$

$$\Rightarrow \quad A = -\frac{1}{3},\ B = \frac{1}{3},\ C = \frac{1}{3}$$

$$\therefore \quad I = -\frac{1}{3}\int\frac{dt}{1+t} + \frac{1}{3}\int\frac{t+1}{1-t+t^2}dt$$

$$= -\frac{1}{3}\int\frac{dt}{1+t} + \frac{1}{6}\int\frac{2t+2}{t^2-t+1}dt$$

$$= -\frac{1}{3}\int\frac{dt}{1+t} + \frac{1}{6}\int\frac{(2t-1)+3}{t^2-t+1}dt$$

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Notes

$$= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{(2t-1)dt}{t^2-t+1} + \frac{1}{2}\int \frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}.\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$= -\frac{1}{3}\log|1+\tan\theta| + \frac{1}{6}\log|\tan^2\theta - \tan\theta + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C$$

CHECK YOUR PROGRESS 26.8

Evaluate the following :

1. (a) $\int \sqrt{4x^2-5}\ dx$ (b) $\int \sqrt{x^2+3x}\ dx$

   (c) $\int \sqrt{3-2x-2x^2}\ dx$

2. (a) $\int \frac{x+1}{(x-2)(x-3)}\ dx$ (b) $\int \frac{x}{x^2-16}\ dx$

3. (a) $\int \frac{x^3}{x^2-4}\ dx$ (b) $\int \frac{2x^2+x+1}{(x-1)^2(x+2)}\ dx$

4. $\int \frac{x^2+x+1}{(x-1)^3}\ dx$

5. (a) $\int \frac{\sin x}{\sin 4x}\ dx$ (b) $\int \frac{1-\cos x}{\cos x(1+\cos x)}\ dx$

LET US SUM UP

- Integration is inverse of differention
- Standard form of some indefinite integrals

  (a) $\int x^n\ dx \quad = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

(b) $\int \frac{1}{x} dx$ $= \log |x| + C$

(c) $\int \sin x \, ax$ $= -\cos x + C$

(d) $\int \cos x \, dx$ $= \sin x + C$

(e) $\int \sec^2 x \, dx$ $= \tan x + C$

(f) $\int \operatorname{cosec}^2 x \, dx$ $= -\cot x + C$

(g) $\int \sec x \tan x \, dx$ $= \sec x + C$

(h) $\int \operatorname{cosec} x \cot x \, dx$ $= -\operatorname{cosec} x + C$

(i) $\int \frac{1}{\sqrt{1-x^2}} dx$ $= \sin^{-1} x + C$

(j) $\int \frac{1}{1+x^2} dx$ $= \tan^{-1} x + C$

(k) $\int \frac{1}{x\sqrt{x^2-1}} dx$ $= \sec^{-1} x + C$

(l) $\int e^x \, dx$ $= e^x + C$

(m) $\int a^x \, dx$ $= \frac{a^x}{\log a} + C \; (a > 0 \text{ and } a \neq 1)$

- Properties of indefinite integrals

(a) $\int [f(x) \pm g(x)] dx$ $= \int f(x) dx \pm \int g(x) dx$

(b) $\int k f(x) dx$ $= k \int f(x) dx$

(i) $\int (ax+b)^n dx$ $= \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C \; (n \neq -1)$

(ii) $\int \frac{1}{ax+b} dx$ $= \frac{1}{a} \log |ax+b| + C$

(iii) $\int \sin(ax+b) dx$ $= \frac{-1}{a} \cos(ax+b) + C$

(iv) $\int \cos(ax+b) dx$ $= \frac{1}{a} \sin(ax+b) + C$

(v) $\int \sec^2(ax+b) dx$ $= \frac{1}{a} \tan(ax+b) + C$

(vi) $\int \operatorname{cosec}^2(ax+b)dx \qquad = -\frac{1}{a}\cot(ax+b)+C$

(vii) $\int \sec(ax+b)\tan(ax+b)\,dx \qquad = \frac{1}{a}\sec(ax+b)+C$

(viii) $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx \qquad = -\frac{1}{a}\operatorname{cosec}(ax+b)+C$

(xi) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}+C$

$\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+C$

- (i) $\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$

  (ii) $\int \cot x\,dx = \log|\sin x| + C$

  (iii) $\int \sec x\,dx = \log|\sec x + \tan x| + C$

  (iv) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$

  (i) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2}\log\left|\frac{a+x}{a-x}\right|+C$ (ii) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

  (iii) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$ (iv) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+C$

  (v) $\int \frac{1}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|+C$

  (vi) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right|+C$

- (i) $\int \frac{x^2+1}{x^4+1}dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)+C$

  (ii) $\int \frac{x^2-1}{x^4+1}dx = \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+C$

  (iii) $\int \frac{x^2}{x^4+1}dx = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{2}}\right)+\log\left[\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right]+C$

(iv) $$\int \frac{1}{x^4+1}dx = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

- (i) $$\int \frac{x^2+1}{x^4+x^2+1}dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right) + C$$

(ii) $$\int \frac{x^2-1}{x^4+x^2+1}dx = \frac{1}{2}\log\left(\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right) + C$$

(iii) $$\int \frac{1}{x^4+x^2+1}dx = \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot x-1}{\sqrt{2}}\right)\right] + C$$

(iv) $$\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx = \sqrt{2}\,\sin^{-1}(\sin x-\cos x) + C$$

- Integral of the product of two functions

$$\text{I function} \times \text{II function} - \int\left[\text{Derivative of I function} \times \text{Integral of II function}\right]dx$$

- $$\int e^x\left[f(x)+f'(x)\right]dx = e^x f(x) + C$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2-a^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right] + C$$

$$\int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int\sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + C$$

- Rational fractions are of following two types :

(i) Proper, where degree of variable of numerator < denominator.

(ii) Improper, where degree of variable of numerator $\geq$ denominator.

- If g(x) is a proper fraction $\frac{f(x)}{g(x)}$ can be resolved into real factors, then $\frac{f(x)}{g(x)}$ can be written in the following form :

Factors in denominator	Corresponding partial fraction
$ax + b$	$\frac{A}{ax+b}$
$(ax + b)^2$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$
$(ax + b)^3$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$
$ax^2 + bx + C$	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2 + bx + c)^2$	$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}$

where A, B, C, D are arbitrary constants.

SUPPORTIVE WEBSITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

Integrate the following functions w.r.t. x :

1. $\frac{\sin^3 x+\cos^3 x}{\sin^2 x\ \cos^2 x}$
2. $\sqrt{1+\sin 2x}$
3. $\frac{\cos 2x}{\cos^2 x\sin^2 x}$
4. $(\tan x\ -\cot x)^2$
5. $\frac{4}{1+x^2}-\frac{1}{\sqrt{1-x^2}}$
6. $\frac{2\sin^2 x}{1+\cos 2x}$
7. $\frac{2\cos^2 x}{1-\cos 2x}$
8. $\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2$
9. $\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2$
10. $\cos(7x-\pi)$
11. $\sin(3x+4)$
12. $\sec^2(2x+b)$

13. $\int \frac{dx}{\sin x - \cos x}$

14. $\int \frac{1}{(1+x^2)\tan^{-1} x} dx$

15. $\int \frac{\operatorname{cosec} x}{\log\left(\tan \frac{x}{2}\right)} dx$

16. $\int \frac{\cot x}{3 + 4\log \sin x} dx$

17. $\int \frac{dx}{\sin 2x \, \log \tan x}$

18. $\int \frac{e^x + 1}{e^x - 1} dx$

19. $\int \sec^4 x \, \tan x \, dx$

20. $\int e^x \, \sin e^x \, dx$

21. $\int \frac{x \, dx}{\sqrt{2x^2 + 3}}$

22. $\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$

23. $\int \sqrt{25 - 9x^2} \, dx$

24. $\int \sqrt{2ax - x^2} \, dx$

25. $\int \sqrt{3x^2 + 4} \, dx$

26. $\int \sqrt{1 + 9x^2} \, dx$

27. $\int \frac{x^2 \, dx}{\sqrt{x^2 - a^2}}$

28. $\int \frac{dx}{\sin^2 x + 4\cos^2 x}$

29. $\int \frac{dx}{2 + \cos x}$

30. $\int \frac{dx}{x^2 - 6x + 13}$

31. $\int \frac{dx}{1 + 3\sin^2 x}$

32. $\int \frac{x^2}{x^2 - a^2} dx$

33. $\int \frac{dx}{1 - 4\cos^2 x}$

34. $\int \frac{\sin x}{\sin 3x} dx$

35. $\int \frac{dx}{1 - 4\cos^2 x}$

36. $\int \sec^2(ax + b) dx$

37. $\int \frac{dx}{x(2 + \log x)}$

38. $\int \frac{x^5}{1 + x^6} dx$

39. $\int \frac{\cos x - \sin x}{\sin x + \cos x} dx$

40. $\int \frac{\cot x}{\log \sin x} dx$

41. $\int \frac{\sec^2 x}{a + b\tan x} dx$

42. $\int \frac{\sin x}{1 + \cos x} dx$

43. $\int \cos^2 x \, dx$

44. $\int \sin^3 x \, dx$

45. $\int \sin 5x \sin 3x \, dx$

46. $\int \sin^2 x \cos^3 x \, dx$

47. $\int \sin^4 x \, dx$

48. $\int \frac{1}{1 + \sin x} dx$

49. $\int \tan^3 x \, dx$

50. $\int \frac{\cos x - \sin x}{1 + \sin 2x} \, dx$

51. $\int \frac{\operatorname{cosec}^2 x}{1 + \cot x} \, dx$

52. $\int \frac{1 + x + \cos 2x}{x^2 + \sin 2x + 2x} \, dx$

53. $\int \frac{\sec\theta \operatorname{cosec}\theta \, d\theta}{\log \tan\theta}$

54. $\int \frac{\cot\theta \, d\theta}{\log \sin\theta}$

55. $\int \frac{dx}{1 + 4x^2}$

56. $\int \frac{1 - \tan\theta}{1 + \tan\theta} \, d\theta$

57. $\int \frac{1}{x^2} e^{-\frac{1}{x}} \, dx$

58. $\int \frac{\sin x \cos x \, dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

59. $\int \frac{dx}{\sin x + \cos x}$

60. $\int e^x \left(\cos^{-1} x - \frac{1}{\sqrt{1 - x^2}} \right) dx$

61. $\int e^x \left(\frac{\sin x + \cos x}{\cos^2 x} \right) dx$

62. $\int \tan^{-1} \sqrt{\frac{1 - \cos x}{1 + \cos x}} \, dx$

63. $\int \cos \left[2 \cot^{-1} \left(\sqrt{\frac{1 - x}{1 + x}} \right) \right] dx$

64. $\int \frac{\sin^{-1} x}{(1 - x^2)^{\frac{3}{2}}} \, dx$

65. $\int \sqrt{x} \log x \, dx$

66. $\int e^x (1 + x) \log(xe^x) \, dx$

67. $\int \frac{\log x}{(1 + x)^2} \, dx$

68. $\int e^x (1 + x) \log(xe^x) \, dx$

69. $\int \cos(\log x) \, dx$

70. $\int \log(x + 1) \, dx$

71. $\int \frac{x^2 + 1}{(x - 1)^2 (x + 3)} \, dx$

72. $\int \frac{\sin\theta \cos\theta}{\cos^2\theta - \cos\theta - 2} \, d\theta$

73. $\int \frac{dx}{x(x^5 + 1)}$

74. $\int \frac{x^2 + 1}{(x^2 + 2)(2x^2 + 1)} \, dx$

75. $\int \frac{\log x}{x(1 + \log x)(2 + \log x)} \, dx$

76. $\int \frac{dx}{1 - e^x}$

ANSWERS

CHECK YOUR PROGRESS 26.1

1. $\frac{2}{7}x^{\frac{2}{7}}+1,\ \frac{2}{7}x^{\frac{7}{2}}+2,\ \frac{2}{7}x^{\frac{7}{2}}+3,\ \frac{2}{7}x^{\frac{7}{2}}+4,\ \frac{2}{7}x^{\frac{7}{2}}+5$

2. (a) $\frac{x^6}{6}+C$ (b) $\sin x+C$ (c) 0

3. (a) $\frac{x^7}{7}+C$ (b) $\frac{1}{6x^6}+C$ (c) $\log|x|+C$

(d) $\dfrac{\left(\frac{3}{5}\right)^x}{\log\left(\frac{3}{5}\right)}+C$ (e) $\frac{3}{4}x^{\frac{4}{3}}+C$ (f) $\frac{-1}{8x^8}+C$

(g) $2\sqrt{x}+C$ (h) $9x^{\frac{1}{9}}+C$

4. (a) $-\text{cosec}\,\theta+C$ (b) $\sec\theta+C$
(c) $\tan\theta+C$ (d) $-\cot\theta+C$

CHECK YOUR PROGRESS 26.2

1. (a) $\frac{x^2}{2}+\frac{1}{2}x+C$ (b) $-x+\tan^{-1}x+C$

(c) $x^{10}-\frac{2}{3}x^{\frac{3}{2}}+2\sqrt{x}+C$ (d) $-\frac{1}{x^5}-\frac{3}{4x^4}+\frac{2}{3x^3}+\frac{7}{x}-8x+C$

(e) $\frac{x^3}{3}-x-\tan^{-1}x+C$ (f) $\frac{x^2}{2}+4x+4\log x+C$

2. (a) $\frac{1}{2}\tan x+C$ (b) $\tan x-x+C$

(c) $-2\,\text{cosec}\,x+C$ (d) $-\frac{1}{2}\cot x+C$

(e) $-\sec x+C$ (f) $-\cot x+\text{cosec}\,x+C$

3. (a) $\sqrt{2}\sin x+C$ (b) $-\sqrt{2}\cos x+C$

(c) $-\frac{1}{2}\cot x+C$

4. (a) $\frac{2}{3}(x+2)^{\frac{3}{2}}+C$ (b) $\frac{-17}{16}\frac{1}{(x+1)^{16}}+C$

CHECK YOUR PROGRESS 26.3

1. (a) $\frac{1}{5}\cos(4-5x)+C$ (b) $\frac{1}{3}\tan(2+3x)+C$

(c) $\log\left|\sec\left(x+\frac{\pi}{4}\right)+\tan\left(x+\frac{\pi}{4}\right)\right|+C$

(d) $\frac{1}{4}\sin(4x+5)+C$

(e) $\frac{1}{3}\sec(3x+5)+C$ (f) $-\frac{1}{5}\operatorname{cosec}(3+5x)+C$

2. (a) $\frac{1}{12(3-4x)^3}+C$ (b) $\frac{1}{5}(x+1)^5+C$

(c) $-\frac{1}{77}(4-7x)^{11}+C$ (d) $\frac{1}{16}(4x-5)^4+C$

(e) $\frac{1}{3}\log|3x-5|+C$ (f) $-\frac{2}{9}\sqrt{5-9x}+C$

(g) $\frac{1}{6}(2x+1)^3+C$ (h) $\log|x+1|+C$

3. (a) $\frac{1}{2}e^{2x+1}+C$ (b) $-\frac{1}{8}e^{3-8x}+C$

(c) $\frac{1}{4e^{(7+4x)}}+C$

4. (a) $\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+C$ (b) $\frac{1}{32}\left(-\frac{3}{2}\cos 2x+\frac{1}{6}\cos 6x\right)+C$

(c) $\frac{1}{2}\left[-\frac{\cos 7x}{7}-\cos x\right]+C$ (d) $\frac{1}{2}\left[\frac{\sin 6x}{6}+\frac{\sin 2x}{2}\right]+C$

CHECK YOUR PROGRESS 26.4

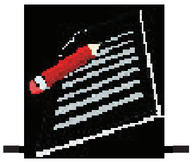
1. (a) $\frac{1}{6}\log|3x^2-2|+C$ (b) $\log|x^2+x+1|+C$

(c) $\log|x^2+9x+30|+C$ (d) $\frac{1}{3}\log|x^3+3x+3|+C$

(e) $\log|x^2+x-5|+C$ (f) $2\log|5+\sqrt{x}|+C$

(g) $\log|8+\log x|+C$

2. (a) $\frac{1}{b}\log|a+be^x|+C$ (b) $\tan^{-1}(e^x)+C$



Notes

CHECK YOUR PROGRESS 26.5

1. (a) $x+\frac{3}{2}\log\left|\frac{x-3}{x+3}\right|+C$ (b) $\tan^{-1}\left(e^x\right)+C$

(c) $\frac{1}{2}\tan^{-1}(x^2)+C$ (d) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right)+C$

(e) $\frac{1}{2}\tan^{-1}(2\tan x)+C$ (f) $\sin^{-1}\left(\frac{x+1}{2}\right)+C$

(g) $\frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$ (h) $\sin^{-1}\left(\frac{x+2}{3}\right)+C$

(i) $\frac{1}{2\sqrt{3}}\sec^{-1}\frac{x}{2}+C$ (j) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2\theta-1}{\sqrt{2}\tan\theta}\right)+C$

(k) $\log\left|e^x+\sqrt{1+e^{2x}}\right|+C$ (l) $\sin^{-1}x-\sqrt{1-x^2}+C$

(m) $\sin^{-1}\left(\frac{x-a}{a}\right)+C$ (n) $\frac{1}{4}\sin^{-1}\left(\frac{4}{3}x^3\right)+C$

(o) $\sqrt{x^2+1}+\log\left|x-\sqrt{x^2+1}\right|+C$ (p) $\frac{1}{2}\log\left|\frac{2x+\sqrt{9+4x^2}}{2}\right|+C$

(q) $-\frac{1}{2}\log\left|2\cos\theta+\sqrt{4\cos^2\theta-1}\right|+C$

(r) $\log\left|\tan x+\sqrt{\tan^2 x-4}\right|+C$

(s) $\tan^{-1}\left(\frac{x+2}{1}\right)+C$

CHECK YOUR PROGRESS 26.6

1. (a) $-x\cos x+\sin x+C$

(b) $\frac{1}{2}\left(1+x^2\right)\sin 2x+\frac{x\cos 2x}{2}-\frac{\sin 2x}{4}+C$

(c) $\frac{-x\cos 2x}{2}+\frac{1}{2}\frac{\sin 2x}{2}+C$

2. (a) $x\tan x-\log|\sec x|-x+C$

(b) $\frac{1}{6}x^3-\frac{1}{4}x^2\sin 2x-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+C$

3. (a) $\dfrac{x^4 \log 2x}{4} - \dfrac{x^4}{16} + C$ (b) $\left(x - \dfrac{x^3}{3}\right)\log x - x + \dfrac{x^3}{9} + C$

(c) $x(\log x)^2 - 2x \log x + 2x + C$

4. (a) $\dfrac{x^{1-n}}{1-n} \log x - \dfrac{x^{1-n}}{(1-n)^2} + C$ (b) $\log x \,.\, [\log(\log x) - 1] + C$

5. (a) $e^{3x}\left[\dfrac{x^2}{3} - \dfrac{2x}{9} + \dfrac{2}{27}\right] + C$ (b) $x\dfrac{e^{4x}}{4} - \dfrac{e^{4x}}{16} + C$

6. $\dfrac{x^2}{2}\left[(\log x)^2 - \log x + \dfrac{1}{2}\right] + C$

7. (a) $x \sec^{-1} x - \log\left|x + \sqrt{x^2 - 1}\right| + C$

(b) $\dfrac{x^2}{2}\cot^{-1} x + \dfrac{x}{2} + \dfrac{1}{2}\cot^{-1} x + C$

CHECK YOUR PROGRESS 26.7

1. (a) $e^x \sec x + C$ (b) $e^x \log|\sec x + \tan x| + C$

2. (a) $\dfrac{1}{x} e^x + C$ (b) $e^x \sin^{-1} x + C$

3. $\dfrac{e^x}{(1+x)^2} + C$ 4. $\dfrac{e^x}{1+x} + C$

5. $x \tan \dfrac{x}{2} + C$

6. $\dfrac{1}{5} e^x (\sin 2x - 2\cos 2x) + C$

CHECK YOUR PROGRESS 26.7

1. (a) $x\sqrt{x^2 - \dfrac{5}{4}} - \dfrac{5}{4}\log\left|x + \sqrt{x^2 - \dfrac{5}{4}}\right| + C$

(b) $\dfrac{(2x+3)}{4}\sqrt{x^2 + 3x} - \dfrac{9}{8}\log\left|\left(x + \dfrac{3}{2}\right) + \sqrt{x^2 + 3x}\right| + C$

(c) $\dfrac{1}{4}(2x+1)\sqrt{3 - 2x - 2x^2} + \dfrac{7}{4\sqrt{2}}\sin^{-1}\left(\dfrac{2x+1}{\sqrt{7}}\right) + C$

2. (a) $4\log|x-3|-3\log|x-2|+C$

(b) $\frac{1}{2}\log|x-4|+\log|x+4|+C$

3. (a) $\frac{x^2}{2}-2[\log|x-2|+\log|x+2|]+C$

(b) $\frac{11}{9}\log|x-1|+\frac{7}{9}\log(x+2)-\frac{4}{3(x-1)}+C$

4. $\log|x-1|-\frac{3}{(x-1)}-\frac{3}{2(x-1)^2}+C$

5. (a) $\frac{1}{8}\log|1-\sin x|-\frac{1}{8}|1+\sin x|$

$$-\frac{1}{4\sqrt{2}}\log\left|1-\sqrt{2}\sin x\right|+\frac{1}{4\sqrt{2}}\log\left|1+\sqrt{2}\sin x\right|+C$$

(b) $\log|\sec x+\tan x|-2\tan\frac{x}{2}+C$

TERMINAL EXERCISE

1. $\sec x - \operatorname{cosec} x + C$
2. $\sin x - \cos x + C$
3. $-\cot x - \tan x + C$
4. $\tan x - \cot x - 4x + C$
5. $4\tan^{-1}x - \sin^{-1}x + C$
6. $\tan x - x + C$
7. $-\cot x - x + C$
8. $x - \cos x + C$
9. $x + \cos x + C$
10. $\frac{\sin(7x-\pi)}{7}+C$
11. $\frac{-\cos(3x+4)}{3}+C$
12. $\frac{\tan(2x+b)}{2}+C$
13. $\frac{1}{\sqrt{2}}\log\left|\operatorname{cosec}\left(x-\frac{\pi}{4}\right)-\cot\left(x-\frac{\pi}{4}\right)\right|+C$
14. $\log|\tan^{-1}x|+C$
15. $\log\left|\log\tan\frac{x}{2}\right|+C$
16. $\frac{1}{4}\log\left|3+4\log\sin x\right|+C$
17. $\frac{1}{2}\log\left|\log\tan x\right|+C$
18. $2\log|e^{\frac{x}{2}}-e^{\frac{-x}{2}}|+C$
19. $\frac{1}{4}\sec^4 x+C$

20. $-\cos\ c^{x} + C$

21. $\dfrac{\sqrt{2x^2+3}}{2} + C$

22. $2\sqrt{\tan x} + C$

23. $\dfrac{1}{6}x\sqrt{(25-9x^2)} + \dfrac{25}{6}\sin^{-1}\left(\dfrac{3}{5}x\right) + C$

24. $\dfrac{1}{2}(x-a)\sqrt{2ax-x^2} + \dfrac{1}{2}a^2\sin^{-1}\left(\dfrac{x-a}{a}\right) + C$

25. $\dfrac{x\sqrt{3x^2+4}}{2} + \dfrac{2}{\sqrt{3}}\log\left|\dfrac{\sqrt{3}x+\sqrt{x^2+4}}{2}\right| + C$

26. $\dfrac{x\sqrt{9x^2+1}}{2} + \dfrac{1}{6}\log\left|3x+\sqrt{1+9x^2}\right| + C$

27. $\left[\dfrac{1}{2}x\sqrt{x^2-a^2} + \dfrac{1}{2}a^2\log\left|x+\sqrt{x^2-a^2}\right|\right] + C$

28. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{\tan x}{2}\right) + C$

29. $\dfrac{2}{\sqrt{3}}\tan^{-1}\left[\dfrac{\tan\left(\dfrac{x}{2}\right)}{\sqrt{3}}\right] + C$

30. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x-3}{2}\right) + C$

31. $\dfrac{1}{2}\tan^{-1}(2\tan x) + C$

32. $x + \dfrac{a}{2}\log\left|\dfrac{x-a}{x+a}\right| + C$

33. $\dfrac{1}{12}\log\left|\dfrac{\sqrt{9+x^4}-3}{\sqrt{9+x^4}+3}\right| + C$

34. $\dfrac{1}{2\sqrt{3}}\log\left|\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$

35. $\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{\tan x-\sqrt{2}}{\tan x+\sqrt{2}}\right| + C$

36. $\dfrac{1}{a}\tan(ax+b) + C$

37. $\log|(2+\log x)| + C$

38. $\dfrac{1}{6}\log(1+x^6) + C$

39. $\log|\sin x + \cos x| + C$

40. $\log|\log(\sin x)| + C$

41. $\dfrac{1}{b}\log|a+b\tan x| + C$

42. $-\log|1+\cos x| + C$

43. $\dfrac{1}{2}\,\dfrac{\sin 2x}{2} + \dfrac{1}{2}x + C$

44. $-\cos x + \dfrac{\cos^3 x}{3} + C$

45. $\dfrac{1}{2}\dfrac{\sin 2x}{2} - \dfrac{1}{2}\dfrac{\sin 8x}{8} + C$

46. $\dfrac{1}{3}\sin^3 x - \dfrac{\sin^5 x}{5} + C$

47. $\dfrac{1}{32}[12x - 8\sin 2x + \sin 4x] + C$

48. $\tan x - \sec x + C$

49. $\dfrac{\tan^2 x}{2} + \log|\cos x| + C$

50. $\dfrac{-1}{\cos x + \sin x} + C$

51. $\log\left|\dfrac{1}{1+\cot x}\right| + C$

52. $\dfrac{1}{2}\log\left|x^2 + \sin 2x + 2x\right| + C$

53. $\log|\tan\theta| + C$

54. $\log|\log\sin\theta| + C$

55. $\dfrac{1}{2}\tan^{-1} 2x$

56. $\log|\cos\theta + \sin\theta| + C$

57. $e^{-\frac{1}{x}} + C$

58. $\dfrac{1}{2(a^2 - b^2)}\log\left|a^2\sin^2 x + b^2\cos^2 x\right| + C$

59. $\dfrac{1}{\sqrt{2}}\log\left|\sec\left(x - \dfrac{\pi}{4}\right) + \tan\left(x - \dfrac{\pi}{4}\right)\right| + C$

60. $e^x \cos^{-1} x + C$

61. $e^x \sec x + C$

62. $\dfrac{1}{4}x^2 + C$

63. $-\dfrac{1}{2}x^2 + C$

64. $\dfrac{x\sin^{-1} x}{\sqrt{1-x^2}} + \dfrac{1}{2}\log|1 - x^2| + C$

65. $\dfrac{2}{3}x^{\frac{3}{2}}\left(\log x - \dfrac{2}{3}\right) + C$

66. $xe^x\left[\log\left(xe^x\right) - 1\right] + C$

67. $-\dfrac{1}{1+x}\log|x| + \log|x| - \log|x+1| + C$

68. $\dfrac{1}{2}e^x - \dfrac{e^x}{10}(2\sin 2x + \cos 2x) + C$

69. $\dfrac{x}{2}[\cos(\log x) + \sin(\log x)] + C$

70. $x\log|x+1| - x + \log|x+1| + C$

71. $\dfrac{3}{8}\log|x-1| - \dfrac{1}{2(x-1)} + \dfrac{5}{8}\log|x+3| + C$

72. $-\frac{2}{3}\log|\cos\theta-2|-\frac{1}{3}\log|\cos\theta+1|+C$

73. $\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right|+C$

74. $\frac{1}{3\sqrt{2}}\left[\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\tan^{-1}\left(\sqrt{2}x\right)\right]+C$

75. $\log\left|\frac{(2+\log x)^2}{1+\log x}\right|+C$

76. $\log\left|\frac{e^x}{1-e^x}\right|+C$

Integration

26.9 Reduction Formulae

There are many functions whose integrals cannot be reduced to one or the other of the well known standard forms of integration. However, in some cases these integrals can be connected algebraically with integrals of other expressions which are directly integrable or which may be easier to integrate than the original functions. Such connecting algebraic relations are called reduction formulae. The formulae connect an integral with another which is of the same type, but is of lower degreee or order or at any rate relatively easier to integrate. In this section, we illustrate the method of integration by successive reduction.

26.9.1 Reduction formula for $\int x^n e^{ax} dx$ n being a positive integer

Let $I_n = \int x^n e^{ax} dx$

On using formula for integration by parts, we get

$$I_n = \frac{x^n e^{ax}}{a} - \int n\, x^{n-1} \frac{e^{ax}}{a} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

This is called a reduction formula for $\int x^n e^{ax} dx$. Now I_{n-1} in turn can be connected to I_{n-2}. By successive reduction of n, the original integral I_n finally depends on I_0 where

$$I_0 = \int e^{ax} dx = \frac{e^{ax}}{a}$$

Example 1

$$\int x^3 e^{5x} \, dx$$

Sol: We take $a = 5$ and use the reduction formula for $n = 3, 2, 1$ in that order.

Then we have

$$I_3 = \int x^3 e^{5x} \, dx = \frac{x^3 e^{5x}}{5} - \frac{3}{5} I_2$$

$$I_2 = \frac{x^2 e^{5x}}{5} - \frac{2}{5} I_1$$

$$I_1 = \frac{x e^{5x}}{5} - \frac{1}{5} I_0$$

and $I_0 = \dfrac{e^{5x}}{5} + c$

Hence $I_3 = \dfrac{x^3 e^{5x}}{5} - \dfrac{3}{5^2} x^2 e^{5x} + \dfrac{6}{5^3} x e^{5x} - \dfrac{6}{5^4} e^{5x} + c.$

1. Theorem: Reduction formula for $\int \sin^n x \, dx$ for an integer $n \geq 2$

Proof: Let $I_n = \int \sin^n x \, dx$

$$I_n = \int \sin^{n-1} x \sin x \, dx$$

$$= \int \sin^{n-1} x \frac{d}{dx} (-\cos x) \, dx$$

$$= \sin^{n-1} x \, (-\cos x) - \int (n-1) \sin^{n-2} x \cos x \, (-\cos x) \, dx$$

$$= -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \, (1 - \sin^2 x) \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

Hence $I_n = \dfrac{-\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}$

This is called reduction formula for $\int \sin^n x \; dx$

If n is even, after succesive reduction, we get

$$I_0 = \int (\sin x)^0 \; dx = x + c_1$$

If n is odd, after successive reduction, we get

$$I_1 = \int (\sin x)^1 \; dx = -\cos x + c_2$$

Example 2

Evaluate $\int \sin^4 x \, dx$.

Sol: On using the reduction formula for $\int \sin^4 x \, dx$ with $n = 4$ and 2 in that order we have

$$I_4 = \int \sin^4 x \, dx$$

$$= -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} I_2$$

$$= -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\left[-\frac{\sin x \cos x}{2} + \frac{1}{2} I_0\right]$$

$$= -\frac{\sin^3 x \cos x}{4} - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + c.$$

Notes :

(1) Reduction formula for $\int \text{Tan}^n x \; dx$ for an integer $n \geq 2$.

Let $I_n = \int \text{Tan}^n x = \dfrac{\text{Tan}^{n-1} x}{n-1} - I_{n-2}$

where n is even, I_n will finally depend on

$$I_0 = \int dx = x + c_1$$

when n is odd, I_n will finally depend on

$$I_1 = \int \text{Tan} x \, dx = \log |\sec x| + c_2$$

(2) Reduction formula for $\int \sec^n x \, dx$ for an integer $n \geq 2$

Let $I_n = \int \sec^n x \, dx = \dfrac{1}{n-1} \sec^{n-2} x \; \text{Tan}\, x + \dfrac{n-2}{n-1} I_{n-2}.$

when n is even the last integral to which I_n can be reduced is $I_0 = \int dx = x + c_1$

When n is odd, the ultimate integral is I_1.

which is $I_1 = \int \sec x \, dx = \log|\sec x + \tan x| + c_2$

Example 3

$$\int \sec^5 x \, dx$$

Sol : On using $n = 5$ in the above reduction formula

$$I_5 = \int \sec^5 x \, dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} I_3$$

$$= \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \frac{\sec x \operatorname{Tan} x}{2} + \frac{3}{8} I_1$$

$$= \frac{\sec^3 x \tan x}{4} + \frac{3}{8} \sec x \operatorname{Tan} x + \frac{3}{8} \log|\operatorname{Sec} x + \operatorname{Tan} x| + c.$$

Check Your Progress 1

I. Evaluate the following integrals.

1. $\int x^2 e^{-3x} \, dx$
2. $\int x^3 e^{ax} \, dx$

II. Evaluate the following integrals.

1. $\int \operatorname{Tan}^4 x \, dx$
2. $\int \cos^4 x \, dx$
3. $\int \sec^3 x \, dx$

III 1. If $I_n = \int \cos^n x \, dx$. Then show that

$$I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$$

Let Us Sum Up

1. If $I_n = \int x^n e^{ax}\, dx$, Then

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1} \quad \text{For a positive inter } n.$$

2. If $I_n = \int \sin^n x\, dx$ then

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}, \text{ for an inter } n \geq 2.$$

3. If $I_n = \int \cos^n x\, dx$, Then

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}, \text{ for an integer } n \geq 2$$

4. If $I_n = \int \text{Tan}^n x\, dx$, Then

$$I_n = \frac{\text{Tan}^{n-1} x}{n-1} - I_{n-2}, \text{ for an integer } n \geq 2.$$

Supportive Web sites

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

Answers

Check Your Progress 1

I (1) $-\frac{e^{-3x}}{27}(9x^2+6x+2)+c$

(2) $\frac{e^{ax}}{a^4}(a^3x^3-3a^2x^2+6ax-6)+c$

II(1) $-\frac{\sin^3 x \cos x}{4} - \frac{3}{8}\sin x \cos x + \frac{3}{8}x + c$

(2) $\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + c$

27

DEFINITE INTEGRALS

In the previous lesson we have discussed the anti-derivative, i.e., integration of a function. The very word integration means to have some sort of summation or combining of results.

Now the question arises : Why do we study this branch of Mathematics? In fact the integration helps to find the areas under various laminas when we have definite limits of it. Further we will see that this branch finds applications in a variety of other problems in Statistics, Physics, Biology, Commerce and many more.

In this lesson, we will define and interpret definite integrals geometrically, evaluate definite integrals using properties and apply definite integrals to find area of a bounded region.

OBJECTIVES

After studying this lesson, you will be able to :

- define and interpret geometrically the definite integral as a limit of sum;
- evaluate a given definite integral using above definition;
- state fundamental theorem of integral calculus;
- state and use the following properties for evaluating definite integrals :

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (ii) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

(iii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

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(vi) $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if $f(2a - x) = f(x)$

$= 0$ if $f(2a - x) = -f(x)$

(vii) $\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if f is an even function of x

$= 0$ if f is an odd function of x.

- apply definite integrals to find the area of a bounded region.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of integration
- Area of a bounded region

27.1 DEFINITE INTEGRAL AS A LIMIT OF SUM

In this section we shall discuss the problem of finding the areas of regions whose boundary is not familiar to us. (See Fig. 27.1)

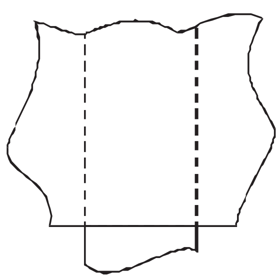


Fig. 27.1

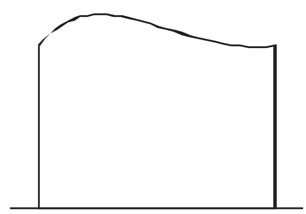


Fig. 27.2

Let us restrict our attention to finding the areas of such regions where the boundary is not familiar to us is on one side of x-axis only as in Fig. 27.2.

This is because we expect that it is possible to divide any region into a few subregions of this kind, find the areas of these subregions and finally add up all these areas to get the area of the whole region. (See Fig. 27.1)

Now, let f (x) be a continuous function defined on the closed interval [a, b]. For the present, assume that all the values taken by the function are non-negative, so that the graph of the function is a curve above the x-axis (See. Fig.27.3).

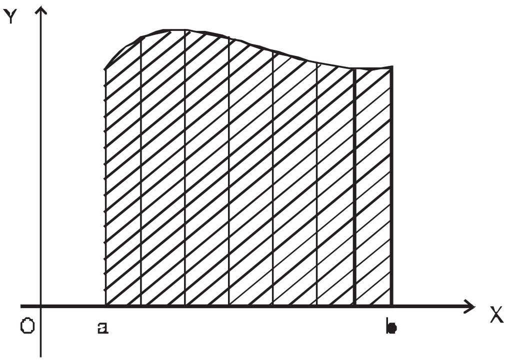


Fig. 27.3

Consider the region between this curve, the x-axis and the ordinates $x = a$ and $x = b$, that is, the shaded region in Fig.27.3. Now the problem is to find the area of the shaded region.

In order to solve this problem, we consider three special cases of $f(x)$ as rectangular region , triangular region and trapezoidal region.

The area of these regions = base × average height

In general for any function $f(x)$ on [a, b]

Area of the bounded region (shaded region in Fig. 27.3) = base × average height

The base is the length of the domain interval [a, b]. The height at any point x is the value of $f(x)$ at that point. Therefore, the average height is the average of the values taken by f in [a, b]. (This may not be so easy to find because the height may not vary uniformly.) Our problem is how to find the average value of f in [a,b].

27.1.1 Average Value of a Function in an Interval

If there are only finite number of values of f in [a,b], we can easily get the average value by the formula.

Average value of f in $[a,b] = \dfrac{\text{Sum of the values of f in } [a,b]}{\text{Numbers of values}}$

But in our problem, there are infinite number of values taken by f in [a, b]. How to find the average in such a case? The above formula does not help us, so we resort to estimate the average value of f in the following way:

First Estimate : Take the value of f at 'a' only. The value of f at a is f (a). We take this value, namely f (a), as a rough estimate of the average value of f in [a,b].

Average value of f in [a, b] (first estimate) = f (a) (i)

Second Estimate : Divide [a, b] into two equal parts or sub-intervals.

Let the length of each sub-interval be h, $h = \dfrac{b-a}{2}$.

Take the values of f at the left end points of the sub-intervals. The values are f (a) and f (a + h)

(Fig. 27.4)

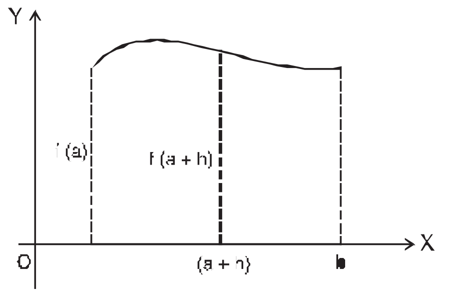


Fig. 27.4

Take the average of these two values as the average of f in [a, b].

Average value of f in [a, b] (Second estimate)

$$= \frac{f(a) + f(a+h)}{2}, \quad h = \frac{b-a}{2} \tag{ii}$$

This estimate is expected to be a better estimate than the first.

Proceeding in a similar manner, divide the interval [a, b] into n subintervals of length h (Fig. 27.5), $h = \frac{b-a}{n}$

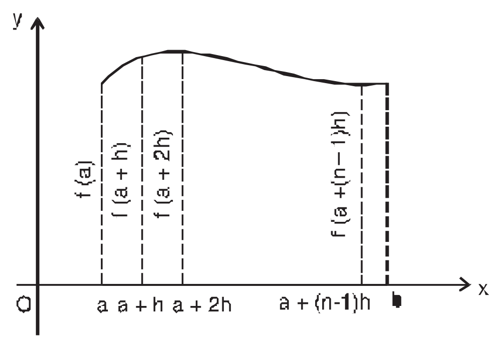


Fig. 27.5

Take the values of f at the left end points of the n subintervals.

The values are f (a), f (a + h),......,f [a + (n-1) h]. Take the average of these n values of f in [a, b].

Average value of f in [a, b] (nth estimate)

$$= \frac{f(a) + f(a+h) + \dots\dots\dots + f(a + (n-1)h)}{n}, \quad h = \frac{b-a}{n} \tag{iii}$$

For larger values of n, (iii) is expected to be a better estimate of what we seek as the average value of f in [a, b]

Thus, we get the following sequence of estimates for the average value of f in [a, b]:

$$f(a)$$

$$\frac{1}{2}\left[f(a)+f(a+h)\right], \qquad h=\frac{b-a}{2}$$

$$\frac{1}{3}\left[f(a)+f(a+h)+f(a+2h)\right], \qquad h=\frac{b-a}{3}$$

....

....

$$\frac{1}{n}\left[f(a)+f(a+h)+........+f\{a+(n-1)h\}\right], \quad h=\frac{b-a}{n}$$

As we go farther and farther along this sequence, we are going closer and closer to our destination, namely, the average value taken by f in [a, b]. Therefore, it is reasonable to take the limit of these estimates as the average value taken by f in [a, b]. In other words,

Average value of f in [a, b]

$$\lim_{n\to\infty}\frac{1}{n}\left\{f(a)+f(a+h)+f(a+2h)+.....+f[a+(n-1)h]\right\},$$

$$h=\frac{b-a}{n} \qquad \text{(iv)}$$

It can be proved that this limit exists for all continuous functions f on a closed interval [a, b].

Now, we have the formula to find the area of the shaded region in Fig. 27.3, The base is $(b-a)$ and the average height is given by (iv). The area of the region bounded by the curve f (x), x-axis, the ordinates x = a and x = b

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left\{f(a)+f(a+h)+f(a+2h)+.....+f[a+(n-1)h]\right\},$$

$$\lim_{n\to 0}\frac{1}{n}\left[f(a)+f(a+h)+........+f\{a+(n-1)h\}\right], \; h=\frac{b-a}{n} \qquad \text{(v)}$$

We take the expression on R.H.S. of (v) as the definition of a **definite integral.** This integral is denoted by

$$\int_a^b f(x)\,dx$$

read as integral of f (x) from a to b'. The numbers a and b in the symbol $\int_a^b f(x)\,dx$ are called respectively the lower and upper limits of integration, and f (x) is called the integrand.

Note : In obtaining the estimates of the average values of f in [a, b], we have taken the left end points of the subintervals. Why left end points?

Why not right end points of the subintervals? We can as well take the right end points of the subintervals throughout and in that case we get

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\{f(a+h)+f(a+2h)+\ldots+f(b)\},\quad h=\frac{b-a}{n}$$

$$= \lim_{h\to 0} h\left[f(a+h)+f(a+2h)+\ldots\ldots+f(b)\right] \qquad \text{(vi)}$$

Example 27.1 Find $\int_1^2 x\,dx$ as the limit of sum.

Solution : By definition,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots\ldots+f\{a+(n-1)h\}\right],$$

$$h=\frac{b-a}{n}$$

Here a = 1, b = 2, f (x) = x and $h=\frac{1}{n}$.

$$\therefore \qquad \int_1^2 x\,dx = \lim_{n\to\infty}\frac{1}{n}\left[f(1)+f\left(1+\frac{1}{n}\right)+\ldots\ldots+f\left(1+\frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{1}{n}\right)+\left(1+\frac{2}{n}\right)\ldots\ldots+\left(1+\frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\underbrace{1+1+\ldots\ldots+1}_{n\text{times}}+\left(\frac{1}{n}+\frac{2}{n}+\ldots+\ldots+\frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}(1+2+\ldots\ldots+(n-1))\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[n+\frac{(n-1).n}{n.2}\right]$$

$$\left[\text{Since } 1+2+3+\ldots+(n-1)=\frac{(n-1).n}{2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{3n-1}{2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{3}{2}-\frac{1}{2n}\right]=\frac{3}{2}$$

Example 27.2 Find $\int_0^2 e^x dx$ as limit of sum.

Solutions : By definition

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\ldots+f\{a+(n-1)h\}\right]$$

where $\quad h = \dfrac{b-a}{n}$

Here $a = 0, b = 2, f(x) = e^x$ and $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

$$\therefore \quad \int_0^2 e^x dx = \lim_{h\to 0} h\left[f(0)+f(h)+f(2h)+\ldots\ldots+f(n-1)h\right]$$

$$= \lim_{h\to 0} h\left[e^0+e^h+e^{2h}+\ldots\ldots+e^{(n-1)h}\right]$$

$$= \lim_{h\to 0} h\left[e^0\left(\frac{\left(e^h\right)^n-1}{e^h-1}\right)\right]$$

$$\left[\text{Since} \quad a+ar+ar^2+\ldots\ldots+ar^{n-1} = a\left(\frac{r^n-1}{r-1}\right)\right]$$

$$= \lim_{h\to 0} h\left[\frac{e^{nh}-1}{e^h-1}\right] = \lim_{h\to 0}\frac{h}{h}\left[\frac{e^2-1}{\left(\frac{e^h-1}{h}\right)}\right] \qquad (\because nh = 2)$$

$$= \lim_{h\to 0}\frac{e^2-1}{\frac{e^h-1}{h}} = \frac{e^2-1}{1}$$

$$= e^2 - 1 \qquad \left[\because \lim_{h\to 0}\frac{e^h-1}{h} = 1\right]$$

In examples 27.1 and 27.2 we observe that finding the definite integral as the limit of sum is quite difficult. In order to overcome this difficulty we have the fundamental theorem of integral calculus which states that

Theorem 1 : If f is continuous in [a, b] and F is an antiderivative of f in [a, b] then

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \ldots\ldots(1)$$

The difference F (b) – F (a) is commonly denoted by $\left[F(x)\right]_a^b$ so that (1) can be written as

$$\int_a^b f(x)\,dx = F(x)\big]_a^b \text{ or } \left[F(x)\right]_a^b$$

In words, the theorem tells us that

$$\int_a^b f(x)\,dx = \text{(Value of antiderivative at the upper limit b)}$$
$$- \text{(Value of the same antiderivative at the lower limit a)}$$

Example 27.3 Find $\int_1^2 x\,dx$

Solution : $$\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

Example 27.4 Evaluate the following

(a) $\int_0^{\frac{\pi}{2}} \cos x\,dx$ (b) $\int_0^2 e^{2x}\,dx$

Solution : We know that

$$\int \cos x dx = \sin x + c$$

$\therefore$ $$\int_0^{\frac{\pi}{2}} \cos x dx = [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

(b) $$\int_0^2 e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_0^2, \qquad \left[\because \int e^x dx = e^x\right]$$

$$= \left(\frac{e^4 - 1}{2}\right)$$

Theorem 2 : If f and g are continuous functions defined in [a, b] and c is a constant then,

(i) $\int_a^b c f(x) dx = c\int_a^b f(x) dx$

(ii) $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$

(iii) $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$

Example 27.5 Evaluate $\int_0^2 \left(4x^2 - 5x + 7\right) dx$

Solution : $\int_0^2 \left(4x^2 - 5x + 7\right) dx = \int_0^2 4x^2 dx - \int_0^2 5x\, dx + \int_0^2 7\, dx$

$$= 4\int_0^2 x^2 dx - 5\int_0^2 x dx + 7\int_0^2 1\, dx$$

$$= 4.\left[\frac{x^3}{3}\right]_0^2 - 5\left[\frac{x^2}{2}\right]_0^2 + 7[x]_0^2$$

$$= 4.\left(\frac{8}{3}\right) - 5\left(\frac{4}{2}\right) + 7(2)$$

$$= \frac{32}{3} - 10 + 14$$

$$= \frac{44}{3}$$

CHECK YOUR PROGRESS 27.1

1. Find $\int_0^5 (x+1)\, dx$ as the limit of sum.

2. Find $\int_{-1}^{1} e^x\, dx$ as the limit of sum.

3. Evaluate (a) $\int_0^{\frac{\pi}{4}} \sin x\, dx$ (b) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\, dx$

   (c) $\int_0^1 \frac{1}{1+x^2} dx$ (d) $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$

27.2 EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION

The principal step in the evaluation of a definite integral is to find the related indefinite integral.

In the preceding lesson we have discussed several methods for finding the indefinite integral. One of the important methods for finding indefinite integrals is the method of substitution. When we use substitution method for evaluation the definite integrals, like

$$\int_2^3 \frac{x}{1+x^2} dx, \quad \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx,$$

the steps could be as follows :

(i) Make appropriate substitution to reduce the given integral to a known form to integrate. Write the integral in terms of the new variable.

(ii) Integrate the new integrand with respect to the new variable.

(iii) Change the limits accordingly and find the difference of the values at the upper and lower limits.

Note : If we don't change the limit with respect to the new variable then after integrating resubstitute for the new variable and write the answer in original variable. Find the values of the answer thus obtained at the given limits of the integral.

Example 27.6 Evaluate $\int_2^3 \frac{x}{1+x^2}\,dx$

Solution : Let $1 + x^2 = t$

$$2x\,dx = dt \qquad \text{or} \qquad x\,dx = \frac{1}{2}dt$$

When x = 2, t = 5 and x =3, t = 10. Therefore, 5 and 10 are the limits when t is the variable.

Thus $$\int_2^3 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_5^{10}\frac{1}{t}\,dt$$

$$= \frac{1}{2}[\log t]_5^{10}$$

$$= \frac{1}{2}[\log 10 - \log 5]$$

$$= \frac{1}{2}\log 2$$

Example 27.7 Evaluate the following :

(a) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$ (b) $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta}\,d\theta$ (c) $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$

Solution : (a) Let $\cos x = t$ then $\sin x\,dx = -\,dt$

When x = 0, t =1 and $x = \frac{\pi}{2}, t = 0$. As x varies from 0 to $\frac{\pi}{2}$, t varies from 1 to 0.

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_1^0 \frac{1}{1+t^2}\,dt = -\left[\tan^{-1} t\right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[0 - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

(b) $$I = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\left(\sin^2\theta + \cos^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1 - 2\sin^2\theta\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta\, d\theta}{1 - 2\sin^2\theta\left(1 - \sin^2\theta\right)}$$

Let $\quad \sin^2\theta = t$

Then $\quad 2\sin\theta\cos\theta\, d\theta = dt$ i.e. $\quad \sin 2\theta\, d\theta = dt$

When $\theta = 0, t = 0$ and $\theta = \frac{\pi}{2}, t = 1$. As θ varies from 0 to $\frac{\pi}{2}$, the new variable t varies from 0 to 1.

$$\therefore \qquad I = \int_0^1 \frac{1}{1 - 2t(1 - t)}\, dt$$

$$= \int_0^1 \frac{1}{2t^2 - 2t + 1}\, dt$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{t^2 - t + \frac{1}{4} + \frac{1}{4}}\, dt$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\, dt$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$= \left[\tan^{-1}1 - \tan^{-1}(-1)\right]$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

(c) We know that $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} \frac{1}{5+4\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{5+\dfrac{4\left(1-\tan^2\left(\frac{x}{2}\right)\right)}{\left(1+\tan^2\left(\frac{x}{2}\right)\right)}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{9+\tan^2\left(\frac{x}{2}\right)}\,dx \qquad (1)$$

Let $\qquad \tan\frac{x}{2} = t$

Then $\qquad \sec^2\frac{x}{2}\,dx = 2dt$ when $x = 0, \quad t = 0$, when $x = \frac{\pi}{2}, \ t = 1$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{1}{5+4\cos x}\,dx = 2\int_0^1 \frac{1}{9+t^2}\,dt \qquad \text{[From (1)]}$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^1 = \frac{2}{3}\left[\tan^{-1}\frac{1}{3}\right]$$

27.3 SOME PROPERTIES OF DEFINITE INTEGRALS

The definite integral of f (x) between the limits a and b has already been defined as

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{Where } \frac{d}{dx}[F(x)] = f(x),$$

where a and b are the lower and upper limits of integration respectively. Now we state below some important and useful properties of such definite integrals.

(i) $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ $\qquad$ (ii) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$ $\qquad$ where a<c<b.

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(v) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(vi) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vii) $\int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \end{cases}$

(viii) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function of } x \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function of } x \end{cases}$

Many of the definite integrals may be evaluated easily with the help of the above stated properties, which could have been very difficult otherwise.

The use of these properties in evaluating definite integrals will be illustrated in the following examples.

Example 27.8 Show that

(a) $\int_0^{\frac{\pi}{2}} \log|\tan x|\,dx = 0$ (b) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx = \pi$

Solution : (a) Let $I = \int_0^{\frac{\pi}{2}} \log|\tan x|\,dx$ ....(i)

Using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$I = \int_0^{\frac{\pi}{2}} \log\left(\tan\left(\frac{\pi}{2}-x\right)\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\cot x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\tan x)^{-1} dx$$

$$= -\int_0^{\frac{\pi}{2}} \log \tan x \, dx$$

$$= -I \qquad \text{[Using (i)]}$$

$\therefore \qquad 2I = 0$

i.e. $\qquad I = 0 \qquad$ or $\qquad \int_0^{\frac{\pi}{2}} \log|\tan x| dx = 0$

(b) $\qquad \int_0^{\pi} \frac{x}{1+\sin x} dx$

Let $\qquad I = \int_0^{\pi} \frac{x}{1+\sin x} dx \qquad$ (i)

$$\therefore \qquad I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx \qquad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx\right]$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx \qquad \text{(ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\pi} \frac{x + \pi - x}{1+\sin x} dx = \pi\int_0^{\pi} \frac{1}{1+\sin x} dx$$

or $\qquad 2I = \pi\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx$

$$= \pi\int_0^{\pi} \left(\sec^2 x - \tan x \sec x\right) dx$$

$$= \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi[(\tan\pi - \sec\pi) - (\tan 0 - \sec 0)]$$

$$= \pi[0 - (-1) - (0-1)]$$

$$= 2\pi$$

$\therefore \qquad I = \pi$

Example 27.9 Evaluate

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{1+\sin x\cos x}\,dx$

Solution : (a) Let $\displaystyle I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ (i)

Also
$$I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

(Using the property $\displaystyle\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$).

$$=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$=\int_0^{\frac{\pi}{2}} 1.dx$$

$$=[x]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$$

$\therefore$ $$I=\frac{\pi}{4}$$

i.e. $$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx=\frac{\pi}{4}$$

(b) Let $\displaystyle I=\int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{1+\sin x\cos x}\,dx$ (i)

Then $$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} + \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x + \cos x - \sin x}{1 + \sin x \cos x} dx$$

$$= 0$$

$$\therefore \qquad I = 0$$

Example 27.10 Evaluate (a) $\int_{-a}^{a} \frac{xe^{x^2}}{1+x^2} dx$ (b) $\int_{-3}^{3} |x+1|\,dx$

Solution : (a) Here $\qquad f(x) = \dfrac{xe^{x^2}}{1+x^2}$

$$\therefore \qquad f(-x) = -\frac{xe^{x^2}}{1+x^2}$$

$$= -f(x)$$

$\therefore$ $f(x)$ is an odd function of x.

$$\therefore \qquad \int_{-a}^{a} \frac{xe^{x^2}}{1+x^2} dx = 0$$

(b) $\int_{-3}^{3} |x+1|\,dx$

$$|x+1| = \begin{cases} x+1, \text{if } x \geq -1 \\ -x-1, \text{if } x < -1 \end{cases}$$

$$\therefore \quad \int_{-3}^{3} |x+1|\,dx = \int_{-3}^{-1} |x+1|\,dx + \int_{-1}^{3} |x+1|\,dx, \text{ using property (iii)}$$

$$= \int_{-3}^{-1} (-x-1)\,dx + \int_{-1}^{3} (x+1)\,dx$$

$$= \left[\frac{-x^2}{2} - x\right]_{-3}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{3}$$

$$= -\frac{1}{2} + 1 + \frac{9}{2} - 3 + \frac{9}{2} + 3 - \frac{1}{2} + 1 = 10$$

Example 27.11 Evaluate $\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$

Solution : Let $I = \int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$ (i)

Also $I = \int_0^{\frac{\pi}{2}} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx,$ [Using property (iv)]

$$= \int_0^{\frac{\pi}{2}} \log(\cos x)\,dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} [\log(\sin x) + \log(\cos x)]\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin x \cos x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \int_0^{\frac{\pi}{2}} \log(2)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \frac{\pi}{2}\log 2 \qquad \text{(iii)}$$

Again, let $I_1 = \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx$

Put $2x = t$ $\Rightarrow$ $dx = \frac{1}{2}dt$

When $x = 0, t = 0$ and $x = \frac{\pi}{2}, t = \pi$

$$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\,dt$$

$$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log(\sin t)\,dt, \qquad \text{[using property (vi)]}$$

$$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log(\sin x)\,dt \qquad \text{[using property (i)]}$$

$\therefore$ $I_1 = I,$ [from (i)] .....(iv)

Putting this value in (iii), we get

$$2I = I - \frac{\pi}{2}\log 2 \quad \Rightarrow \quad I = -\frac{\pi}{2}\log 2$$

Hence, $\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx = -\frac{\pi}{2}\log 2$

CHECK YOUR PROGRESS 27.2

Evaluate the following integrals :

1. $\int_0^1 xe^{x^2}dx$

2. $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin x}$

3. $\int_0^1 \frac{2x+3}{5x^2+1}dx$

4. $\int_{-5}^{5} |x+2|\,dx$

5. $\int_0^2 x\sqrt{2-x}\,dx$

6. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}dx$

7. $\int_0^{\frac{\pi}{2}} \log \cos x\,dx$

8. $\int_{-a}^{a} \frac{x^3 e^{x^4}}{1+x^2}dx$

9. $\int_0^{\frac{\pi}{2}} \sin 2x \log \tan x\,dx$

10. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x+\cos x}dx$

27.4 APPLICATIONS OF INTEGRATION

Suppose that f and g are two continuous functions on an interval [a, b] such that $f(x) \le g(x)$ for $x \in [a,\ b]$ that is, the curve y = f (x) does not cross under the curve y = g (x) over [a, b]. Now the question is how to find the area of the region bounded above by y = f (x), below by y = g (x), and on the sides by x = a and x = b.

Again what happens when the upper curve y = f (x) intersects the lower curve y = g (x) at either the left hand boundary x = a , the right hand boundary x = b or both?

27.4.1 Area Bounded by the Curve, x-axis and the Ordinates

Let AB be the curve y = f (x) and CA, DB the two ordinates at x = a and x = b respectively. Suppose y = f (x) is an increasing function of x in the interval $a \le x \le b$.

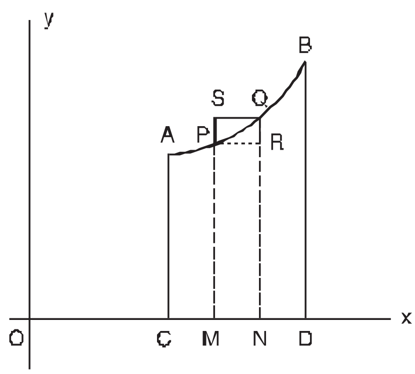


Fig.27.6

Let P (x, y) be any point on the curve and $Q(x + \delta x,\ y + \delta y)$ a neighbouring point on it. Draw their ordinates PM and QN.

Here we observe that as x changes the area (ACMP) also changes. Let

$$A = \text{Area (ACMP)}$$

Then the area (ACNQ)= $A + \delta A$.

The area (PMNQ)=Area (ACNQ)– Area (ACMP)

$$= A + \delta A - A = \delta A.$$

Complete the rectangle PRQS. Then the area (PMNQ) lies between the areas of rectangles PMNR and SMNQ, that is

δA lies between $y\ \delta x$ and $(y + \delta y)\delta x$

$\Rightarrow \quad \dfrac{\delta A}{\delta x}$ lies between y and $(y + \delta y)$

In the limiting case when $Q \to P, \delta x \to 0 \text{ and } \delta y \to 0$.

$\therefore \quad \lim\limits_{\delta x \to 0} \dfrac{\delta A}{\delta x}$ lies between y and $\lim\limits_{\delta y \to 0} (y + \delta y)$

$$\therefore \quad \frac{dA}{dx} = y$$

Integrating both sides with respect to x, from x = a to x = b, we have

$$\int_a^b y\,dx = \int_a^b \frac{dA}{dx}\cdot dx = [A]_a^b$$

$$= (\text{Area when } x = b) - (\text{Area when } x = a)$$

$$= \text{Area (ACDB)} - 0$$

$$= \text{Area (ACDB)}.$$

Hence Area (ACDB) $= \int_a^b f(x)\,dx$

The area bounded by the curve y = f (x), the x-axis and the ordinates x = a, x = b is

$$\int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx$$

where y = f (x) is a continuous single valued function and y does not change sign in the interval $a \le x \le b$.

Example 27.12 Find the area bounded by the curve y = x, x-axis and the lines x =0, x = 2.

Solution : The given curve is y = x

$\therefore$ Required area bounded by the curve, x-axis and the ordinates $x = 0,\ x = 2$ (as shown in Fig. 27.7)

is $\int_0^2 x\,dx$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= 2 - 0 = 2 \text{ square units}$$

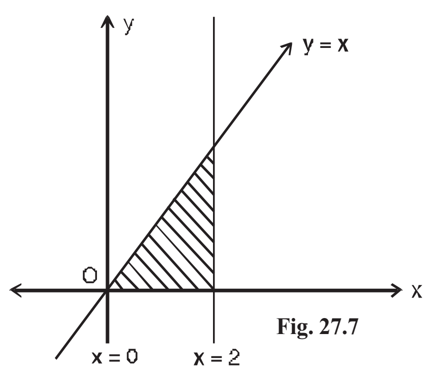


Fig. 27.7

Example 27.13 Find the area bounded by the curve $y = e^x$, x-axis and the ordinates x = 0 and x = a > 0.

Solution : The given curve is $y = e^x$.

$\therefore$ Required area bounded by the curve, x-axis and the ordinates x = 0, x = a is

$$\int_0^a e^x\,dx$$

$$= \left[e^x\right]_0^a$$

$$= (e^a - 1) \text{ square units}$$

Example 27.14 Find the area bounded by the curve $y = c\cos\left(\frac{x}{c}\right)$, x-axis and the ordinates $x = 0$, $x = a$, $2a \le c\pi$.

Solution : The given curve is $y = c\cos\left(\frac{x}{c}\right)$

$$\therefore \qquad \text{Required area} = \int_0^a y\,dx$$

$$= \int_0^a c\cos\left(\frac{x}{c}\right)dx$$

$$= c^2\left[\sin\left(\frac{x}{c}\right)\right]_0^a$$

$$= c^2\left(\sin\left(\frac{a}{c}\right) - \sin 0\right)$$

$$= c^2 \sin\left(\frac{a}{c}\right) \text{ square units}$$

Example 27.15 Find the area enclosed by the circle $x^2 + y^2 = a^2$, and x-axis in the first quadrant.

Solution : The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre and radius are (0, 0) and a respectively. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the x-axis and the ordinates $x = 0$ and $x = a$.

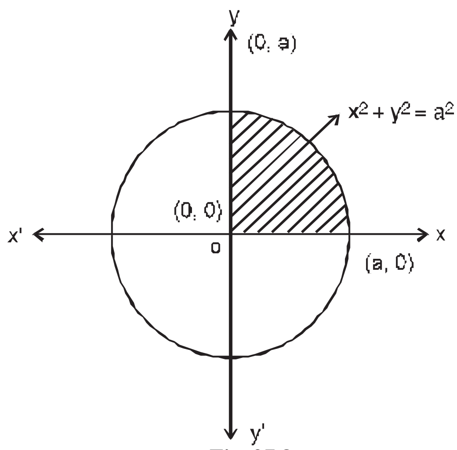


Fig. 27.8

$$\therefore \qquad \text{Required area} = \int_0^a y\,dx$$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx,$$

($\because$ y is positive in the first quadrant)

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1} 1 - 0 - \frac{a^2}{2}\sin^{-1} 0$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} \qquad \left(\because \sin^{-1} 1 = \frac{\pi}{2}, \sin^{-1} 0 = 0\right)$$

$$= \frac{\pi a^2}{4} \text{ square units}$$

Example 27.16 Find the area bounded by the x-axis, ordinates and the following curves :

(i) $xy = c^2, x = a, x = b, \quad a > b > 0$

(ii) $y = \log_e x, x = a, x = b, \quad b > a > 1$

Solution : (i) Here we have to find the area bounded by the x-axis, the ordinates x = a, x = b and the curve

$$xy = c^2 \qquad \text{or} \qquad y = \frac{c^2}{x}$$

$$\therefore \qquad \text{Area} = \int_b^a y\,dx \qquad (\because a > b \text{ given})$$

$$= \int_b^a \frac{c^2}{x}\,dx$$

$$= c^2 \left[\log x\right]_b^a$$

$$= c^2 (\log a - \log b)$$

$$= c^2 \log\left(\frac{a}{b}\right)$$

(ii) Here $\quad y = \log_e x$

$$\therefore \qquad \text{Area} = \int_a^b \log_e x\,dx, \qquad (\because b > a > 1)$$

$$= \left[x \log_e x\right]_a^b - \int_a^b x \cdot \frac{1}{x}\,dx$$

$$= b\log_e b - a\log_e a - \int_a^b dx$$

$$= b\log_e b - a\log_e a - \left[x\right]_a^b$$

$$= b\log_e b - a\log_e a - b + a$$

$$= b(\log_e b - 1) - a(\log_e a - 1)$$

$$= b\log_e\left(\frac{b}{e}\right) - a\log_e\left(\frac{a}{e}\right) \qquad (\because \log_e e = 1)$$

CHECK YOUR PROGRESS 27.3

1. Find the area bounded by the curve $y = x^2$, x-axis and the lines x = 0, x = 2.
2. Find the area bounded by the curve y = 3 x, x-axis and the lines x = 0 and x = 3.

3. Find the area bounded by the curve $y = e^{2x}$, x-axis and the ordinates x = 0, x = a, a > 0.

4. Find the area bounded by the x-axis, the curve $y = c\sin\left(\frac{x}{c}\right)$ and the ordinates x = 0 and x = a, $2a \le c\pi$.

27.4.2. Area Bounded by the Curve x = f (y) between y-axis and the Lines y = c, y = d

Let AB be the curve x = f (y) and let CA, DB be the abscissae at y = c, y = d respectively.

Let P (x, y) be any point on the curve and let $Q(x+\delta x, y+\delta y)$ be a neighbouring point on it. Draw PM and QN perpendiculars on y-axis from P and Q respectively. As y changes, the area (ACMP) also changes and hence clearly a function of y. Let A denote the area (ACMP), then the area (ACNQ) will be $A + \delta A$.

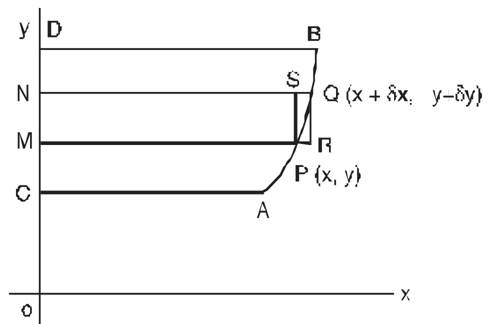


Fig. 27.9

The area (PMNQ) = Area (ACNQ) − Area (ACMP) $= A + \delta A - A = \delta A$.

Complete the rectangle PRQS. Then the area (PMNQ) lies between the area (PMNS) and the area (RMNQ), that is,

δA lies between $x\,\delta y$ and $(x+\delta x)\delta y$

$\Rightarrow \quad \dfrac{\delta A}{\delta y}$ lies between x and $x + \delta x$

In the limiting position when $Q \to P$, $\delta x \to 0$ and $\delta y \to 0$.

$\therefore \quad \lim\limits_{\delta y \to 0} \dfrac{\delta A}{\delta y}$ lies between x and $\lim\limits_{\delta x \to 0}(x + \delta x)$

$$\Rightarrow \quad \frac{dA}{dy} = x$$

Integrating both sides with respect to y, between the limits c to d, we get

$$\int_c^d x\,dy = \int_c^d \frac{dA}{dy}\cdot dy$$

$$= [A]_c^d$$

= (Area when y = d) − (Area when y = c)

= Area (ACDB) − 0

= Area (ACDB)

Hence area (ACDB) $= \int_c^d x\,dy = \int_c^d f(y)\,dy$

The area bounded by the curve x = f (y), the y-axis and the lines y = c and y = d is

$$\int_c^d x\,dy \quad \text{or} \quad \int_c^d f(y)\,dy$$

where x = f (y) is a continuous single valued function and x does not change sign in the interval $c \le y \le d$.

Example 27.17 Find the area bounded by the curve x = y, y-axis and the lines y = 0, y = 3.

Solution : The given curve is x = y.

∴ Required area bounded by the curve, y-axis and the lines y =0, y = 3 is

$$= \int_0^3 x\,dy$$

$$= \int_0^3 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^3$$

$$= \frac{9}{2} - 0$$

$$= \frac{9}{2} \text{ square units}$$

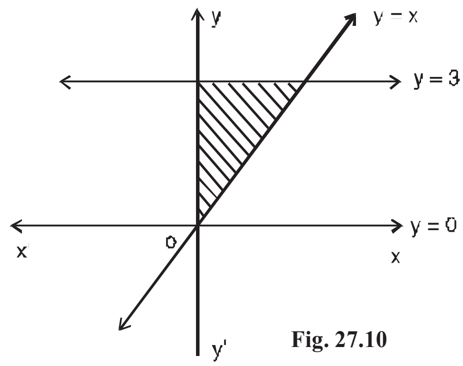


Fig. 27.10

Example 27.18 Find the area bounded by the curve $x = y^2$, y = axis and the lines y = 0, y = 2.

Solution : The equation of the curve is $x = y^2$

∴ Required area bounded by the curve, y-axis and the lines y =0, y = 2

$$= \int_0^2 y^2 dy = \left[\frac{y^3}{3}\right]_0^2$$

$$= \frac{8}{3} - 0$$

$$= \frac{8}{3} \text{ square units}$$

Example 27.19 Find the area enclosed by the circle $x^2 + y^2 = a^2$ and y-axis in the first quadrant.

Solution : The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre is (0, 0) and radius a. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the y-axis and the abscissae y = 0, y = a.

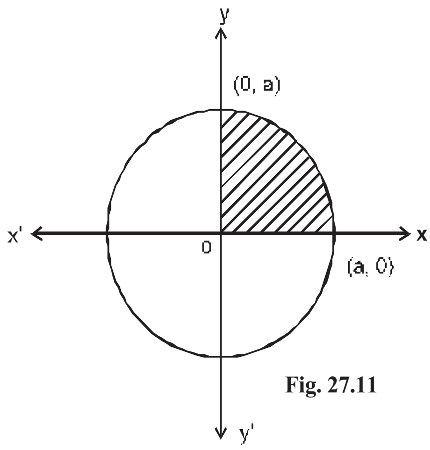


Fig. 27.11

$\therefore$ Required area $= \int_0^a x \, dy$

$$= \int_0^a \sqrt{a^2 - y^2} \, dy$$

(because x is positive in first quadrant)

$$= \left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right) \right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1} 1 - 0 - \frac{a^2}{2}\sin^{-1} 0$$

$$= \frac{\pi a^2}{4} \text{ square units} \qquad \left(\because \sin^{-1} 0 = 0, \sin^{-1} 1 = \frac{\pi}{2}\right)$$

Note : The area is same as in Example 27.14, the reason is the given curve is symmetrical about both the axes. In such problems if we have been asked to find the area of the curve, without any restriction we can do by either method.

Example 27.20 Find the whole area bounded by the circle $x^2 + y^2 = a^2$.

Solution : The equation of the curve is $x^2 + y^2 = a^2$.

The circle is symmetrical about both the axes, so the whole area of the circle is four times the area os the circle in the first quadrant, that is,

Area of circle = 4 × area of OAB

$$= 4 \times \frac{\pi a^2}{4} \text{ (From Example 27.15 and 27.19) } = \pi a^2$$

square units

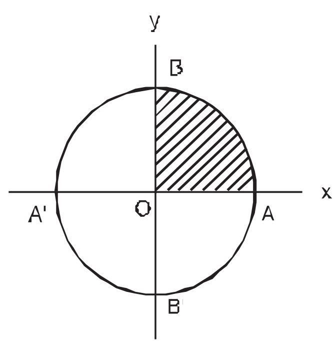


Fig. 27.12

Example 27.21 Find the whole area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution : The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The ellipse is symmetrical about both the axes and so the whole area of the ellipse is four times the area in the first quadrant, that is,

Whole area of the ellipse $= 4 \times$ area (OAB)

In the first quadrant,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \quad \text{or} \quad y = \frac{b}{a}\sqrt{a^2 - x^2}$$

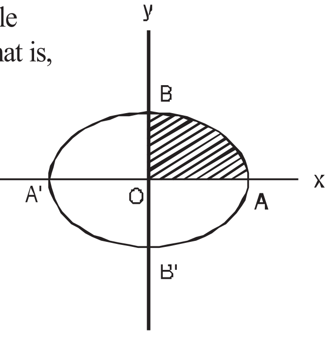


Fig.27.13

Now for the area (OAB), x varies from 0 to a

$$\therefore \qquad \text{Area (OAB)} = \int_0^a y\,dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0\right]$$

$$= \frac{ab\pi}{4}$$

Hence the whole area of the ellipse

$$= 4 \times \frac{ab\pi}{4}$$

$$= \pi ab. \text{ square units}$$

27.4.3 Area between two Curves

Suppose that f (x) and g (x) are two continuous and non-negative functions on an interval [a, b] such that $f(x) \geq g(x)$ for all $x \in [a, b]$ that is, the curve y = f (x) does not cross under the curve y = g (x) for $x \in [a, b]$. We want to find the area bounded above by y = f (x), below by y = g (x), and on the sides by x = a and x = b.

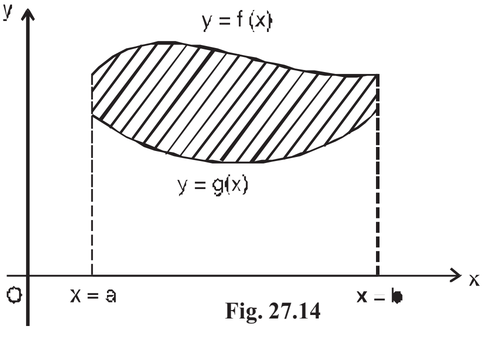


Fig. 27.14

Let A= [Area under y = f (x)] – [Area under y = g (x)] .....(1)

Now using the definition for the area bounded by the curve $y = f(x)$, x-axis and the ordinates x = a and x = b, we have

Area under

$$y = f(x) = \int_a^b f(x)\,dx \qquad .....(2)$$

Similarly,Area under $y = g(x) = \int_a^b g(x)\,dx$ .....(3)

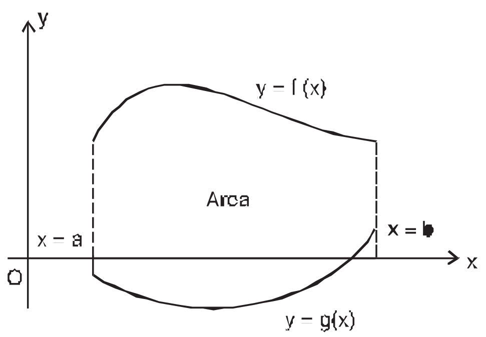
Using equations (2) and (3) in (1), we get

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx \qquad .....(4)$$

What happens when the function g has negative values also? This formula can be extended by translating the curves f (x) and g (x) upwards until both are above the x-axis. To do this let-m be the minimum value of g (x) on [a, b] (see Fig. 27.15).

Since $\quad g(x) \geq -m \quad \Rightarrow \quad g(x) + m \geq 0$



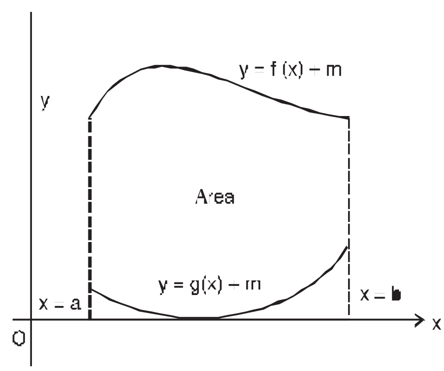


Fig. 27.15 **Fig. 27.16**

Now, the functions $g(x) + m$ and $f(x) + m$ are non-negative on [a, b] (see Fig. 27.16). It is intuitively clear that the area of a region is unchanged by translation, so the area A between f and g is the same as the area between $g(x) + m$ and $f(x) + m$. Thus,

A = [area under $y = [f(x) + m]$] – [area under $y = [g(x) + m]$] .....(5)

Now using the definitions for the area bounded by the curve y = f (x), x-axis and the ordinates x = a and x = b, we have

$$\text{Area under } y = f(x) + m = \int_a^b [f(x) + m]\,dx \qquad .....(6)$$

and Area under $y = g(x) + m = \int_a^b [g(x) + m]\,dx$ (7)

The equations (6), (7) and (5) give

$$A = \int_a^b [f(x) + m]\,dx - \int_a^b [g(x) + m]\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

which is same as (4) Thus,

If f (x) and g (x) are continuous functions on the interval [a, b], and $f(x) \geq g(x), \forall\, x \in [a, b]$, then the area of the region bounded above by y = f (x), below by y = g (x), on the left by x = a and on the right by x = b is

$$= \int_a^b [f(x) - g(x)]\,dx$$

Example 27.22 Find the area of the region bounded above by y = x + 6, bounded below by $y = x^2$, and bounded on the sides by the lines x = 0 and x = 2.

Solution : $y = x + 6$ is the equation of the straight line and $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and origin the vertex. Also the region is bounded by the lines $x = 0$ and $x = 2$.

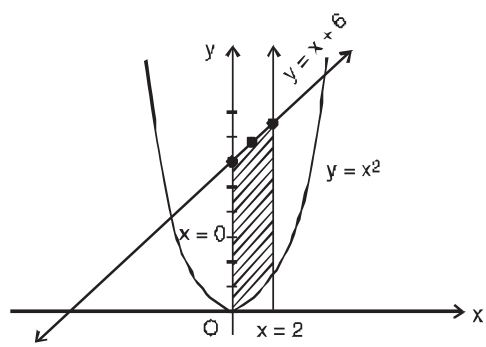


Fig. 27.17

Thus, $$A = \int_0^2 (x + 6)\,dx - \int_0^2 x^2\,dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{34}{3} - 0$$

$$= \frac{34}{3} \text{ square units}$$

If the curves intersect then the sides of the region where the upper and lower curves intersect reduces to a point, rather than a vertical line segment.

Example 27.23 Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution : We know that $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and vertex is origin and $y = x + 6$ is the equation of the straight line which makes an angle 45° with the x-axis and having the intercepts of -6 and 6 with the x and y axes respectively. (See Fig. 27.18).

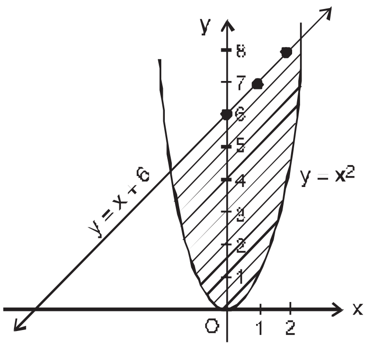


Fig. 27.18

A sketch of the region shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. These two curves intersect at two points, say A and B. Solving these two equations we get

$$x^2 = x + 6 \quad \Rightarrow \quad x^2 - x - 6 = 0$$

$$\Rightarrow \quad (x-3)(x+2) = 0 \quad \Rightarrow \quad x = 3, -2$$

When $x = 3$, $y = 9$ and when $x = -2$, $y = 4$

$$\therefore \text{ The required area } = \int_{-2}^{3} \left[(x+6) - x^2 \right] dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^{3}$$

$$= \frac{27}{2} - \left(-\frac{22}{3} \right)$$

$$= \frac{125}{6} \text{ square units}$$

Example 27.24 Find the area of the region enclosed between the curves $y = x^2$ and $y = x$.

Solution : We know that $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and vertex is origin. y = x is the equation of the straight line passing through the origin and making an angle of 45° with the x-axis (see Fig. 27.19).

A sketch of the region shows that the lower boundary is $y = x^2$ and the upper boundary is the line y = x. These two curves intersect at two points O and A. Solving these two equations, we get

$$x^2 = x$$

$$\Rightarrow \quad x(x-1) = 0$$

$$\Rightarrow \quad x = 0, 1$$

Here $f(x) = x, g(x) = x^2, a = 0$ and $b = 1$

Therefore, the required area

$$= \int_0^1 \left(x - x^2\right) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ square units}$$

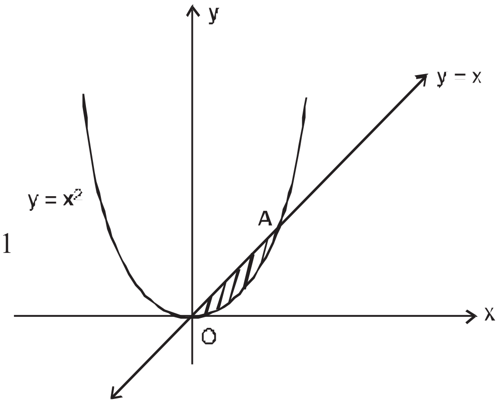


Fig. 27.19

Example 27.25 Find the area bounded by the curves $y^2 = 4x$ and y = x.

Solution : We know that $y^2 = 4x$ the equation of the parabola which is symmetric about the x-axis and origin is the vertex. y = x is the equation of the straight line passing through origin and making an angle of 45° with the x-axis (see Fig. 27.20).

A sketch of the region shows that the lower boundary is y = x and the upper boundary is $y^2 = 4x$. These two curves intersect at two points O and A. Solving these two equations, we get

$$\frac{y^2}{4} - y = 0$$

$$\Rightarrow \quad y(y-4) = 0$$

$$\Rightarrow \quad y = 0, 4$$

When y = 0, x = 0 and when y = 4, x = 4.

Here $f(x) = (4x)^{\frac{1}{2}}, g(x) = x, a = 0, b = 4$

Therefore, the required area is

$$= \int_0^4 \left(2x^{\frac{1}{2}} - x\right) dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^4$$

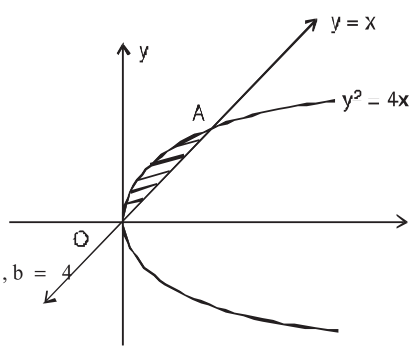


Fig. 27.20

$$= \frac{32}{3} - 8$$

$$= \frac{8}{3} \quad \text{square units}$$

Example 27.26 Find the area common to two parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

Solution : We know that $y^2 = 4ax$ and $x^2 = 4ay$ are the equations of the parabolas, which are symmetric about the x-axis and y-axis respectively.

Also both the parabolas have their vertices at the origin (see Fig. 27.19).

A sketch of the region shows that the lower boundary is $x^2 = 4ay$ and the upper boundary is $y^2 = 4ax$. These two curves intersect at two points O and A. Solving these two equations, we have

$$\frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow \quad x\left(x^3 - 64a^3\right) = 0$$

$$\Rightarrow \quad x = 0, 4a$$

Hence the two parabolas intersect at point (0, 0) and (4a, 4a).

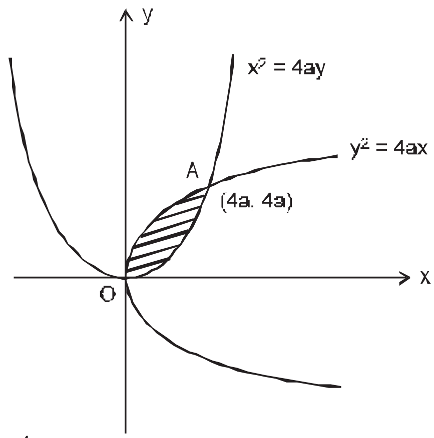


Fig. 27.21

Here $f(x) = \sqrt{4ax}, g(x) = \frac{x^2}{4a}, a = 0 \text{ and } b = 4a$

Therefore, required area

$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] dx$$

$$= \left[\frac{2.2\sqrt{a}\,x^{\frac{3}{2}}}{3} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16}{3}a^2 \text{ square units}$$

CHECK YOUR PROGRESS 27.4

1. Find the area of the circle $x^2 + y^2 = 9$

2. Find the area of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

3. Find the area of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

4. Find the area bounded by the curves $y^2 = 4ax$ and $y = \frac{x^2}{4a}$

5. Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$.

6. Find the area enclosed by the curves $y = x^2$ and $y = x + 2$

LET US SUM UP

- If f is continuous in [a, b] and F is an anti derivative of f in [a, b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- If f and g are continuous in [a, b] and c is a constant, then

(i) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

(ii) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(iii) $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

- The area bounded by the curve y = f (x), the x-axis and the ordinates

$$x = a, x = b \text{ is } \int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx$$

where $y = f(x)$ is a continuous single valued function and y does not change sign in the interval $a \le x \le b$

- If f (x) and g (x) are continuous functions on the interval [a, b] and $f(x) \geq g(x)$, for all $x \in [a,b]$, then the area of the region bounded above by y = f (x), below by y = g (x), on the left by x = a and on the right by x = b is

$$\int_a^b [f(x) - g(x)]dx$$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

Evaluate the following integrals (1 to 5) as the limit of sum.

1. $\int_a^b x\,dx$
2. $\int_a^b x^2\,dx$
3. $\int_a^b \sin x\,dx$
4. $\int_a^b \cos x\,dx$
5. $\int_0^2 \left(x^2+1\right)dx$

Evaluate the following integrals (6 to 25)

6. $\int_0^2 \sqrt{a^2 - x^2}\,dx$
7. $\int_0^{\frac{\pi}{2}} \sin 2x\,dx$
8. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\,dx$
9. $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$
10. $\int_0^1 \sin^{-1} x\,dx$
11. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$
12. $\int_3^4 \frac{1}{x^2-4}\,dx$
13. $\int_0^{\pi} \frac{1}{5+3\cos\theta}\,d\theta$
14. $\int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx$
15. $\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$
16. $\int_0^2 x\sqrt{x+2}\,dx$
17. $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\cos^5\theta\,d\theta$

18. $\int_0^{\pi} x \log \sin x \, dx$

19. $\int_0^{\pi} \log(1 + \cos x)\, dx$

20. $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

21. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$

22. $\int_0^{\frac{\pi}{4}} \log(1 + \tan x)\, dx$

23. $\int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x \, dx$

24. $\int_0^{2} x\left(x^2 + 1\right)^3 dx$

ANSWERS

CHECK YOUR PROGRESS 27.1

1. $\frac{35}{2}$ 2. $e - \frac{1}{e}$

3. (a) $\frac{\sqrt{2}-1}{\sqrt{2}}$ (b) 2 (c) $\frac{\pi}{4}$ (d) $\frac{64}{3}$

CHECK YOUR PROGRESS 27.2

1. $\frac{e-1}{2}$ 2. $\frac{2}{3}\tan^{-1}\frac{1}{3}$ 3. $\frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$

4. 29 5. $\frac{24\sqrt{2}}{15}$ 6. $\frac{\pi}{4}$ 7. $-\frac{\pi}{2}\log 2$

8. 0 9. 0 10. $\frac{1}{2}\left[\frac{\pi}{2} - \log 2\right]$

CHECK YOUR PROGRESS 27.3

1. $\frac{8}{3}$ sq. units 2. $\frac{27}{2}$ sq. units 3. $\frac{e^{2a}-1}{2}$ sq. units

4. $c^2\left(1 - \cos\frac{a}{c}\right)$

CHECK YOUR PROGRESS 27.4

1. 9π sq. units 2. 6π sq. units 3. 20π sq. units

4. $\frac{16}{3}a^2$ sq. units 5. $\frac{16}{3}$ sq. units 6. $\frac{9}{2}$ sq. units

TERMINAL EXERCISE

1. $\frac{b^2 - a^2}{2}$ 2. $\frac{b^3 - a^3}{3}$ 3. $\cos a - \cos b$

4. $\sin b - \sin a$ 5. $\frac{14}{3}$ 6. $\frac{\pi a^2}{4}$

7. 1	8. $\frac{1}{2}\log 2$	9. $\frac{\pi}{4}$
10. $\frac{\pi}{2}-1$	11. $\frac{\pi}{2}$	12. $\frac{1}{4}\log\frac{5}{3}$
13. $\frac{\pi}{4}$	14. $1-\log 2$	15. $\frac{2}{3}$
16. $\frac{16}{15}\left(2+\sqrt{2}\right)$	17. $\frac{64}{231}$	18. $-\frac{\pi^2}{2}\log 2$
19. $-\pi\log 2$	20. $\frac{\pi^2}{4}$	21. $\frac{1}{\sqrt{2}}\log\left(1+\sqrt{2}\right)$
22. $\frac{\pi}{8}\log 2$	23. $\frac{1}{96}$	24. 78

27.5 Reduction formulae

In this section, we derive some reduction formulae for the evaluation of definite integrals of $\sin^n x, \cos^n x, \sin^m x, \cos^n x$ between 0 and $\frac{\pi}{2}$ for positive integers m, n.

The following theorem gives a useful formula to evaluate the definite integral of $\sin^n x$ between 0 and $\frac{\pi}{2}$, when n is an integer $n \geq 2$.

27.5.1 Theorem

Let n be an integer greater than or equal to 2. Then

$$\int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \dfrac{n-1}{n} \dfrac{n-3}{n-2} \ldots\ldots \dfrac{1}{2} \dfrac{\pi}{2}, & \text{If } n \text{ is even} \\ \dfrac{n-1}{n} \dfrac{n-3}{n-2} \ldots\ldots \dfrac{2}{3}, & \text{If } n \text{ is odd} \end{cases}$$

Note: 1. $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$

Example 1. Find

(1) $\int_0^{\pi/2} \sin^4 x \, dx$

(ii) $\int_0^{\pi/2} \sin^7 x \, dx$

(iii) $\int_0^{\pi/2} \cos^8 x \, dx$

Sol: (i) $\int_0^{\pi/2} \sin^4 x\, dx = \frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2}$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{16}\pi$$

(ii) $\int_0^{\pi/2} \sin^7 x\, dx = \frac{7-1}{7} \cdot \frac{7-3}{7-2} \cdot \frac{7-5}{7-4}$

$$= \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{16}{35}.$$

(iii) $\int_0^{\pi/2} \cos^8 x\, dx = \frac{8-1}{8} \cdot \frac{8-3}{8-2} \cdot \frac{8-5}{8-4} \cdot \frac{8-7}{8-6} \cdot \frac{\pi}{2}$

$$= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{35}{256}\pi.$$

27.5.2 Theorem : Let m and n be positive integers. Then

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \begin{cases} \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}. \text{ If } 1 \neq n \text{ is odd} \\ \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdots\cdot \frac{1}{2} \cdot \frac{\pi}{2} \\ \qquad \text{If } n \text{ is even and } m \text{ is even} \\ \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdots\cdot \frac{2}{3} \\ \qquad \text{If } n \text{ is even and } 1 \neq m \text{ is odd.} \end{cases}$$

Example 2

Evaluate the following definite integrals

(i) $\int_0^{\pi/2} \sin^4 x \cos^5 x\, dx$

(ii) $\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx$

(iii) $\int_0^{\pi/2} \sin^6 x \cos^4 x\, dx$

Sol :

(i) $\int_0^{\pi/2} \sin^4 x \cos^5 x\, dx = \frac{5-1}{4+5} \cdot \frac{5-3}{4+5-2} \cdot \frac{1}{4+1}$

$$= \frac{4}{9} \cdot \frac{2}{7} \cdot \frac{1}{5} = \frac{8}{315}$$

(ii) $\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx = \frac{4-1}{5+4} \cdot \frac{4-3}{5+4-2} \cdot \frac{5-1}{5} \cdot \frac{5-3}{5-2}$

$$= \frac{3}{9} \cdot \frac{1}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{315}$$

(iii) $\int_0^{\pi/2} \sin^6 x \cos^4 x\, dx = \frac{4-1}{6+4} \cdot \frac{4-3}{6+4-2} \cdot \frac{6-1}{6} \cdot \frac{6-3}{6-2} \cdot \frac{6-5}{6-4} \cdot \frac{\pi}{2}$

$$= \frac{3}{10} \cdot \frac{1}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3}{512} \pi.$$

Example 3.

Find $\int_0^{2\pi} \sin^4 x \cos^6 x\, dx$

Sol: Let $f(x) = \sin^4 x \cos^6 x$

Since $f(2\pi - x) = f(\pi - x) = f(x)$, it follows from previous theorem

$$\int_0^{2\pi} f(x)\, dx = 2\int_0^{\pi} \sin^4 x \cos^6 x\, dx$$

$$= 4 \int_0^{\pi/2} \sin^4 x \cos^6 x\, dx$$

$$= 4 \cdot \frac{6-1}{4+6} \cdot \frac{6-3}{4+6-2} \cdot \frac{6-5}{4+6-4} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3}{128} \pi.$$

Example 4

Find $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^4 x\, dx$

Sol: Let $f(x) = \sin^2 x \cos^4 x$ since f is even we have

$$\int_{-\pi/2}^{\pi/2} f(x)\, dx = 2 \int_{0}^{\pi/2} f(x)\, dx$$

Hence $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^4 x\, dx = 2 \int_{0}^{\pi/2} \sin^2 x \cos^4 x\, dx$

$$= 2.\frac{4-1}{2+4} \cdot \frac{4-3}{2+4-2} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 2.\frac{3}{6} \cdot \frac{1}{4} \cdot \frac{\pi}{4} = \frac{\pi}{16}.$$

Check Your Progress

I. Find the values of the following integrals.

1. $\int_{0}^{\pi/2} \sin^{10} x\, dx$
2. $\int_{0}^{\pi/2} \sin^4 x \cos^4 x\, dx$
3. $\int_{0}^{\pi/2} \cos^7 x \sin^2 x\, dx$
4. $\int_{0}^{\pi/4} \sin^3 \theta\, d\theta$

Let Us Sum Up

1. If n is an integer $n \geq 2$, then

$$\int_{0}^{\pi/2} \sin^n x\, dx = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \cdots\cdots \frac{1}{2} \cdot \frac{\pi}{2}, & \text{If } n \text{ is even} \\ \frac{n-1}{n} \frac{n-3}{n-2} \cdots\cdots \frac{2}{3}. & \text{If } n \text{ is odd} \end{cases}$$

2. $\int_0^{\pi/2} \sin^3 x \, dx = \int_0^{\pi/2} \cos^n x \, dx$, n is a positive integer.

3. If m and n are positive intgers then

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1} \cdot \text{ If } 1 \neq n \text{ is odd}$$

$$\frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdots \frac{1}{2} \frac{\pi}{2}$$

If n is even and m is even

$$\frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdots \frac{2}{3}$$

If n is even and $1 \neq m$ is odd.

Supportive Websites

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

Answers

Check Your Progress

(1) $\frac{63}{512} \pi$

(2) $\frac{3}{256} \pi$

(3) $\frac{16}{315}$

(4) $\frac{2}{3} - \frac{5}{6\sqrt{2}}$

Numerical Integration

Introduction

Integral calculus arose from the ara problem. Before Newton, and Leibnitz developed a systematic method to evaluate a difinite integral bassed on the Fundamental theorem of calculus, problems of finding areas, volumes and lengths of curves were so difficult that only a genius could attempt to solve tham. But now, one can solve such challenging problems with greater ease. In this chapter we study the area bounded by the graph of a non-negative continuous function $y = f(x)$ defined on a closed interval $[a, b]$ and the x-axis as an application of the definite integral.

Objectives

From this chapter we can get to know the following points.

- The area bounded between the two curves
- lengths, volumes and areas of the curve
- When the integration becomes difficult we can find the approximate value with simpson or Trapezoidel rule.

27.6 Determination of plane areas involving second degree curves

In chapter 9, it was observed that if $y = f(x)$ is non-negative continuous function defined on $[a, b]$, then the area under the graph of f betwen the ordinates $x = a, x = b$ and the X-axis is given by the value of the definite integral $\int_a^b f(x)\,dx$. In this section we will give different possible ways of calculating such areas depending on the nature of the integrand.

27.6.1 Area under curves

(i) If $f : [a, b] \to [0, \infty)$ is continuous, then the area A of the region bounded by the curve $y = f(x)$, the X-axis and the lines $x = a$ and $x = b$ is given by $A = \int_a^b f(x)\,dx$ which is shown in fig 27.2 graphically.

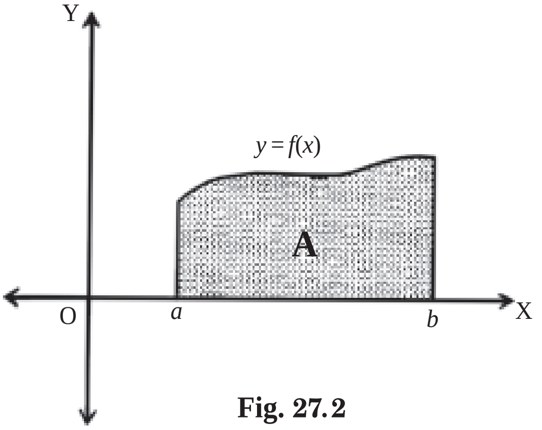


Fig. 27.2

(ii) Let $f : [a, b] \to (-\infty, 0]$ be continuous. Then the graphs of $y = f(x)$ and $y = -f(x)$ in $[a, b]$ are symmetric about the X-axis. So, the area bounded by the graph of $y = f(x)$, the X-axis and the lines $x = a, x = b$ is same as the area bounded by the graph of $y = -f(x)$, the X-axis and the lines $x = a, x = b$ which is, hence, given by $A = -\int_a^b f(x)\,dx$. This is shown graphically in the fig. 27.3.

From (i) and (ii) we observe that $A = \left| \int_a^b f(x)\,dx \right|$

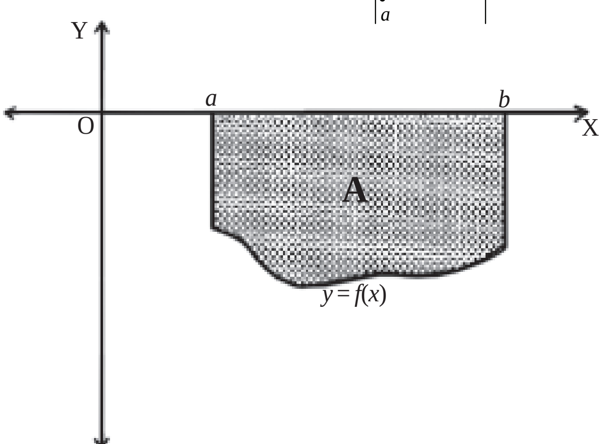


Fig. 27.3

(iii) Let $f:[a\,,b] \to R$ be continuous and $f(x) \geq 0$ for all $x \in [a\,,c]$ and $f(x) \leq 0$ for all $x \in [c, b]$ where $a < c < b$. Then the area of the region bounded by the curve $y = f(x)$, X-axis, and the lines. $x = a, x = b$ is given byb

$$A = \int_a^c f(x)\,dx + \int_c^b -f(x)\,dx$$

$$= \int_a^c f(x)\,dx - \int_c^b f(x)\,dx \qquad \ldots(1)$$

Which is shown in fig 27.4 graphically.

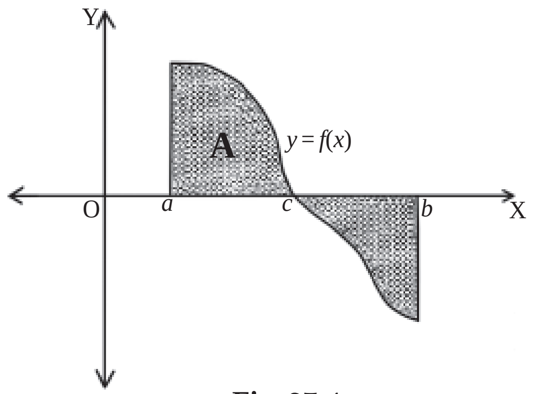


Fig. 27.4

Note: (1) can be written as $A = \left| \int_a^c f(x)\,dx \right| + \left| \int_c^b f(x)\,dx \right|$

(iv) Let $f:[a,b] \to \mathrm{R}$, $g:[a,b] \to \mathrm{R}$ be continuous and $f(x) \leq g(x)$ for all $x \in [a, b]$. Then the area of the region bounded by the curves $y = f(x), y = g(x)$]and the lines $x = a, x = b$ is given by

$$A = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$

In case $g(x) \leq f(x)$ for all $x \in [a, b]$ then the area A is given by

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$A = \left| \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right|$$

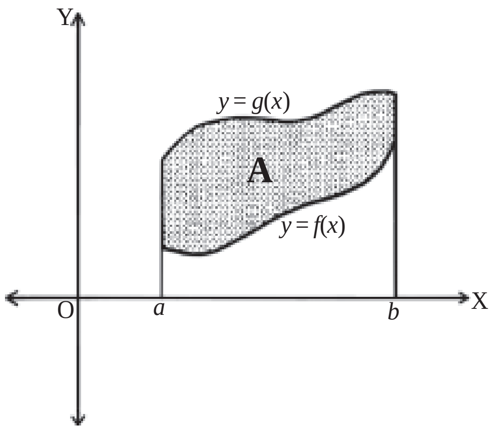


Fig. 27.5

Check Your Progress 1

Find the area of the region enclosed by the given curves.

I 1. $y = \cos x,\ y = 1 - \frac{2x}{\pi}$

2. $y = x^3 + 3,\ y = 0,\ x = -1,\ x = 2$

3. $x = 4 - y^2,\ x = 0$

II (1) $x^2 = 4y,\ x = 2,\ y = 0$

(2) $y^2 = 3x,\ x = 3$

(3) $y = x^2,\ y = 2x$

27.7 Trapezoidal rule and Simpson's rule - Simple applications

As already explained in the introduction, sometimes it is extremely difficult or even impossible to evaluate a definite integral $\int_a^b f(x)\,dx$ even if f is continuous on $[a, b]$. In such cases we take a set of numerical values of the integrand f in the interval $[a, b]$ and evaluate the definite integral $\int_a^b f(x)\,dx$ approximately. This process of finding the approximate value of a definite integral is called Numerical Integration. If the integrand is a function of single variable, then this process is known as quadrature.

There are several methods for approximating a definite integral. We discuss only two methods namely Trapezoidal rule and simpson's rule.

27.7.1 The Trapezoidal rule

Let $f : [a, b] \to [0, \infty)$ be continuous and $P = \{a = x_0, x_1, x_2 \ldots x_n = b\}$ be a partition of $[a, b]$ into n equal subintervals, each of length $h = \dfrac{b-a}{n}$ Let $y_0, y_1, y_2 \ldots y_n$ be the values of f at $x = x_0, x_1, x_2 \ldots x_n$ respectively (see fig. 27.6).

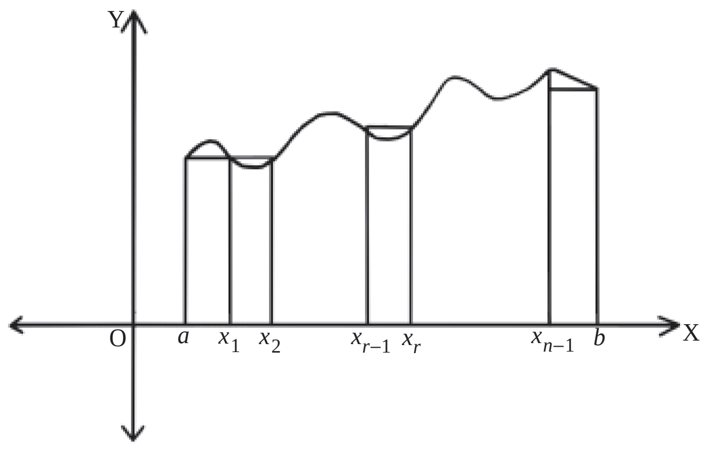


Fig. 27.6

$$\int_a^b f(x)\,dx \cong \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \ldots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

i.e., $\displaystyle\int_a^b f(x)\,dx \cong \frac{h}{2}$ [(Sum of the first and last ordinates) + 2(Sum of the remaining ordinates)]

27.7.2 Working rule for Trapezoidal rule

Step 1 : Note down a, b and n, the number of divisions

Step 2 : Calculate $h = \dfrac{b-a}{n}$

Step 3 : Compute the values of f at $x = x_0(= a), x = x_1, x = x_2 ..., x = x_n (= b)$ and denote them as $y_0, y_1, y_2 ... y_n.$ respectively.

Step 4 : Write down the Trapezoidal rule for the division specified in the question and substitute the values obtained in Step 3 in the rule (1) and sumplify.

Example 1

Find the approximate vale of $\int_1^9 x^2\,dx$ using trapezoidal rule with 4 equal intervals.

Sol : $n = 4, a = 1, b = 9$

$$h = \frac{9-1}{4} = 2$$

$$y = x^2$$

x	1	3	5	7	9
y	1	9	25	49	81

Required Area $= \frac{2}{2}[(1 + 81) + 2(9 + 25 + 49)]$

$= 82 + 166$

$= 248$

27.7.3 Simpson's rule of evaluation of $\int_a^b f(x)\,dx$ approximately

This is another rule for approximate integration results using parabolas instead of straight line segments to approximate a curve. As before, we divide $[a, b]$ into n subintervals of equal length $h = \frac{b-a}{n}$, but this time we assume that n is even. Here we approximate the curve $y = f(x) \geq 0$ passing through three successive points of subdivision by a parabola as shown in fig 27.7.

Let $y_r = f(x_r)$ for $r = 0, 1, 2, ... n$; and P_r be the point on the curve (x_r, y_r).

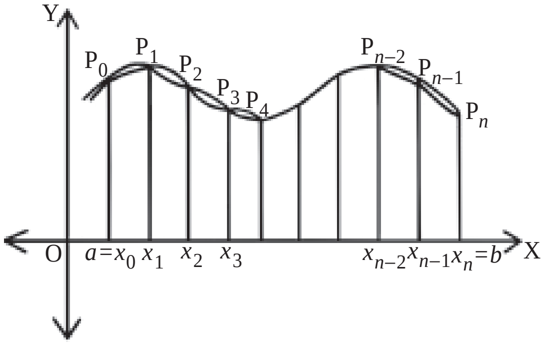


Fig. 27.7

For simplifying the calculations, we first consider the case where $x_0 = -h, x_1 = 0$ and $x_2 = h$ (see fig. 27.8). The equation of the parabola passing through P_0, P_1 and P_2 is of the form

$y = Ax^2 + Bx + C$ and therefore the area under the parabola from $x = -h$ to $x = h$

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = 2\int_{0}^{h} (Ax^2 + C)\,dx = \frac{h}{3}(2Ah^2 + 6C).$$

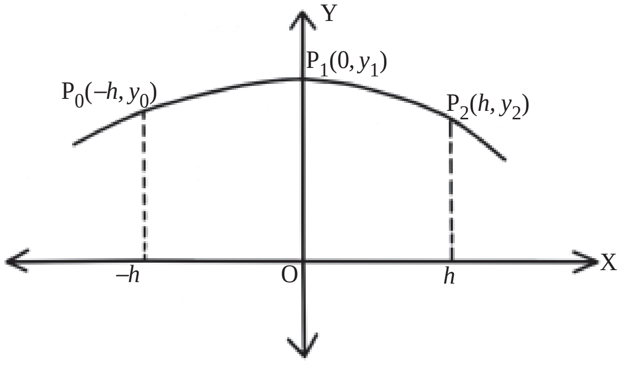


Fig. 27.8

But since the parabola passes through $P_0(-h, y_0), P_1(0, y_1)$ and $P_2(h, y_2)$ we have

$$y_0 = A\,h^2 - Bh + C,$$

$$y_1 = C$$

Hence $y_2 = Ah^2 + Bh + C$

Thus, the area under the parabola passing through P_0, P_1 and P_2 is $\frac{h}{3}(y_0 + 4y_1 + y_2)$.

Similarly, the area under the parabola through P_2, P_3 and P_4 from $x = x_2$ to $x = x_4$ is $\frac{h}{3}(y_2 + 4y_3 + y_4)$. If we compute the areas under all the parabolas in the above manner and add the results, we obtain

$$\int_a^b f(x)\,dx \cong \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_1 + 4y_3 + y_4) + ...$$

$$+ (y_{n-2} + 4y_{n-1} + y_n)] \qquad \ldots(2)$$

i.e.,

$$\int_a^b f(x)\,dx \cong \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + y_6 ... + y_{n-2})]$$

$$= \frac{h}{3}\text{ [(Sum of the first and last ordinates) + 4(Sum of the even ordinates)}$$

$$\text{+ 2(Sum of all other odd ordinates execluding first and last)]}$$

Although we have derived this approximation for the case $f(x) \geq 0$, it is a reasonable approximation for any continuous function f and is called Simpson's Rule or Simpson's one-third rule after the English mathematician Thomas Simpson.

Example 2

Find the approximate value of π from $\int_0^1 \frac{1}{1+x^2}dx$ using sumpson's rule by dividing [0, 1] in to 4 equal parts.

Sol: $n = 4$, $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

$$y = \frac{1}{1+x^2}$$

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y	1	$\frac{16}{17}$	$\frac{4}{5}$	$\frac{16}{25}$	$\frac{1}{2}$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_{2n}) + 4(y_1 + y_3 + ... + y_{2n-1}) + 2(y_2 + y_4 + ... + y_{2n-2})]$$

$$= \frac{1}{12}\left[\left(1+\frac{1}{2}\right) + 4\left(\frac{16}{17}+\frac{16}{25}\right) + 2\left(\frac{4}{5}\right)\right]$$

$$= \frac{1}{12}\left[1.5 + 4(0.9412 + 0.64) + 1.6\right]$$

$$= \frac{1}{12}[3.1 + 6.3248]$$

$$= \frac{9.4248}{12}$$

$$= 0.7854$$

But $\int_0^1 \frac{dx}{1+x^2} = \left[\mathrm{Tan}^{-1} x\right]_0^1 = \mathrm{Tan}^{-1} 1 - \mathrm{Tan}^{-1} 0 = \frac{\pi}{4}$

$\therefore \frac{\pi}{4} = 0.7854 \Rightarrow \pi = 3.1416$

Check Your Progress 2

I. 1. Calculate the approximate value of $\int_0^1 (1+ x^2)\, dx$ by dividing $[0, 1]$ into 5 equal parts using Trapezoidal rule.

2. Calculate the approximate value of $\int_1^3 \frac{1}{x}\, dx$, by suing simpson's rule and taking $n = 4$.

3. Calculate the approximate value of $\int_1^5 \frac{dx}{1+x}\, dx$ by taking $n = 4$ in the simpson's rule.

II. 1. Calculate the approximate value of $\int_{-3}^{3} x^4 \ dx$ using

(i) Trapezoidal rule (ii) Simpson's rule by dividing [–3, 3] fin to 6 equal parts.

2. Use simson's rule to evaluate $\int_{1}^{7} \frac{1}{x} \ dx$ approximately by taking $n = 6$ and hence find the approximate value of $\log_e^7$ to three places of decimals.

3. Calculate the approximate value of $\int_{0}^{6} \sqrt{1+x^2} \ dx$, by taking $n = 6$ in sumpson's rule.

4. Evaluate $\int_{0}^{1} \sqrt{1+x^3} \ dx$ approximately by dividing [0, 1] into 10 subintervals of the same length by (i) Trapezoidal rule (ii) Simposon's rule.

Let Us Sum Up

1. Let $f(x) \geq 0$ for all $x \in [a, b]$ and

$p = \{a = x_0, x_1, x_2, ...x_r, ... x_n = b]$ be a partition of $[a, b]$ into n equal subintervals, each of length $h = \dfrac{b-a}{n}$. If $f(x_r) = y_r$, for $0 \leq r \leq n$ then

(i) Trapezoidal rule

$$\int_a^b f(x)\,dx \cong \frac{h}{3} \ [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2\ (y_2 + y_4 + y_6 + ... + y_{n-2})]$$

2. Simposon's one third rule (for even n)

$$\int_a^b f(x)\,dx \cong \frac{h}{3} \ [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2\ (y_2 + y_4 + y_6 + ... + y_{n-2})]$$

3. $$\int_a^b f(x)\,dx \cong \frac{h}{3}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})$$

Supportive Websites

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

Answers

Check Your Progress 1

I. 1. $2 - \frac{\pi}{2}$ 2. $\frac{51}{4}$

3. $\frac{32}{3}$

II. (1) $\frac{2}{3}$

(2) 1 2

(3) $\frac{4}{3}$

Check Your Progress 2

I. (1) 1.34

(2) 1.1

(3) 1.1

(4) 0.695635

I. (1) 115

(2) 1.959

(3) 19.5011

(4) (i) 1.1122

(ii) 1.11132

28

DIFFERENTIAL EQUATIONS

Having studied the concept of differentiation and integration, we are now faced with the question where do they find an application.

In fact these are the tools which help us to determine the exact takeoff speed, angle of launch, amount of thrust to be provided and other related technicalities in space launches.

Not only this but also in some problems in Physics and Bio-Sciences, we come across relations which involve derivatives.

One such relation could be $\frac{ds}{dt} = 4.9\ t^2$ where s is distance and t is time. Therefore, $\frac{ds}{dt}$ represents velocity (rate of change of distance) at time t.

Equations which involve derivatives as their terms are called differential equations. In this lesson, we are going to learn how to find the solutions and applications of such equations.

OBJECTIVES

After studying this lesson, you will be able to :

- define a differential equation, its order and degree;
- determine the order and degree of a differential equation;
- form differential equation from a given situation;
- illustrate the terms "general solution" and "particular solution" of a differential equation through examples;
- solve differential equations of the following types :

  (i) $\frac{dy}{dx} = f(x)$ (ii) $\frac{dy}{dx} = f(x)g(y)$

  (iii) $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (iv) $\frac{dy}{dx} + P(x)y = Q(x)$ (v) $\frac{d^2y}{dx^2} = f(x)$

- find the particular solution of a given differential equation for given conditions.

EXPECTED BACKGROUND KNOWLEDGE

- Integration of algebraic functions, rational functions and trigonometric functions

28.1 DIFFERENTIAL EQUATIONS

As stated in the introduction, many important problems in Physics, Biology and Social Sciences, when formulated in mathematical terms, lead to equations that involve derivatives. Equations which involve one or more differential coefficients such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ (or differentials) etc. and independent and dependent variables are called differential equations.

For example,

(i) $\frac{dy}{dx} = \cos x$ (ii) $\frac{d^2y}{dx^2} + y = 0$ (iii) $xdx + ydy = 0$

(iv) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$ (vi) $y = \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

28.2 ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

Order : It is the order of the highest derivative occurring in the differential equation.

Degree : It is the degree of the highest order derivative in the differential equation after the equation is free from negative and fractional powers of the derivatives. For example,

	Differential Equation	**Order**	**Degree**
(i)	$\frac{dy}{dx} = \sin x$	One	One
(ii)	$\left(\frac{dy}{dx}\right)^2 + 3y^2 = 5x$	One	Two
(iii)	$\left(\frac{d^2s}{dt^2}\right)^2 + t^2\left(\frac{ds}{dt}\right)^4 = 0$	Two	Two
(iv)	$\frac{d^3v}{dr^3} + \frac{2}{r}\frac{dv}{dr} = 0$	Three	One
(v)	$\left(\frac{d^4y}{dx^4}\right)^2 + x^3\left(\frac{d^3y}{dx^3}\right)^5 = \sin x$	Four	Two

Example 28.1 Find the order and degree of the differential equation :

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 0$$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = -\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$$

The term $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$ has fractional index. Therefore, we first square both sides to remove fractional index.

Squaring both sides, we have

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

Hence each of the order of the diferential equation is 2 and the degree of the differential equation is also 2.

Note : Before finding the degree of a differential equation, it should be free from radicals and fractions as far as derivatives are concerned.

28.3 LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation in which the dependent variable and all of its derivatives occur only in the first degree and are not multiplied together is called a **linear differential equation**. A differential equation which is not linear is called non-linear differential equation . For example, the differential equations

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{and} \quad \cos^2 x \frac{d^3y}{dx^3} + x^3 \frac{dy}{dx} + y = 0 \text{ are linear.}$$

The differential equation $\left(\frac{dy}{dx}\right)^2 + \frac{y}{x} = \log x$ is non-linear as degree of $\frac{dy}{dx}$ is two.

Further the differential equation $y \frac{d^2y}{dx^2} - 4 = x$ is non-linear because the dependent variable y and its derivative $\frac{d^2y}{dx^2}$ are multiplied together.

28.4 FORMATION OF A DIFFERENTIAL EQUATION

Consider the family of all straight lines passing through the origin (see Fig. 28.1).

This family of lines can be represented by

$$y = mx \qquad .....(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = m \qquad .....(2)$$

From (1) and (2), we get

$$y = x\frac{dy}{dx} \qquad .....(3)$$

So $y = mx$ and $y = x\frac{dy}{dx}$ represent the same family.

Clearly equation (3) is a differential equation.

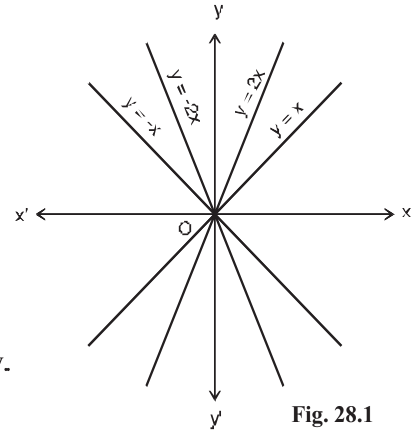


Fig. 28.1

Working Rule : To form the differential equation corresponding to an equation involving two variables, say x and y and some arbitrary constants, say, a, b, c, etc.

(i) Differentiate the equation as many times as the number of arbitrary constants in the equation.

(ii) Eliminate the arbitrary constants from these equations.

Remark

If an equation contains n arbitrary constants then we will obtain a differential equation of n^{th} order.

Example 28.2 Form the differential equation representing the family of curves.

$$y = ax^2 + bx. \qquad ......(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = 2ax + b \qquad ......(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = 2a \qquad ......(3)$$

$$\Rightarrow \qquad a = \frac{1}{2}\frac{d^2y}{dx^2} \qquad ......(4)$$

(The equation (1) contains two arbitrary constants. Therefore, we differentiate this equation two times and eliminate 'a' and 'b').

On putting the value of 'a' in equation (2), we get

$$\frac{dy}{dx} = x\frac{d^2y}{dx^2} + b$$

$$\Rightarrow \qquad b = \frac{dy}{dx} - x\frac{d^2y}{dx^2} \qquad ......(5)$$

Substituting the values of 'a' and 'b' given in (4) and (5) above in equation (1), we get

$$y = x^2\left(\frac{1}{2}\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)$$

or
$$y = \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}$$

or
$$y = x\frac{dy}{dx} - \frac{x^2}{2}\frac{d^2y}{dx^2}$$

or
$$\frac{x^2}{2}\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

which is the required differential equation.

Example 28.3 Form the differential equation representing the family of curves

$$y = a\cos(x + b).$$

Solution : $y = a\cos(x + b)$ .....(1)

Differentiating both sides, we get

$$\frac{dy}{dx} = -a\sin(x + b) \qquad ....(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -a\cos(x + b) \qquad .....(3)$$

From (1) and (3), we get

$$\frac{d^2y}{dx^2} = -y \quad \text{or} \quad \frac{d^2y}{dx^2} + y = 0$$

which is the required differential equation.

Example 28.4 Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.

Solution : As the centre lies on the x-axis, its coordinates will be (a, 0).

Since each circle passes through the origin, its radius is a.

Then the equation of any circle will be

$$(x - a)^2 + y^2 = a^2 \qquad (1)$$

To find the corresponding differential equation, we differentiate equation (1) and get

$$2(x - a) + 2y\frac{dy}{dx} = 0$$

or
$$x - a + y\frac{dy}{dx} = 0$$

or
$$a = y\frac{dy}{dx} + x$$

Substituting the value of 'a' in equation (1), we get

$$\left(x - y\frac{dy}{dx} - x\right)^2 + y^2 = \left(y\frac{dy}{dx} + x\right)^2$$

or
$$\left(y\frac{dy}{dx}\right)^2 + y^2 = x^2 + \left(y\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

or
$$y^2 = x^2 + 2xy\frac{dy}{dx}$$

which is the required differential equation.

Remark

If an equation contains one arbitrary constant then the corresponding differential equation is of the first order and if an equation contains two arbitrary constants then the corresponding differential equation is of the second order and so on.

Example 28.5 Assuming that a spherical rain drop evaporates at a rate proportional to its surface area, form a differential equation involving the rate of change of the radius of the rain drop.

Solution : Let r(t) denote the radius (in mm) of the rain drop after t minutes. Since the radius is decreasing as t increases, the rate of change of r must be negative.

If V denotes the volume of the rain drop and S its surface area, we have

$$V = \frac{4}{3}\pi r^3 \qquad .....(1)$$

and
$$S = 4\pi r^2 \qquad .....(2)$$

It is also given that

$$\frac{dV}{dt} \propto S$$

or
$$\frac{dV}{dt} = -kS$$

or
$$\frac{dV}{dr}.\frac{dr}{dt} = -kS \qquad .....(3)$$

Using (1), (2) and (3) we have

$$4\pi r^2.\frac{dr}{dt} = -4k\pi r^2$$

or $$\frac{dr}{dt} = k$$

which is the required differential equation.

CHECK YOUR PROGRESS 28.1

1. Find the order and degree of the differential equation

$$y = x\frac{dy}{dx} + \frac{1}{\frac{dy}{dx}}$$

2. Write the order and degree of each of the following differential equations.

(a) $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$ (b) $y = 2x\frac{dy}{dx} + x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

(c) $\sqrt{1 - x^2}dx + \sqrt{1 - y^2}dy = 0$ (d) $\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 4 = 0$

3. State whether the following differential equations are linear or non-linear.

(a) $\left(xy^2 - x\right)dx + \left(y - x^2y\right)dy = 0$ (b) $dx + dy = 0$

(c) $\frac{dy}{dx} = \cos x$ (d) $\frac{dy}{dx} + \sin^2 y = 0$

4. Form the differential equation corresponding to

$(x - a)^2 + (y - b)^2 = r^2$ by eliminating 'a' and 'b'.

5. (a) Form the differential equation corresponding to

$y^2 = m\left(a^2 - x^2\right)$

(b) Form the differential equation corresponding to

$y^2 - 2ay + x^2 = a^2$, where a is an arbitrary constant.

(c) Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-3x}$ where A and B are arbitrary constants.

(d) Find the differential equation of all straight lines passing through the point (3,2).

(e) Find the differential equation of all the circles which pass through origin and whose centres lie on y-axis.

28.5 GENERAL AND PARTICULAR SOLUTIONS

Finding solution of a differential equation is a reverse process. Here we try to find an equation which gives rise to the given differential equation through the process of differentiations and elimination of constants. The equation so found is called the primitive or the solution of the differential equation.

Notes

Remarks

(1) If we differentiate the primitive, we get the differential equation and if we integrate the differential equation, we get the primitive.

(2) Solution of a differential equation is one which satisfies the differential equation.

Example 28.6 Show that $y = C_1 \sin x + C_2 \cos x$, where C_1 and C_2 are arbitrary constants, is a solution of the differential equation :

$$\frac{d^2y}{dx^2} + y = 0$$

Solution : We are given that

$$y = C_1 \sin x + C_2 \cos x \qquad .....(1)$$

Differentiating both sides of (1), we get

$$\frac{dy}{dx} = C_1 \cos x - C_2 \sin x \qquad .....(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -C_1 \sin x - C_2 \cos x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we get

$$\frac{d^2y}{dx^2} + y = C_1 \sin x + C_2 \cos x + (-C_1 \sin x - C_2 \cos x)$$

or $$\frac{d^2y}{dx^2} + y = 0$$

In integration, the arbitrary constants play important role. For different values of the constants we get the different solutions of the differential equation.

A solution which contains as many as arbitrary constants as the order of the differential equation is called the **General Solution** or complete primitive.

If we give the particular values to the arbitrary constants in the general solution of differential equation, the resulting solution is called a **Particular Solution**.

Remark

General Solution contains as many arbitrary constants as is the order of the differential equation.

Example 28.7 Show that $y = cx + \frac{a}{c}$ (where c is a constant) is a solution of the differential equation.

$$y = x\frac{dy}{dx} + a\frac{dx}{dy}$$

Solution : We have $y = cx + \frac{a}{c}$ .....(1)

Differentiating (1), we get

$$\frac{dy}{dx} = c \qquad \Rightarrow \qquad \frac{dx}{dy} = \frac{1}{c}$$

On substituting the values of $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in R.H.S of the differential equation, we have

$$x(c) + a\left(\frac{1}{c}\right) = cx + \frac{a}{c} = y$$

$\Rightarrow$ R.H.S. = L.H.S.

Hence $y = cx + \frac{a}{c}$ is a solution of the given differential equation.

Example 28.8 If $y = 3x^2 + C$ is the general solution of the differential equation $\frac{dy}{dx} - 6x = 0$, then find the particular solution when y = 3, x = 2.

Solution : The general solution of the given differential equation is given as

$$y = 3x^2 + C \qquad ....(1)$$

Now on substituting y = 3, x = 2 in the above equation , we get

$$3 = 12 + C \qquad \text{or} \qquad C = -9$$

By substituting the value of C in the general solution (1), we get

$$y = 3x^2 - 9$$

which is the required particular solution.

28.6 TECHNIQUES OF SOLVING A DIFFERENTIAL EQUATION

28.6.1 When Variables are Separable

(i) Differential equation of the type $\frac{dy}{dx} = f(x)$

Consider the differential equation of the type $\frac{dy}{dx} = f(x)$

or $\qquad dy = f(x)\,dx$

On integrating both sides, we get

$$\int dy = \int f(x)\,dx$$

$$y = \int f(x)\,dx + c$$

where c is an arbitrary constant. This is the general solution.

Note : It is necessary to write c in the general solution, otherwise it will become a particular solution.

Example 28.9 Solve

$$(x+2)\frac{dy}{dx} = x^2 + 4x - 5$$

Solution : The given differential equation is $(x+2)\frac{dy}{dx} = x^2 + 4x - 5$

or $\quad \dfrac{dy}{dx} = \dfrac{x^2 + 4x - 5}{x + 2} \quad$ or $\quad \dfrac{dy}{dx} = \dfrac{x^2 + 4x + 4 - 4 - 5}{x + 2}$

or $\quad \dfrac{dy}{dx} = \dfrac{(x+2)^2}{x+2} - \dfrac{9}{x+2} \quad$ or $\quad \dfrac{dy}{dx} = x + 2 - \dfrac{9}{x+2}$

or $\quad dy = \left(x + 2 - \dfrac{9}{x+2}\right)dx \qquad$ .....(1)

On integrating both sides of (1), we have

$$\int dy = \int\left(x + 2 - \frac{9}{x+2}\right)dx \quad \text{or} \quad y = \frac{x^2}{2} + 2x - 9\log|x+2| + c,$$

where c is an arbitrary constant, is the required general solution.

Example 28.10 Solve

$$\frac{dy}{dx} = 2x^3 - x$$

given that y = 1 when x = 0

Solution : The given differential equation is $\dfrac{dy}{dx} = 2x^3 - x$

or $\quad dy = (2x^3 - x)\,dx \qquad$ ...(1)

On integrating both sides of (1), we get

$$\int dy = \int (2x^3 - x)\,dx \quad \text{or} \quad y = 2.\frac{x^4}{4} - \frac{x^2}{2} + C$$

or $\quad y = \dfrac{x^4}{2} - \dfrac{x^2}{2} + C \qquad$ .....(2)

where C is an arbitrary constant.

Since y = 1 when x = 0, therefore, if we substitute these values in (2) we will get

$$1 = 0 - 0 + C \quad \Rightarrow \quad C = 1$$

Now, on putting the value of C in (2), we get

$$y = \frac{1}{2}\left(x^4 - x^2\right) + 1 \quad \text{or} \qquad y = \frac{1}{2}x^2\left(x^2 - 1\right) + 1$$

which is the required particular solution.

(ii) Differential equations of the type $\frac{dy}{dx} = f(x) \cdot g(y)$

Consider the differential equation of the type

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

or $$\frac{dy}{g(y)} = f(x)\,dx \qquad .....(1)$$

In equation (1), x's and y's have been separated from one another. Therefore, this equation is also known differential equation with variables separable.

To solve such differential equations, we integrate both sides and add an arbitrary constant on one side.

To illustrate this method, let us take few examples.

Example 28.11 Solve

$$\left(1 + x^2\right)dy = \left(1 + y^2\right)dx$$

Solution : The given differential equation

$$\left(1 + x^2\right)dy = \left(1 + y^2\right)dx$$

can be written as $$\frac{dy}{1 + y^2} = \frac{dx}{1 + x^2}$$ (Here variables have been seperated)

On integrating both sides of (1), we get

$$\int \frac{dy}{1 + y^2} = \int \frac{dx}{1 + x^2}$$

or $$\tan^{-1} y = \tan^{-1} x + C$$

where C is an arbitrary constant.

This is the required solution.

Example 28.12 Solve

$$\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$$

Solution : The given differential equation

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$$\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$$

can be written as $$x^2(1-y)\frac{dy}{dx} + y^2(1+x) = 0$$

or $$\frac{(1-y)}{y^2}dy = \frac{-(1+x)}{x^2}dx$$

(Variables separated) ....(1)

If we integrate both sides of (1), we get

$$\int\left(\frac{1}{y^2} - \frac{1}{y}\right)dy = \int\left(-\frac{1}{x^2} - \frac{1}{x}\right)dx$$

where C is an arbitrary constant.

or $$-\frac{1}{y} - \log|y| = \frac{1}{x} - \log|x| + C$$

or $$\log\left|\frac{x}{y}\right| = \frac{1}{x} + \frac{1}{y} + C$$

Which is the required general solution.

Example 28.13 Find the particular solution of

$$\frac{dy}{dx} = \frac{2x}{3y^2+1}$$

when y (0) = 3 (i.e. when x = 0, y = 3).

Solution : The given differential equation is

$$\frac{dy}{dx} = \frac{2x}{3y^2+1} \text{ or } \left(3y^2+1\right)dy = 2x\,dx \text{ (Variables separated)} \quad ....(1)$$

If we integrate both sides of (1), we get

$$\int\left(3y^2+1\right)dy = \int 2x\,dx,$$

where C is an arbitrary constant.

$$y^3 + y = x^2 + C \quad ....(2)$$

It is given that, $y(0) = 3$.

$\therefore$ on substituting y = 3 and x = 0 in (2), we get

$$27 + 3 = C$$

$\therefore$ $$C = 30$$

Thus, the required particular solution is

$$y^3 + y = x^2 + 30$$

28.6.2 Homogeneous Differential Equations

Consider the following differential equations :

(i) $y^2 + x^2 \dfrac{dy}{dx} = xy\dfrac{dy}{dx}$ (ii) $\left(x^3 + y^3\right)dx - 3xy^2dy = 0$

(iii) $\dfrac{dy}{dx} = \dfrac{x^3 + xy^2}{y^2x}$

In equation (i) above, we see that each term except $\dfrac{dy}{dx}$ is of degree 2

[as degree of y^2 is 2, degree of x^2 is 2 and degree of xy is 1 + 1 =2]

In equation (ii) each term except $\dfrac{dy}{dx}$ is of degree 3.

In equation (iii) each term except $\dfrac{dy}{dx}$ is of degree 3, as it can be rewritten as

$$y^2x\frac{dy}{dx} = x^3 + x\,y^2$$

Such equations are called **homogeneous equations**.

Remarks

Homogeneous equations do not have constant terms.

For example, differential equation

$$\left(x^2 + 3yx\right)dx - \left(x^3 + x\right)dy = 0$$

is not a homogeneous equation as the degree of the function except $\dfrac{dy}{dx}$ in each term is not the same. [degree of x^2 is 2, that of 3yx is 2, of x^3 is 3, and of x is 1]

28.6.3 Solution of Homogeneous Differential Equation :

To solve such equations, we proceed in the following manner :

(i) write one variable = v. (the other variable).
 (i.e. either y = vx or x = vy)

(ii) reduce the equation to separable form

(iii) solve the equation as we had done earlier.

Example 28.14 Solve

$$\left(x^2 + 3xy + y^2\right)dx - x^2dy = 0$$

Solution : The given differential equation is

$$\left(x^2 + 3xy + y^2\right)dx - x^2dy = 0$$

or $$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2} \quad .....(1)$$

It is a homogeneous equation of degree two. (Why?)

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ From (1), we have

$$v + x\frac{dv}{dx} = \frac{x^2 + 3x.vx + (vx)^2}{x^2} \quad \text{or} \quad v + x\frac{dv}{dx} = x^2\left[\frac{1 + 3v + v^2}{x^2}\right]$$

or $$v + x\frac{dv}{dx} = 1 + 3v + v^2 \quad \text{or} \quad x\frac{dv}{dx} = 1 + 3v + v^2 - v$$

or $$x\frac{dv}{dx} = v^2 + 2v + 1 \quad \text{or} \quad \frac{dv}{v^2 + 2v + 1} = \frac{dx}{x}$$

or $$\frac{dv}{(v+1)^2} = \frac{dx}{x} \quad .....(2)$$

Further on integrating both sides of (2), we get

$$\frac{-1}{v+1} + C = \log|x|,$$ where C is an arbitrary constant.

On substituting the value of v, we get

$$\frac{x}{y+x} + \log|x| = C$$ which is the required solution.

28.6.4 Differential Equation

Consider the equation

$$\frac{dy}{dx} + Py = Q \quad .....(1)$$

where P and Q are functions of x. This is linear equation of order one.

To solve equation (1), we first multiply both sides of equation (1) by $e^{\int P dx}$ (called integrating factor) and get

$$e^{\int Pdx}\frac{dy}{dx} + Pye^{\int Pdx} = Qe^{\int Pdx}$$

or $$\frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx} \quad .....(2)$$

$$\left[\because \frac{d}{dx}\left(ye^{\int Pdx}\right) = e^{\int Pdx}\frac{dy}{dx} + Py.e^{\int Pdx}\right]$$

On integrating, we get

$$ye^{\int P dx} = \int Q e^{\int P dx} dx + C \qquad .....(3)$$

where C is an arbitrary constant,

or $$y = e^{-\int P dx}\left[\int Q e^{\int P dx} dx + C\right]$$

Note : $e^{\int P dx}$ is called the integrating factor of the equation and is written as I.F in short.

Remarks

(i) We observe that the left hand side of the linear differential equation (1) has become $\frac{d}{dx}\left(ye^{\int P dx}\right)$ after the equation has been multiplied by the factor $e^{\int P dx}$.

(ii) The solution of the linear differential equation

$$\frac{dy}{dx} + Py = Q$$

P and Q being functions of x only is given by

$$ye^{\int P dx} = \int Q\left(e^{\int P dx}\right) dx + C$$

(iii) The coefficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing the equation by it throughout.

(iv) Some differential equations become linear differential equations if y is treated as the independent variable and x is treated as the dependent variable.

For example, $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y only, is also a linear differential equation of the first order.

In this case $\qquad$ $\text{I.F.} = e^{\int P dy}$

and the solution is given by

$$x\,(\text{I.F.}) = \int Q.(\text{I.F.})dy + C$$

Example 28.15 Solve

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x}$$

Solution : Here $P = \frac{1}{x}, Q = e^{-x}$ (Note that both P an Q are functions of x)

I.F. (Integrating Factor) $e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x \qquad (x > 0)$

On multiplying both sides of the equation by I.F., we get

$$x \cdot \frac{dy}{dx} + y = x \cdot e^{-x} \quad \text{or} \quad \frac{d}{dx}(y \cdot x) = xe^{-x}$$

Integrating both sides, we have

$$yx = \int xe^{-x}dx + C$$

where C is an arbitrary constant

or $\quad xy = -xe^{-x} + \int e^{-x}dx + C$

or $\quad xy = -xe^{-x} - e^{-x} + C$

or $\quad xy = -e^{-x}(x+1) + C$

or $\quad y = -\left(\frac{x+1}{x}\right)e^{-x} + \frac{C}{x}$

Note: In the solution $x > 0$.

Example 28.16 Solve :

$$\sin x \frac{dy}{dx} + y\cos x = 2\sin^2 x \cos x$$

Solution : The given differential equation is

$$\sin x \frac{dy}{dx} + y\cos x = 2\sin^2 x \cos x$$

or $$\frac{dy}{dx} + y\cot x = 2\sin x\cos x \qquad .....(1)$$

Here $\quad P = \cot x, Q = 2\sin x\cos x$

$$\text{I.F.} = e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$$

On multiplying both sides of equation (1) by I.F., we get ($\sin x > 0$)

$$\frac{d}{dx}(y\sin x) = 2\sin^2 x\cos x$$

Further on integrating both sides, we have

$$y\sin x = \int 2\sin^2 x \cos x\, dx + C$$

where C is an arbitrary constant ($\sin x > 0$)

or $\quad y\sin x = \frac{2}{3}\sin^3 x + C,\quad$ which is the required solution.

Example 28.17 Solve $\quad \left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$

Solution : The given differential equation is

$$\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$$

or $$\frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$$

or $$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \qquad \text{....(1)}$$

which is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are the functions of y only.

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Multiplying both sides of equation (1) by I.F., we get

$$\frac{d}{dy}\left(x e^{\tan^{-1} y}\right) = \frac{\tan^{-1} y}{1+y^2}\left(e^{\tan^{-1} y}\right)$$

On integrating both sides, we get

or $$\left(e^{\tan^{-1} y}\right) x = \int e^t . t dt + C,$$

where C is an arbitrary constant and $t = \tan^{-1} y$ and $dt = \frac{1}{1+y^2} dy$

or $$\left(e^{\tan^{-1} y}\right) x = te^t - \int e^t + C$$

or $$\left(e^{\tan^{-1} y}\right) x = te^t - e^t + C$$

or $$\left(e^{\tan^{-1} y}\right) x = \tan^{-1} y \, e^{\tan^{-1} y} - e^{\tan^{-1} y} + C \qquad \text{(on putting } t = \tan^{-1} y\text{)}$$

or $$x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$$

CHECK YOUR PROGRESS 28.2

1. (i) Is $y = \sin x$ a solution of $\frac{d^2y}{dx^2} + y = 0$?

   (ii) Is $y = x^3$ a solution of $x\frac{dy}{dx} - 4y = 0$?

2. Given below are some solutions of the differential equation $\frac{dy}{dx} = 3x$.

   State which are particular solutions and which are general solutions.

(i) $2y = 3x^2$ (ii) $y = \frac{3}{2}x^2 + 2$

(iii) $2y = 3x^2 + C$ (iv) $y = \frac{3}{2}x^2 + 3$

3. State whether the following differential equations are homogeneous or not ?

(i) $\frac{dy}{dx} = \frac{x^2}{1+y^2}$ (ii) $\left(3xy + y^2\right)dx + \left(x^2 + xy\right)dy = 0$

(iii) $(x+2)\frac{dy}{dx} = x^2 + 4x - 9$ (iv) $\left(x^3 - yx^2\right)dy + \left(y^3 + x^3\right)dx = 0$

4. (a) Show that y = a sin 2x is a solution of $\frac{d^2y}{dx^2} + 4y = 0$

(b) Verify that $y = x^3 + ax^2 + c$ is a solution of $\frac{d^3y}{dx^3} = 6$

5. The general solution of the differential equation

$\frac{dy}{dx} = \sec^2 x$ is $y = \tan x + C$.

Find the particular solution when

(a) $x = \frac{\pi}{4}, y = 1$ (b) $x = \frac{2\pi}{3}, y = 0$

6. Solve the following differential equations :

(a) $\frac{dy}{dx} = x^5 \tan^{-1}\left(x^3\right)$ (b) $\frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x$

(c) $\left(1+x^2\right)\frac{dy}{dx} = x$ (d) $\frac{dy}{dx} = x^2 + \sin 3x$

7. Find the particular solution of the equation $e^x \frac{dy}{dx} = 4$, given that y = 3 , when x = 0

8. Solve the following differential equations :

(a) $\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$ (b) $\frac{dy}{dx} = xy + x + y + 1$

(c) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$ (d) $\frac{dy}{dx} = e^{x-y} + e^{-y}x^2$

9. Find the particular solution of the differential equation

$\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that y = 0 when x = 0

10. Solve the following differential equations :

(a) $\left(x^2 + y^2\right)dx - 2xy\,dy = 0$ (b) $x\frac{dy}{dx} + \frac{y^2}{x} = y$

(c) $\dfrac{dy}{dx} = \dfrac{\sqrt{x^2 - y^2} + y}{x}$ (d) $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\left(\dfrac{y}{x}\right)$

11. Solve : $\dfrac{dy}{dx} + y\sec x = \tan x$

12. Solve the following differential equations :

(a) $\left(1 + x^2\right)\dfrac{dy}{dx} + y = \tan^{-1} x$ (b) $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

(c) $x\log x \dfrac{dy}{dx} + y = 2\log x, x > 1$

13. Solve the following differential equations:

(a) $(x + y + 1)\dfrac{dy}{dx} = 1$

[Hint: $\dfrac{dx}{dy} = x + y + 1$ or $\dfrac{dx}{dy} - x = y + 1$ which is of the form $\dfrac{dx}{dy} + Px = Q$]

(b) $\left(x + 2y^2\right)\dfrac{dy}{dx} = y, y > 0$ [Hint: $y\dfrac{dx}{dy} = x + 2y^2$ or $\dfrac{dx}{dy} - \dfrac{x}{y} = 2y$]

28.7 DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Till now we were dealing with the differential equations of first order. In this section, simple differential equations of second order and third order will be discussed.

28.7.1 Differential Equations of the Type $\dfrac{d^2y}{dx^2} = f(x)$

Consider the differential equation

$$\frac{d^2y}{dx^2} = f(x)$$

It may be noted that it is a differential equation of second order. So its general solution will contain two arbitrary constants.

Now we have, $\dfrac{d^2y}{dx^2} = f(x)$ ....(1)

or $$\frac{d}{dx}\left(\frac{dy}{dx}\right) = f(x)$$

Integrating both sides of (1) , we get

$$\frac{dy}{dx} = \int f(x)\,dx + C_1,$$ where C_1 is an arbitrary constant

Let $$\int f(x)\,dx = \phi(x)$$

Then $$\frac{dy}{dx} = \phi(x) + C_1 \qquad ...(2)$$

Again on integrating both sides of (2), we get

$$y = \int \phi(x)dx + C_1x + C_2,$$

where C_2 is another arbitrary constant. Therefore in order to find the particular solution we need two conditions. [See Example 28.19]

Example 28.18 Solve $\frac{d^2y}{dx^2} = xe^x$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} = xe^x \qquad ...(1)$$

Now integrating both sides of (1), we have

$$\frac{dy}{dx^2} = \int xe^x dx + C_1, \qquad ...(1)$$

where C_1 is an arbitrary constant

or $$\frac{dy}{dx} = xe^x - \int e^x dx + C_1$$

or $$\frac{dy}{dx} = xe^x - e^x + C_1 \qquad ...(2)$$

Again on integrating both sides of (2), we get

$$y = \int \left(xe^x - e^x + C_1\right)dx + C_2,$$

where C_2 is another arbitrary constant.

or $$y = xe^x - \int e^x dx - e^x + C_1x + C_2$$

or $$y = xe^x - 2e^x + C_1x + C_2$$

which is the required general solution.

Example 28.19 Find theparticular solution of the different equation

$$\frac{d^2y}{dx^2} = x^2 + \sin 3x$$

$$\text{for which } y(0)=0 \text{ and } \frac{dy}{dx} = 0 \text{ when } x = 0$$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} = x^2 + \sin 3x$$

Integrating both sides of (1), we have

$$\frac{dy}{dx} = \int \left(x^2 + \sin 3x\right) dx + C_1,$$

where C_1 is an arbitrary constant

or $$\frac{dy}{dx} = \frac{x^3}{3} - \frac{\cos 3x}{3} + C_1 \qquad ...(2)$$

or $$dy = \left(\frac{x^3}{3} - \frac{\cos 3x}{3} + C_1\right) dx$$

Further on integrating both sides of (2), we get

or $$y = \int \left(\frac{x^3}{3} - \frac{\cos 3x}{3} + C_1\right) dx + C_2, \qquad ...(3)$$

Now substituting $y = 0$, when $x = 0$ in (3), we have, $0 = C_2$

Again substituting $\frac{dy}{dx} = 0$ when $x = 0$ in (2) we have

$$0 = -\frac{1}{3} + C_1 \text{ or } C_1 = \frac{1}{3}$$

Putting values of C_1 and C_2 in (3), we get

$$y = \frac{x^4}{12} - \frac{1}{9} \sin 3x + \frac{x}{3}$$

which is the required particular solution.

28.7.2 Differential Equations of the Type $\frac{d^3y}{dx^3} = f(x)$

Consider the differential equation

$$\frac{d^3y}{dx^3} = f(x)$$

This is a differential equation of order three. Its general solution will contain three arbitrary constants. In order to find its particular solution we need three conditions [see Example 28.21].

For solving the differential equation of order three of the form

$\frac{d^3y}{dx^3} = f(x)$, integrate the equation three times.

Example 28.20 Solve $\frac{d^3y}{dx^3} = x^4$

Solution : The given differential equation is

$$\frac{d^3y}{dx^3} = x^4$$

or $$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = x^4 \qquad ...(1)$$

On integrating both sides of (1), we will get

$$\frac{d^2y}{dx^2} = \frac{x^5}{5} + C_1,$$

where C_1 is an arbitrary constat.

or $$\frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{x^5}{5} + C_1 \qquad ...(2)$$

Further on integrating both sides of (2), we have

$$\frac{dy}{dx} = \frac{x^6}{30} + C_1x + C_2 \qquad ...(3)$$

where C_2 is another arbitrary constant.

On integrating both sides of (3), we have

$$y = \frac{x^7}{7(30)} + C_1\frac{x^2}{2} + C_2x + C_3,$$

where C_3 is also an arbitrary constant.

or $$y = \frac{x^7}{210} + \frac{C_1x^2}{2} + C_2x + C_3$$

Example 28.21 Find the particular solution of the differential equation

$$\frac{d^3y}{dx^3} = xe^x$$

for which $y = 1,\ \frac{dy}{dx} = 2$, when $x = 0$

and $\frac{d^2y}{dx^2} = 0$, when $x = 1$.

Solution : The given differential equation is

$$\frac{d^3y}{dx^3} = xe^x$$

On integrating both sides of (1), we get

$$\frac{d^2y}{dx^2} = \int xe^x dx + C_1$$

where C_1 is an arbitrary constant.

or $$\frac{d^2y}{dx^2} = xe^x - \int e^x dx + C_1$$

$$\frac{d^2y}{dx^2} = xe^x - e^x + C_1 \qquad ...(2)$$

Again on integrating both sides of (2), we will get

$$\frac{dy}{dx} = \int \left(xe^x - e^x + C_1\right) dx + C_2,$$

where C_2 is another arbitrary constant.

or $$\frac{dy}{dx} = xe^x - \int e^x dx - e^x + C_1 x + C_2$$

or $$\frac{dy}{dx} = xe^x - 2e^x + C_1 x + C_2 \qquad ...(3)$$

Further on integrating both sides of (3), we have

$$y = xe^x - e^x - 2e^x + C_1 \frac{x^2}{2} + C_2 x + C_3,$$

where C_3 is also an arbitrary constant.

or $$y = xe^x - 3e^x + C_1 \frac{x^2}{2} + C_2 x + C_3$$

Now we have $y = 1$, $\frac{dy}{dx} = 2$ when $x = 0$ and $\frac{d^2y}{dx^2} = 0$, when $x = 1$

On substituting $\frac{d^2y}{dx^2} = 0,$, $x = 1$ in equation (2), we get

$$0 = e_1 - e_1 + C_1$$

$\therefore$ $$C_1 = 0$$

Now putting $\frac{dy}{dx} = 2$, when $x = 0$ in equation (3), we have

$$2 = 0 - 2e^0 + C_1 \,.\, 0 + C_2$$

$\Rightarrow$ $$2 = -2 + C_2 \Rightarrow C_2 = 4$$

Notes

Now putting $y = 1$, when $x = 0$ in equation (4), we have

$$1 = 0 - 3.e^0 + C_1 . 0 + C_2 . 0 + C_3$$

$\therefore$ $$C_3 = 4$$

Further on substituting values of C1, C2 and C3 in equation (4), we will get

$$y = xe^x - 3e^x + 4x + 4$$

which is the required solution.

CHECK YOUR PROGRESS 28.3

1. Find the particular solution of the differential equation

   $$\frac{d^2y}{dx^2} = \sin^2 x \quad \text{given that } \frac{dy}{dx} = 0, \text{ when } x = \frac{\pi}{2};\ y = 0 \text{ when } x = 0.$$

2. Solve the differential equation

   $$\frac{d^3y}{dx^3} = x + \sin x \quad \text{given that } y = 0,\ \frac{dy}{dx} = -1,\ \frac{d^2y}{dx^2} = 2 \quad \text{when } x = 0$$

3. Solved the following differential equations :

   (a) $\frac{d^2y}{dx^2} = x \sin x$ (b) $(1 - \cos 2x)\frac{d^2y}{dx^2} = 1 + \cos 2x$

   (c) $\frac{d^2y}{dx^2} = \sin^2 x$ (d) $\frac{d^2y}{dx^2} = \cos x - \sin x$

4. Solve the following initial value problems :

   (a) $x\frac{d^2y}{dx^2} = 1$, given that $y = 1$, $\frac{dy}{dx} = 0$, when $x = 1$

   (b) $\frac{d^2y}{dx^2} = 1 + \sin x$, given that $x = 0$, $y = 0$, $\frac{dy}{dx} = 0$

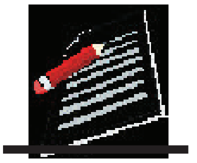
   (c) $\frac{d^2y}{dx^2} = 2$, $x = 1$ when $y = 3$ and $x = -1$ when $y = 1$

5. Solve the follwing differential equations :

   (a) $\frac{d^3y}{dx^3} = x^2 + e^x$ (b) $\frac{d^3y}{dx^3} = e^x + \cos x$ (c) $\frac{d^3y}{dx^3} = 0$

28.8 APPLICATIONS OF DIFFFERENTIAL EQUATIONS

Here we consider the applications of differential equations insome of the problems of Physics, Economics, Biology and other related fields.



28.8.1 Problems of Velogicty and Acceleration

We know that velocitty 'v' and acceleration 'a' of a body at any instant are given by

velocity = rate of change of displacement

$$\therefore \qquad v = \frac{d}{dt}(s), \text{ where s is the displacement.}$$

And acceleration = rate of change of velocity

$$a = \frac{d}{dt}(v)$$

$$= \frac{d}{dt}\left(\frac{d}{dt}(s)\right)$$

$$= \frac{d^2}{dt^2}(s)$$

Example 28.22 A stone is dropped from the top of a tower 44.1 m high. Velocity 'v' at any instant is given by v = 9.8 m/s (metres per second)

If it is given that distance after one second is 4.9m

(i) find the distance after one second is 4.9 m

(ii) value of 't' when stone hits the ground.

Solution : We are given that $v = 9.8\,t$

$$\therefore \qquad \frac{d}{dt}(s) = 9.8\ t$$

On integrating both sides, we get

$$\int ds = \int 9.8t\ dt$$

or $\qquad s = 4.9\,t^2 + C$, where C is an arbitrary constant ...(1)

It is given that s = 4.9 m when t = 1 second. Using this condition we get,

$$C = 0$$

Substituting the value of C in (1), we will get

$$s = 4.9\,t^2 \qquad \text{...(2)}$$

Putting t = 2 in (2), we get

$$s = 4.9 \times 22 \qquad \text{or} \qquad s = 19.6 \text{ m}$$

So it is clear that the distance of the stone in 2 secinds is 19.6 m.

Now when stone hits the ground the distance is 44.1m. Thus, s = 4.1

Putting s = 44.1 in (2), the following will be the outcome.

$$44.1 = 4.9\,t^2$$

or $\quad t^2 = \dfrac{44.1}{4.9}$ $\quad$ or $t_2 = 9$ $\quad$ or $\quad t = 3$

Hence the stone hits the ground after 3 seconds.

28.8.2 Population Growth Problems

Rate of growth of population is proportional to the population i.e. the growth rate $\dfrac{dx}{dt}$ is proportional to x where $x = x(t)$ denotes the number of bacteria present at time t. This fact can be represented by means of the differential equation

$\dfrac{dx}{dt} = kx$ where k is positive constant of proportinality.

Example 28.23 In a certain culture of bacteria the rate of increase is proportional to the number present.

(a) If it is found that the number doubles in 4 hours, how may times is the number expected at the end of 12 hours ?

(b) If there are 10^4 at the end of 3 hours and 4×10^4 at the end of 5 hours, how many were there in the begining ?

Solution : Let x denote the number of bacteria at time t hours. Then,

$$\frac{dx}{dt} = kx, \text{ where k is an arbitrary constant}$$

or
$$\frac{dx}{x} = kdt \qquad ....(1)$$

On integrating both sides, we get

$$\int \frac{dx}{x} = \int kdt + \log C,$$

where C is an arbitrary constant.

or $\qquad \log x = kt + \log C$

Note that the number x is positive.

or $\qquad \log \dfrac{x}{C} = kt \quad$ or $\quad \dfrac{x}{C} = kt$

or $\qquad x = Ce^{kt} \qquad ...(2)$

Assuming that $x = x_0$ at time $t = 0$, we get

$$x_0 = C$$

Putting value of C in (2) we get

$$x = x_0 e^{kt}$$

(a) It is given that when $t = 4$, $x = 2x_0$ substituting these in (3), we get

$$2x_0 = x_0 e^{4k} \quad \text{or} \quad e^{4k} = 2$$

When $t = 12$

$$x = x_0\, e^{12k} \quad \text{or} \quad x = x_0(e^{4k})^3$$

or $x = x_0\,(2)^3$ or $x = 8x_0$

Hence the growth of bacteria is 8 times the original number at the end of 12 hours.

(b) Now it is given that at $t = 3$, $x = 10^4$ and at $t = 5$, xu = 4 104

Substituting these values in (2), we have

$$10^4 = Ce^{3k} \qquad ...(4)$$

and

$$4 \times 10^4 = Ce^{5k} \qquad ...(5)$$

From (4) and (5), we get

$$4 \times Ce^{3k} = Ce^{5k} \quad \text{or} \quad 4 = e^{2k}$$

or $\log 4 = 2k$ or $k = \frac{1}{2}\log 4$

or $k = \log\sqrt{4} = \log 2$

Substituting value of k in equation (4), we get

$$10^4 = Ce^{3\ \log 2} \quad \text{or} \quad 10^4 = Ce^{\log 2^3}$$

or $10_4 = Ce^{3\ \log 2}$ or $10^4 = 8C$

or $\frac{10^4}{8} = C$

Substituting values of C and k in equation (2), we get

$$x = \frac{10^4}{8} e^{t(\log 2)} \qquad ...(6)$$

Substituting $t = 0$ in equation (6), we will get the number of bacteria in beginning.

$\therefore$ $x = \frac{10^4}{8} \times e^0$ or $x = \frac{10^4 \times 1}{8}$

or $x = \frac{25 \times 10^2}{2}$ or $x = 25 \times 50 = 1250$

28.8.3 Newton's Law of Cooling Problems

Newton's law of cooling states that the rate at which the temperature $T = T(t)$ changes in a cooling body is proportional to the difference betweem the instantaneous temperature T of the body and the constant temperature T_0 of the surrounding.

i.e. $\frac{dT}{dt} = -k(T - T_0)$, where $k > 0$. Negative sign indicates that if $T > T_0$, $\frac{dT}{dt}$ is negative, that is the temperature decreases with time.

Example 28.24 A body whose temperature is initially 100°C is allowed to cool in air, whose temperature remains at a constant 20°C. It is given that after 10 minutes the body has cooled to 40°C. Find the temperature of the body after half an hour.

Solution : Let T denote the instantaneous temperature of body and t denote the time. Then according to the Newton's Law of Cooling

$\frac{dT}{dt} = -k(T - 20)$, where $k > 0$, is a constant.

$$\frac{dT}{T - 20} = -k\,dt \qquad ...(1)$$

On integrating both sides of (1), we get $\int \frac{dT}{T - 20} = -\int k\,dt$

$$\log(T - 20) = -kt + \log C, \text{ where C is an arbitrary constant}$$

or $\quad T - 20 = Ce^{-kt}$

or $\quad T = 20 + Ce^{-kt} \qquad ...(2)$

Now substituting $t = 0$, $T = 100$ in equation (2), we get

$$100 = 20 + C$$

or $\quad C = 80$

On sbstituting value of C in equation (2), we get

$$T = 20 + 80e^{-kt} \qquad ...(3)$$

It is given that when $t = 10$, $T = 40$ and on substituting these in (3), we get

$$40 = 20 + 80e^{-10k} \quad \text{or} \quad 80e^{-10t} = 20$$

or $\quad e^{-10k} = \frac{1}{4} \quad$ or $\quad e^{-k} = \left(\frac{1}{4}\right)^{\frac{1}{10}}$

Thus $\quad T = 20 + 80\left(\frac{1}{4}\right)^{\frac{t}{10}}$

Now after half an hour i.e. when $t = 30$, we have

$$T = 20 + 80\left(\frac{1}{4}\right)^3$$

or $\quad T = 20 + 80 \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}$

or $\quad T = 20 + 1.25 \quad$ or $\quad T = 21.25°C$

CHECK YOUR PROGRESS 28.4

1. The population of a certain country is known to increse at the rat of 10% per year. How long does it take for the population to double ?
2. A body at temperature $80^{o}F$ is placed at time $t = 0$ in a medium, the temperature of which is maintained at $50^{o}F$. At the end of 5 minutes, the body has cooled to a temperature of $70^{o}F$. When will the temperature of the body be $60^{o}F$?
3. A particle is moving in a straight line, the veocity v(m/s) at any instant is given by $v = 4t^2$. It is given that the distance of the particle at the end of the article at the end of 1 second is 2 metres. Find the distance of the particle at the end of 3 seconds.
4. The number of bacteria in a yeast culture grows at a rate which is proportional to the number present. If the population of a colony of yeast bacteria triples in 1 hour, how many may be expected at the end of 5 hours ?

Example 28.25 Verify if $y = e^{m \sin^{-1} x}$ is a solution of

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-\frac{dy}{dx}-m^2y=0$$

Solution : We have,

$$y = e^{m \sin^{-1} x} \qquad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx}=\frac{me^{m\sin^{-1}x}}{\sqrt{1-x^2}}=\frac{my}{\sqrt{1-x^2}}$$

or $$\sqrt{1-x^2}\,\frac{dy}{dx}=my$$

Squaring both sides, we get

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2=m^2y^2$$

Differentiating both sides, we get

$$-2x\left(\frac{dy}{dx}\right)^2+2\left(1-x^2\right)\frac{dy}{dx}.\frac{d^2y}{dx^2}=2m^2y\frac{dy}{dx}$$

or $$-x\frac{dy}{dx}+\left(1-x^2\right)\frac{d^2y}{dx^2}=m^2y$$

or $$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$$

Hence $y = e^{m \sin^{-1} x}$ is the solutin of

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

Example 28.26 Find the particular solution of

$$e^{\frac{dy}{dx}} = x+1, \text{ given that } y = 2 \text{ when } x = 0$$

Solution : We are given that

$$e^{\frac{dy}{dx}} = x+1 \qquad \text{or} \qquad \frac{dy}{dx} = \log(x+1)$$

or $\qquad dy = \log(x + 1)\, dx \qquad$ ...(1)

On integrating both sides of equation (1), we get

$$y = \int \log(x+1) \,.\, 1dx + C$$

where C is an arbitrary constant

or $$y = x\log(x+1) - \int \frac{x}{x+1}dx + C$$

or $$y = x\log(x+1) - \int \frac{x+1-1}{x+1}dx + C$$

or $\qquad y = x \log(x+1) - [x - \log(x + 1)] + C$

or $\qquad y = (x + 1) \log(x + 1) - x + C \qquad$ ...(2)

Substituting $y = 2$ and $x = 0$ in equation (2), we get

$$2 = \log 1 - 0 + C \qquad \text{or} \qquad C = 2$$

Putting the value of C in equation (2), we get

$$y = (x + 1) \log(x + 1) - x + 2$$

which is the required solution.

Example 28.27 Find the equation of the curve represented by

$$(y - yx)\, dx + (x + xy)\, dy = 0$$

and passing through the point (1, 1).

Solution : The given differential equation is

$$(y - yx)\, dx + (x + xy)\, dy = 0$$

or $\qquad (x + xy)\, dy = (yx - y)\, dx$

or $\qquad x(1 + y)\, dy = y\,(x - 1)\, dx$

or $$\frac{(1+y)}{y}dy = \frac{x-1}{x}dx \qquad \text{...(1)}$$

Integrating both sides of equation (1), we get

$$\int\left(\frac{1+y}{y}\right)dy = \int\left(\frac{x-1}{x}\right)dx$$

or $$\int\left(\frac{1}{y}+1\right)dy = \int\left(1-\frac{1}{x}\right)dx$$

or $$\log y + y = x - \log x + C$$

Since the curve is passing through the point (1, 1), therefore,

substituting $x = 1,\ y = 1$ in equation (2), we get

$$1 = 1 + C$$

or $$C = 0$$

Thus, the equation of the required curve is

$$\log y + y = x - \log x$$

or $$\log(xy) = x - y$$

Example 28.28 Solve $\dfrac{dy}{dx} = \dfrac{3e^{2x}+3e^{4x}}{e^{x}+e^{-x}}$

Solution : We have $\dfrac{dy}{dx} = \dfrac{3e^{2x}+3e^{4x}}{e^{x}+e^{-x}}$

or $$\frac{dy}{dx} = \frac{3e^{3x}(e^{-x}+e^{x})}{e^{x}+e^{-x}}$$ or $$\frac{dy}{dx} = 3e^{3x}$$

or $$dy = 3e^{3x}\,dx \qquad ...(1)$$

Integrating both sides of (1), we get

$$y = \int 3e^{3x}dx + C$$

where C is an arbitrary constant.

or $$y = 3\frac{e^{3x}}{3} + C$$ or $$y = e^{3x} + C$$

which is required soluton.

Example 28.29 Find the solution of the following differential equation :

$$y - x\frac{dy}{dx} = a\left(y^2 + x^2\frac{dy}{dx}\right)$$

satisfying $x = a,\ y = a$

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Solution : The given differential equation is

$$y - x\frac{dy}{dx} = a\left(y^2 + x^2\frac{dy}{dx}\right)$$

or $$y - ay^2 = x\frac{dy}{dx} + ax^2\frac{dy}{dx} \quad \text{or} \quad \left(y - ay^2\right) = \left(x + ax^2\right)\frac{dy}{dx}$$

or $$\frac{dy}{y - ay^2} = \frac{dx}{x + ax^2} \quad \text{or} \quad \frac{dy}{y(1-ay)} = \frac{dx}{x(1+ax)}$$

or $$\left[\frac{1}{y} + \frac{a}{1-ay}\right]dy = \left[\frac{1}{x} - \frac{a}{1+ax}\right]dx \quad ...(1)$$

Integrating both sides of (1), we get

$$\int\left(\frac{1}{y} + \frac{a}{1-ay}\right)dy = \int\left(\frac{1}{x} - \frac{a}{1+ax}\right)dx + C$$

where C is an arbitrary conswtant

or $$\log y - \frac{a}{a}\log(1-ay) = \log x - \frac{a}{a}\log(1+ax) + \log C_1$$

where $C = \log C_1$ $\quad (x, y > 0),\ (1 - ay) > 0,\ (1 + ax) > 0$

or $\quad \log y - \log(1 - ay) = \log x - \log(1 + ax) + \log C_1$

or $$\log\left(\frac{y}{1-ay}\right) = \log\left(\frac{C_1 x}{1+ax}\right)$$

or $\quad y(1 + ax) = C_1x(1 - ay) \quad ...(2)$

Now using the condition that y = a, where x = a in equation (2), we get

$$a\left(1+a^2\right) = C_1 a\left(1-a^2\right) \quad \text{or} \quad C_1 = \frac{1+a^2}{1-a^2}$$

Substituting value of C_1 in equation (2), we get

$$y\left(1+ax\right)\left(1-a^2\right) = x\left(1-ay\right)\left(1+a^2\right)$$

which is the required solution.

CHECK YOUR PROGRESS 28.5

1. (a) If $y = \tan^{-1}x$, prove that $\left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

   (b) $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

2. Find the particular solution of the differential equation

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y, \text{ given that } y = 0 \text{ when } x = 0$$

3. (a) Find the equation of the curve represented by

$$\frac{dy}{dx} = xy + x + y + 1 \text{ and passing through thepoint } (2, 0)$$

(b) Find the equation of the curve represented by

$$\frac{dy}{dx} = y\cot x = 5e^{\cos x} \text{ and passing through the point } \left(\frac{\pi}{2}, 2\right)$$

4. Solve : $\dfrac{dy}{dx} = \dfrac{4e^{3x} + 4e^{5x}}{e^{x} + e^{-x}}$

5. Solve the following differential equations :

(a) $dx + xdy = e^{-y} \sec^2 y \, dy$

(b) $\left(1 + x^2\right)\dfrac{dy}{dx} - 4x = 3\cot^{-1} x$

(c) $(1 + y)\, xy\, dy = (1 - x^2)(1 - y)\, dx$

LET US SUM UP

- A diffferential equation is an equation involving independent variable, dependent variable and the derivatives of dependent variable (and differentials) with respect to independent variable.
- The order of a differential equation is the order of the highest derivative occurring in it.
- The degree of a differential equation is the degree of thehighest derivative after it has been freed from radicals and fractions as far as the derivatives are concerned.
- A differential equation in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together is called a linear differential equation.
- A linear differential equation is always of the first degree.
- A general solution ofa differential equation is that solution which ontains as many as the number of arbitrary constants as the order of the differential equation.
- A general solution becomes a particular solution when particular values of the arbitrary constants are determined satisfying the given conditions.
- The solution of the diffferential equation of the type $\dfrac{dy}{dx} = f(x)$ is obtained by integrating both sides.
- The solution of the differential equation of the type $\dfrac{dy}{dx} = f(x)\ g(y)$ is obtained after separting the variables and integrating both sides.

- The differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is called homogeneous if $M(x, y)$ and $N(x, y)$ are homogeneous and are the same degree.
- The solution of a homogeneous differential equation is obtained by substituting $y = vx$ or $x = vy$ ad then separating the variables.
- The solution of the first order linear equation $\frac{dy}{dx} + Py = Q$ is

$$ye^{\int P dx} = \int Q\left(e^{\int P dx}\right)dx + C,$$ where C is an arbitrary constant.

The expression $e^{\int P dx}$ is called the integrating factor of the differential equation and is written as I.F. in short.

- The solution of a second order differential equation $\frac{d^2y}{dx^2} = f(x)$ is obtained simply by integrating it twice with respect to x and its general solution contains two arbitrary constants.

SUPPORTIVE WEBSITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. Find the order and degree of the differential equation :

(a) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^4 = 0$ (b) $xdx + ydy = 0$

(c) $\frac{d^4y}{dx^4} - 4\frac{dy}{dx} + 4y = 5\cos 3x$ (d) $\frac{dy}{dx} = \cos x$

(e) $x^2\frac{d^2y}{dx^2} - xy\frac{dy}{dx} = y$ (f) $\frac{d^2y}{dx^2} + y = 0$

(g) $y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ (h) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$

2. Find which of the following equations are linear and which are non-linear

(a) $\frac{dy}{dx} = \cos x$ (b) $\frac{dy}{dx} + \frac{y}{x} = y^2 \log x$

(c) $\left(\frac{d^2y}{dx^2}\right)^3 + x^2\left(\frac{dy}{dx}\right)^2 = 0$ (d) $x\frac{dy}{dx} - 4 = x$

(e) $dx + dy = 0$

3. From the differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating a.

4. Find the differential equation by eliminating a, b, c form

   $y = ax^2 + bx + c$. Write its order and degree.

5. How many constants are contained in the general solution of

   (a) Second order differential equation.

   (b) Differential equation of order three.

   (c) Differential equation of order five.

6. Show that $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

7. Solve the following differential equations :

   (a) $\sin^2 x\frac{dy}{dx} = 3\cos x + 4$ (b) $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$

   (c) $\frac{dy}{dx} + \frac{\cos x\sin y}{\cos y} = 0$ (d) $dy + xydx = xdx$

   (e) $\frac{dy}{dx} + y\tan x = x^m \cos mx$ (f) $\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$

   (g) $\frac{d^2y}{dx^2} = \cos^2 x$ (h) $\frac{d^2y}{dx^2} = \sin^{-1} x$

   (i) $\frac{d^2y}{dx^2} = e^x - \sin x$ (j) $\frac{d^3y}{dx^3} = \cos 3x - \sin 3x$

8. If $\frac{dv}{dt} = g\cos\alpha - kv,\ g, \alpha, k$ being constants. Find v in term of t if v = 0, when t = 0.

9. Solve the differential equation

$$\frac{d^2y}{dx^2} = \log x,\ \text{ given that } y = 1 \text{ and } \frac{dy}{dx} = -1,\ \text{ when } x = 1.$$

10. If the temperature of air is290 K and a substance cools from 370 K to 330 K in 10 minutes, find when the temperature will be 295 K.

ANSWERS

CHECK YOUR PROGRESS 28.1

1. Order is 1 and degree is 2.

2. (a) Order 2, degree 1 (b) Order 1, degree 2
   (c) Order 1, degree 1 (d) Order 2, degree 2

3. (a) Non - linear (b) Linear
   (c) Liinear (d) Non - Linear

4. $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

5. (a) $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$ (b) $\left(x^2-2y^2\right)\left(\frac{dy}{dx}\right)^2-4xy\frac{dy}{dx}-x^2=0$

   (c) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0$ (d) $y=(x-3)\frac{dy}{dx}+2$

   (e) $\left(x^2-y^2\right)\frac{dy}{dx}-2xy=0$

CHECK YOUR PROGRESS 28.2

1. (i) Yes (ii) No
2. (i), (ii) and (iv) are particular solutions (iii) is the general solution
3. (ii), (iv) are homogeneous
5. (a) $y=\tan x$ (b) $y=\tan x+\sqrt{3}$
6. (a) $y=\frac{1}{6}x^6\tan^{-1}(x^3)-\frac{1}{6}x^3+\frac{1}{6}\tan^{-1}(x^3)+C$

   (b) $y=\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+(x-1)e^x+C$

   (c) $y=\frac{1}{2}\log\left|x^2+1\right|+C$ (d) $y=\frac{1}{3}x^3-\frac{1}{3}\cos 3x+C$
7. $y=-4e^{-x}+7$
8. (a) $\log\left|\frac{x}{y}\right|=C+\frac{1}{x}+\frac{1}{y}$ (b) $\log\left|y+1\right|=x+\frac{x^2}{2}+C$

   (c) $\tan x\tan y=C$ (d) $e^y=e^x+\frac{x^3}{3}+C$

9. $3e^{-4y} + 4e^{3x} = 7$

10. (a) $x = C(x^2 - y^2)$ (b) $x + cy = y \log |x|$

(c) $\sin^{-1}\left(\frac{y}{x}\right) = \log|x| + C$ (d) $\tan\frac{y}{2x} = Cx$

11. $y(\sec x + \tan x) = \sec x + \tan x - x + C$

12. (a) $y = \tan^{-1} x - 1 + Ce^{-\tan x}$ (b) $y = \tan x - 1 + Ce^{-\tan x}$

(c) $y = \log x + \frac{C}{\log x}$

13. (a) $x = Ce^y - (y + 2)$ (b) $x = y^2 + Cy$

CHECK YOUR PROGRESS 28.3

1. $y = \frac{1}{4}x^2 - \frac{\pi}{4}x + \frac{1}{8}\cos 2x - \frac{1}{8}$

2. $y = \frac{x^4}{24} + \frac{3}{2}x^2 - x + \cos x - 1$

3. (a) $y = -x \sin x - 2\cos x + C_1x + C_2$ (b) $y = C_1x + C_2 - \log \sin x - \frac{1}{2}x^2$

(c) $y = \frac{1}{4}x^2 + \frac{1}{8}\cos 2x + C_1x + C_2$ (d) $y = -\cos x + \sin x + C_1x + C_2$

4. (a) $y = x(\log x - 1) + 2$ (b) $y = \frac{1}{2}x^2 - \sin x + x$

(c) $y = x^2 + x + 1$

5. (a) $y = e^x + \frac{x^5}{60} + C_1x^2 + C_2x + C_3$ (b) $y = e^x - \sin x + C_1x_2 + C_2x + C_3$

(c) $y = C_1x^2 + C_2x + C_3$

CHECK YOUR PROGRESS 28.4

1. $t = 10 \log 2$ years

2. 13.55 Minutes

3. $12\frac{2}{3}$ minutes

4. 243 Times

CHECK YOUR PROGRESS 28.5

2. $4e^{3x} + 3e^{-4y} = 7$

3. (a) $\log(y+1) = \frac{1}{2}x^2 + x - 4$ (b) $y \sin x + 5e^{\cos x} = 7$

4. $y = \frac{4}{5}e^{5x} + C$

5. (a) $x = e^{-y}(C + \tan y)$ (b) $y = 2\log\left|1+x^2\right| - \frac{3}{2}\left(\cot^{-1}x\right)^2 + C$

(c) $\log x + 2\log|1-y| = \frac{x^2}{2} - \frac{y^2}{2} - 2y + C$

TERMINAL EXERCISE

1. (a) order 2, degree 3 (e) Order 2, degree 1
   (b) Order 1, degree 1 (f) Order 2, degree 1
   (c) Order 4, degree 1 (g) Order 1, degree 2
   (d) Order 1, degree 1 (h) Order 2, degree 1
2. (a), (d), (e) are linear; (b), (c) are non-linear
3. $\left(x^2 - 2y^2\right)\left(\frac{dy}{dx}\right)^2 - 4xy\left(\frac{dy}{dx}\right) - x^2 = 0$
4. $\frac{d^3y}{dx^3} = 0,$ Order 3, degree 1.
5. (a) Two (b) Three (c) Five
7. (a) $y + 3\operatorname{cosec} x + 4\cot x = C$ (b) $e^y = e^x + \frac{x^3}{3} + C$

   (c) $\sin y = Ce^{-\sin x}$ (d) $\log(1-y) + \frac{x^2}{2} = C$

   (e) $y = \frac{x^{m+1}}{m+1}\cos x + C\cos x$ (f) $x = \tan^{-1}y - 1 + Ce^{-\tan^{-1}y}$

   (g) $y = \frac{x^2}{4} - \frac{\cos 2x}{8} + C_1x + C_2$

   (h) $y = \left(\frac{x^2}{4} + \frac{1}{4}\right)\sin^{-1}x + \frac{3}{4}x\sqrt{1-x^2} + C_1x + C_2$

   (i) $y = e^x + \sin x + C_1x + C_2$ (j) $y = -\frac{\sin 3x}{27} - \frac{\cos 3x}{27} + \frac{C_1x^2}{2} + C_2x + C_3$
8. $v = \frac{g\cos\alpha}{k}(1 - e^{-kt})$
9. $y = \frac{x^2\log x}{2} - \frac{3}{4}x^2 + \frac{7}{4}$
10. 40 minutes.

MODULE-V

STATISTICS

29

MEASURES OF DISPERSION

You have learnt various measures of central tendency. Measures of central tendency help us to represent the entire mass of the data by a single value.

Can the **central tendency** describe the data fully and adequately?

In order to understand it, let us consider an example.

The daily income of the workers in two factories are :

Factory A	:	35	45	50	65	70	90	100
Factory B	:	60	65	65	65	65	65	70

Here we observe that in both the groups the mean of the data is the same, namely, 65

(i) In group A, the observations are much more scattered from the mean.

(ii) In group B, almost all the observations are concentrated around the mean.

Certainly, the two groups differ even though they have the same mean.

Thus, there arises a need to differentiate between the groups. We need some other measures which concern with the measure of scatteredness (or spread).

To do this, we study what is known as **measures of dispersion.**

OBJECTIVES

After studying this lesson, you will be able to :

- explain the meaning of dispersion through examples;
- define various measures of dispersion – range, mean deviation, variance and standard deviation;
- calculate mean deviation from the mean of raw and grouped data;
- calculate variance and standard deviation for raw and grouped data; and
- illustrate the properties of variance and standard deviation.

EXPECTED BACKGROUND KNOWLEDGE

- Mean of grouped data
- Median of ungrouped data

29.1 MEANING OF DISPERSION

To explain the meaning of dispersion, let us consider an example.

Two sections of 10 students each in class X in a certain school were given a common test in Mathematics (40 maximum marks). The scores of the students are given below :

Section A :	6	9	11	13	15	21	23	28	29	35
Section B:	15	16	16	17	18	19	20	21	23	25

The average score in section A is 19.

The average score in section B is 19.

Let us construct a dot diagram, on the same scale for section A and section B (see Fig. 29.1)

The position of mean is marked by an arrow in the dot diagram.

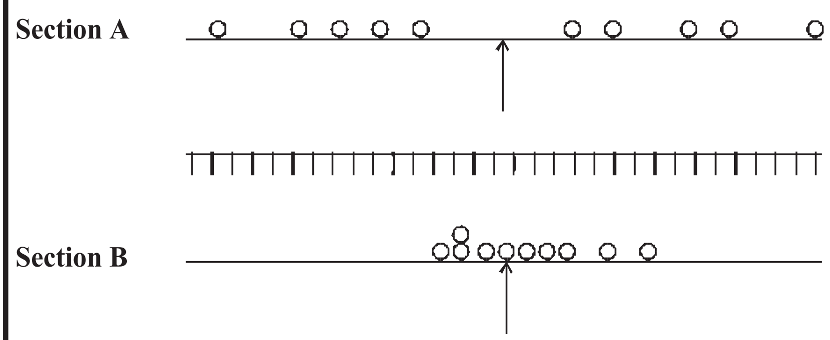

Fig. 29.1

Clearly, the extent of spread or dispersion of the data is different in section A from that of B. The measurement of the scatter of the given data about the average is said to be a measure of dispersion or scatter.

In this lesson, you will read about the following measures of dispersion :

(a) Range

(b) Mean deviation from mean

(c) Variance

(d) Standard deviation

29.2 DEFINITION OF VARIOUS MEASURES OF DISPERSION

(a) **Range :** In the above cited example, we observe that

(i) the scores of all the students in section A are ranging from 6 to 35;

(ii) the scores of the students in section B are ranging from 15 to 25.

The difference between the largest and the smallest scores in section A is 29 (35 – 6)

The difference between the largest and smallest scores in section B is 10 (25 – 15).

Thus, the difference between the largest and the smallest value of a data, is termed as the range of the distribution.

(b) **Mean Deviation from Mean :** In Fig. 29.1, we note that the scores in section B cluster around the mean while in section A the scores are spread away from the mean. Let us take the deviation of each observation from the mean and add all such deviations. If the sum is 'large', the dispersion is 'large'. If, however, the sum is 'small' the dispersion is small.

Let us find the sum of deviations from the mean, i.e., 19 for scores in section A.

Observations (x_i)	Deviations from mean $(x_i - \overline{x})$
6	-13
9	-10
11	-8
13	-6
15	-4
21	$+2$
23	$+4$
28	$+9$
29	$+10$
35	16
190	0

Here, the sum is zero. It is neither 'large' nor 'small'. Is it a coincidence ?

Let us now find the sum of deviations from the mean, i.e., 19 for scores in section B.

Observations (x_i)	Deviations from mean $(x_i - \overline{x})$
15	-4
16	-3
16	-3
17	-2
18	-1
19	0
20	1
21	2
23	4
25	6
190	0

Again, the sum is zero. Certainly it is not a coincidence. In fact, we have proved earlier that **the sum of the deviations taken from the mean is always zero for any set of data.** Why is the sum always zero ?

On close examination, we find that the signs of some deviations are positive and of some other deviations are negative. Perhaps, this is what makes their sum always zero. In both the cases, we get sum of deviations to be zero, so, we cannot draw any conclusion from the sum of deviations. But this can be avoided if we take only the **absolute value of the deviations** and then take their sum.

If we follow this method, we will obtain a measure (descriptor) called the mean deviation from the mean.

The mean deviation is the sum of the absolute values of the deviations from the mean divided by the number of items, (i.e., the sum of the frequencies).

(c) **Variance :** In the above case, we took the absolute value of the deviations taken from mean to get rid of the negative sign of the deviations. Another method is to square the deviations. Let us, therefore, square the deviations from the mean and then take their sum. If we divide this sum by the number of observations (i.e., the sum of the frequencies), we obtain the average of deviations, which is called variance. **Variance is usually denoted by** σ^2.

(d) **Standard Deviation :** If we take the positive square root of the variance, we obtain the root mean square deviation or simply called standard deviation and is denoted by σ.

29.3 MEAN DEVIATION FROM MEAN OF RAW AND GROUPED DATA

Mean Deviation from mean of raw data $= \dfrac{\sum_{i=1}^{n} |x_i - \overline{x}|}{N}$

Mean deviation from mean of grouped data $= \dfrac{\sum_{i=1}^{n} [f_i |x_i - \overline{x}|]}{N}$

where $N = \sum_{i=1}^{n} f_i, \ \overline{x} = \dfrac{1}{N} \sum_{i=1}^{n} (f_i x_i)$

The following steps are employed to calculate the mean deviation from mean.

Step 1 : Make a column of deviation from the mean, namely $x_i - \overline{x}$ (In case of grouped data take x_i as the mid value of the class.)

Step 2 : Take absolute value of each deviation and write in the column headed $|x_i - \overline{x}|$.

For calculating the mean deviation from the mean of raw data use

$$\text{Mean deviation of Mean} = \frac{\sum_{i=1}^{n} |x_i - \overline{x}|}{N}$$

For grouped data proceed to step 3.

Step 3 : Multiply each entry in step 2 by the corresponding frequency. We obtain $f_i (x_i - \overline{x})$ and write in the column headed $f_i |x_i - \overline{x}|$.

Step 4 : Find the sum of the column in step 3. We obtain $\sum_{i=1}^{n}[f_i|x_i - \bar{x}|]$

Step 5 : Divide the sum obtained in step 4 by N.

Now let us take few examples to explain the above steps.

Example 29.1 Find the mean deviation from the mean of the following data :

Size of items x_i	4	6	8	10	12	14	16
Frequency f_i	2	5	5	3	2	1	4

Mean is 10

Solution :

x_i	f_i	$x_i - \bar{x}$	$\lvert x_i - \bar{x}\rvert$	$f_i\lvert x_i - \bar{x}\rvert$
4	2	−5.7	5.7	11.4
6	4	−3.7	3.7	14.8
8	5	−1.7	1.7	8.5
10	3	0.3	0.3	0.9
12	2	2.3	2.3	4.6
14	1	4.3	4.3	4.3
16	4	6.3	6.3	25.2
	21			69.7

Mean deviation from mean $= \dfrac{\sum[f_i|x_i - \bar{x}|]}{21}$

$$= \frac{69.7}{21} = 3.319$$

Example 29.2 Calculate the mean deviation from mean of the following distribution :

Marks	0 − 10	10 − 20	20 − 30	30 − 40	40 − 50
No. of Students	5	8	15	16	6

Mean is 27 marks

Solution :

Marks	Class Marks x_i	f_i	$x_i - \bar{x}$	$\lvert x_i - \bar{x}\rvert$	$f_i\lvert x_i - \bar{x}\rvert$
0 − 10	5	5	−22	22	110
10 − 20	15	8	−12	12	96
20 − 30	25	15	−2	2	30
30 − 40	35	16	8	8	128
40 − 50	45	6	18	18	108
Total		50			472

Mean deviation from Mean $=\dfrac{\sum[f_i|x_i-\bar{x}|]}{N}$

$$=\frac{472}{50}\text{ Marks}=9.44\text{ Marks}$$

CHECK YOUR PROGRESS 29.1

1. The ages of 10 girls are given below :

   3 5 7 8 9 10 12 14 17 18

   What is the range ?

2. The weight of 10 students (in Kg) of class XII are given below :

   45 49 55 43 52 40 62 47 61 58

   What is the range ?

3. Find the mean deviation from mean of the data

   45 55 63 76 67 84 75 48 62 65

   Given mean = 64.

4. Calculate the mean deviation from mean of the following distribution.

Salary (in rupees)	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100
No. of employees	4	6	8	12	7	6	4	3

Given mean = Rs. 57.2

5. Calculate the mean deviation for the following data of marks obtained by 40 students in a test

Marks obtained	20	30	40	50	60	70	80	90	100
No. of students	2	4	8	10	8	4	2	1	1

6. The data below presents the earnings of 50 workers of a factory

Earnings (in rupees) :	1200	1300	1400	1500	1600	1800	2000
No. of workers :	4	6	15	12	7	4	2

   Find mean deviation.

7. The distribution of weight of 100 students is given below :

Weight (in Kg)	50 – 55	55 – 60	60 – 65	65 – 70	70 – 75	75 – 80
No. of students	5	13	35	25	17	5

   Calculate the mean deviation.

8. The marks of 50 students in a particular test are :

Marks	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100
No. of students	4	6	9	12	8	6	4	1

Find the mean deviation for the above data.

29.4 VARIANCE AND STANDARD DEVIATION OF RAW DATA

If there are n observations, $x_1, x_2, \ldots, x_n$, then

$$\text{Variance}\left(\sigma^2\right) = \frac{(x_1-\overline{x})^2 + (x_2-\overline{x})^2 + \ldots.. + (x_n-\overline{x})^2}{n}$$

or $$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\overline{x})^2}{n}; \text{ where } \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The standard deviation, denoted by σ, is the positive square root of σ^2. Thus

$$\sigma = +\sqrt{\frac{\sum_{i=1}^{n}(x_i-\overline{x})^2}{n}}$$

The following steps are employed to calculate the variance and hence the standard deviation of raw data. The mean is assumed to have been calculated already.

Step 1 : Make a column of deviations from the mean, namely, $x_i - \overline{x}$.

Step 2 (check) : Sum of deviations from mean must be zero, i.e., $\sum_{i=1}^{n}(x_i-\overline{x}) = 0$

Step 3: Square each deviation and write in the column headed $(x_i - \overline{x})^2$.

Step 4 : Find the sum of the column in step 3.

Step 5 : Divide the sum obtained in step 4 by the number of observations. We obtain σ^2.

Step 6 : Take the positive square root of σ^2. We obtain σ (Standard deviation).

Example 29.3 The daily sale of sugar in a certain grocery shop is given below :

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
75 kg	120 kg	12 kg	50 kg	70.5 kg	140.5 kg

The average daily sale is 78 Kg. Calculate the variance and the standard deviation of the above data.

Solution : $\overline{x} = 78$ kg (Given)

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
75	-3	9
120	42	1764
12	-66	4356
50	-28	784
70.5	-7.5	56.25
140.5	62.5	3906.25
	0	10875.50

Thus $$\sigma^2 = \frac{\sum_{i=1}(x_i - \overline{x})^2}{n} = \frac{10875.50}{6}$$

$$= 1812.58 \text{ (approx.)}$$

and $$\sigma = 42.57 \text{ (approx.)}$$

Example 29.4 The marks of 10 students of section A in a test in English are given below :

7 10 12 13 15 20 21 28 29 35

Determine the variance and the standard deviation.

Solution : Here $\overline{x} = \frac{\sum x_i}{10} = \frac{190}{10} = 19$

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
7	-12	144
10	-9	81
12	-7	49
13	-6	36
15	-4	16
20	$+1$	1
21	$+2$	4
28	$+9$	81
29	$+10$	100
35	$+16$	256
	0	768

Thus $$\sigma^2 = \frac{768}{10} = 76.8,$$

and $$\sigma = +\sqrt{76.8} = 8.76 \text{ (approx)}$$

CHECK YOUR PROGRESS 29.2

1. The salary of 10 employees (in rupees) in a factory (per day) is

   50 60 65 70 80 45 75 90 95 100

   Calculate the variance and standard deviation.

2. The marks of 10 students of class X in a test in English are given below :

   9 10 15 16 18 20 25 30 32 35

   Determine the variance and the standard deviation.

3. The data on relative humidity (in %) for the first ten days of a month in a city are given below:

   90 97 92 95 93 95 85 83 85 75

   Calculate the variance and standard deviation for the above data.

4. Find the standard deviation for the data

   4 6 8 10 12 14 16

5. Find the variance and the standard deviation for the data

   4 7 9 10 11 13 16

6. Find the standard deviation for the data.

   40 40 40 60 65 65 70 70 75 75 75 80 85 90 90 100

29.5 STANDARD DEVIATION AND VARIANCE OF RAW DATA AN ALTERNATE METHOD

If $\overline{x}$ is in decimals, taking deviations from $\overline{x}$ and squaring each deviation involves even more decimals and the computation becomes tedious. We give below an alternative formula for computing σ^2. In this formula, we by pass the calculation of $\overline{x}$.

We know
$$\sigma^2 = \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{n}$$

$$= \sum_{i=1}^{n} \frac{x_i^2 - 2x_i\overline{x} + \overline{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \frac{2\overline{x}\sum_{i=1}^{n} x_i}{n} + \overline{x}^2$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \overline{x}^2 \qquad \left(\because \overline{x} = \frac{\sum x_i}{n} \right)$$

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i.e. $$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n}$$

And $$\sigma = +\sqrt{\sigma^2}$$

The steps to be employed in calculation of σ^2 and, hence σ by this method are as follows :

Step 1 : Make a column of squares of observations i.e. x_i^2.

Step 2 : Find the sum of the column in step 1. We obtain $\sum_{i=1}^{n} x_i^2$

Step 3 : Substitute the values of $\sum_{i=1}^{n} x_i^2$, n and $\sum_{i=1}^{n} x_i$ in the above formula. We obtain σ^2.

Step 4 : Take the positive sauare root of σ^2. We obtain σ.

Example 29.5 We refer to Example 29.4 of this lesson and re-calculate the variance and standard deviation by this method.

Solution :

x_i	x_i^2
7	49
10	100
12	144
13	169
15	225
20	400
21	441
28	784
29	841
35	1225
190	4378

$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n}$$

$$= \frac{4378 - \frac{(190)^2}{10}}{10}$$

$$= \frac{4378 - 3610}{10}$$

$$= \frac{768}{10} = 76.8$$

and $$\sigma = +\sqrt{76.8} = 8.76 \text{ (approx)}$$

We observe that we get the same value of σ^2 and σ by either methods.

29.6 STANDARD DEVIATION AND VARIANCE OF GROUPED DATA : METHOD - I

We are given k classes and their corresponding frequencies. We will denote the variance and the standard deviation of grouped data by σ_g^2 and σ_g respectively. The formulae are given below :

$$\sigma_g^2 = \frac{\sum_{i=1}^{K}\left[f_i\,(x_i - \overline{x})^2\right]}{N}, \qquad N = \sum_{i=1}^{K} f_i$$

and $$\sigma_g = +\sqrt{\sigma_g^2}$$

The following steps are employed to calculate σ_g^2 and, hence σ_g : (The mean is assumed to have been calculated already).

Step 1 : Make a column of class marks of the given classes, namely x_i

Step 2 : Make a column of deviations of class marks from the mean, namely, $x_i - \overline{x}$. Of course the sum of these deviations need not be zero, since x_i's are no more the original observations.

Step 3 : Make a column of squares of deviations obtained in step 2, i.e., $(x_i - \overline{x})^2$ and write in the column headed by $(x_i - \overline{x})^2$.

Step 4 : Multiply each entry in step 3 by the corresponding frequency.
We obtain $f_i\,(x_i - \overline{x})^2$.

Step 5 : Find the sum of the column in step 4. We obtain $\sum_{i=1}^{k}\left[f_i\,(x_i - \overline{x})^2\right]$

Step 6 : Divide the sum obtained in step 5 by N (total no. of frequencies). We obtain σ_g^2.

Step 7 : $\sigma_g = +\sqrt{\ _g^2}$

Example 29.6 In a study to test the effectiveness of a new variety of wheat, an experiment was performed with 50 experimental fields and the following results were obtained :

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Yield per Hectare (in quintals)	Number of Fields
31 – 35	2
36 – 40	3
41 – 45	8
46 – 50	12
51 – 55	16
56 – 60	5
61 – 65	2
66 – 70	2

The mean yield per hectare is 50 quintals. Determine the variance and the standard deviation of the above distribution.

Solution :

Yield per Hectare (in quintal)	No. of Fields	Class Marks	$(x_i - \overline{x})$	$(x_i - \overline{x})^2$	$f_i(x_i - \overline{x})^2$
31 – 35	2	33	– 17	289	578
36 – 40	3	38	– 12	144	432
41 – 45	8	43	– 7	49	392
46 – 50	12	48	– 2	4	48
51 – 55	16	53	+3	9	144
56 – 60	5	58	+8	64	320
61 – 65	2	63	+13	169	338
66 – 70	2	68	+18	324	648
Total	50				2900

Thus $\sigma_g^2 = \dfrac{\sum_{i=1}^{n}\left[f_i(x_i - \overline{x})^2\right]}{N} = \dfrac{2900}{50} = 58$ and $\sigma_g = +\sqrt{58} = 7.61$ (approx)

29.7 STANDARD DEVIATION AND VARIANCE OF GROUPED DATA : METHOD - II

If $\overline{x}$ is not given or if $\overline{x}$ is in decimals in which case the calculations become rather tedious, we employ the alternative formula for the calculation of σ_g^2 as given below:

$$\sigma_g^2 = \frac{\sum_{i=1}^{k}\left[f_i x_i^2\right] - \dfrac{\left(\sum_{i=1}^{k}[f_i x_i]\right)^2}{N}}{N}, \quad N = \sum_{i=1}^{k} f_i$$

and $\sigma_g = +\sqrt{\sigma_g^2}$

The following steps are employed in calculating σ_g^2, and, hence σ_g by this method:

Step 1 : Make a column of class marks of the given classes, namely, x_i.

Step 2 : Find the product of each class mark with the corresponding frequency. Write the product in the column $x_i f_i$.

Step 3 : Sum the entries obtained in step 2. We obtain $\sum_{i=1}^{k}(f_i x_i)$.

Step 4 : Make a column of squares of the class marks of the given classes, namely, x_i^2.

Step 5 : Find the product of each entry in step 4 with the corresponding frequency. We obtain $f_i x_i^2$.

Step 6 : Find the sum of the entries obtained in step 5. We obtain $\sum_{i=1}^{k}\left(f_i x_i^2\right)$.

Step 7 : Substitute the values of $\sum_{i=1}^{k}\left(f_i x_i^2\right)$, N and $\left(\sum_{i=1}^{k}(f_i x_i)\right)$ in the formula and obtain σ_g^2.

Step 8 : $\sigma_g = +\sqrt{\sigma_g^2}$.

Example 29.7 Determine the variance and standard deviation for the data given in Example 29.6 by this method.

Solution :

Yields per Hectare (in quintals)	f_i	x_i	$f_i x_i$	x_i^2	$f_i x_i^2$
31 – 35	2	33	66	1089	2178
36 – 40	3	38	114	1444	4332
41 – 45	8	43	344	1849	14792
46 – 50	12	48	576	2304	27648
51 – 55	16	53	848	2809	44944
56 – 60	5	58	290	3364	16820
61 – 65	2	63	126	3969	7938
66 – 77	2	68	136	4624	9248
Total	50		2500		127900

Substituting the values of $\sum_{i=1}^{k}\left(f_i x_i^2\right)$, N and $\sum_{i=1}^{k}\left(f_i x_i\right)$ in the formula, we obtain

$$\sigma_g^2 = \frac{127900 - \frac{(2500)^2}{50}}{50}$$

$$= \frac{2900}{50}$$

$$= 58$$

and $\sigma_g = +\sqrt{58}$

$= 7.61$ (approx.)

Again, we observe that we get the same value of σ_g^2, by either of the methods.

CHECK YOUR PROGRESS 29.3

1. In a study on effectiveness of a medicine over a group of patients, the following results were obtained :

Percentage of relief	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100
No. of patients	10	10	25	15	40

Find the variance and standard deviation.

2. In a study on ages of mothers at the first child birth in a village, the following data were available :

Age (in years) at first child birth	18 – 20	20 – 22	22 – 24	24 – 26	26 – 28	28 – 30	30 – 32
No. of mothers	130	110	80	74	50	40	16

Find the variance and the standard deviation.

3. The daily salaries of 30 workers are given below:

Daily salary (In Rs.)	0 – 50	50 – 100	100 – 150	150 – 200	200 – 250	250 – 300
No. of workers	3	4	5	7	8	3

Find variance and standard deviation for the above data.

29.8 STANDARD DEVIATION AND VARIANCE: STEP DEVIATION METHOD

In Example 29.7, we have seen that the calculations were very complicated. In order to simplify the calculations, we use another method called the step deviation method. In most of the frequency distributions, we shall be concerned with the equal classes. Let us denote, the class size by h.

Now we not only take the deviation of each class mark from the arbitrary chosen 'a' but also divide each deviation by h. Let

$$u_i = \frac{x_i - a}{h} \qquad .....(1)$$

Then $$x_i = hu_i + a \qquad .....(2)$$

We know that $$\bar{x} = h\bar{u} + a \qquad .....(3)$$

Subtracting (3) from (2) , we get

$$x_i - \bar{x} = h(u_i - \bar{u}) \qquad .....(4)$$

In (4) , squaring both sides and multiplying by f_i and summing over k, we get

$$\sum_{i=1}^{k}\left[f_i(x_i - \bar{x})^2\right] = h^2\sum_{i=1}^{k}\left[f_i(u_i - \bar{u})^2\right] \qquad .....(5)$$

Dividing both sides of (5) by N, we get

$$\frac{\sum_{i=1}^{k}\left[f_i(x_i - \bar{x})^2\right]}{N} = \frac{h^2}{N}\sum_{i=1}^{k}\left[f_i(u_i - \bar{u})^2\right]$$

i.e. $$\sigma_x^2 = h^2\sigma_u^2 \qquad .....(6)$$

where $\sigma_x{}^2$ is the variance of the original data and $\sigma_u{}^2$ is the variance of the coded data or coded variance. $\sigma_u{}^2$ can be calculated by using the formula which involves the mean, namely,

$$\sigma_u^2 = \frac{1}{N}\sum_{i=1}^{k}\left[f_i(u_i - \bar{u})^2\right], \qquad N = \sum_{i=1}^{k} f_i \qquad .....(7)$$

or by using the formula which does not involve the mean, namely,

$$\sigma_u^2 = \frac{\sum_{i=1}^{k}\left[f_iu_i^2\right] - \frac{\left(\sum_{i=1}^{k}[f_iu_i]\right)^2}{N}}{N}, \qquad N = \sum_{i=1}^{k} f_i$$

Example 29.8 We refer to the Example 29.6 again and find the variance and standard deviation using the coded variance.

Solution : Here h = 5 and let a = 48.

Yield per Hectare (in quintal)	Number of fields f_i	Class marks x_i	$u_i = \frac{x_i - 48}{5}$	f_iu_i	$u_i{}^2$	$f_iu_i{}^2$
31 – 35	2	33	– 3	– 6	9	18
36 – 40	3	38	– 2	– 6	4	12

41 – 45	8	43	– 1	– 8	1	8
46 – 50	12	48	0	0	0	0
51 – 55	16	53	+1	16	1	16
56 – 60	5	58	+2	10	4	20
61 – 65	2	63	+3	6	9	18
66 – 70	2	68	+4	8	16	32
Total	50			20		124

Thus

$$\sigma_u^2 = \frac{\sum_{i=1}^{k} f_i u_i^2 - \frac{\left(\sum_{i=1}^{k} f_i u_i\right)^2}{N}}{N}$$

$$= \frac{124 - \frac{(20)^2}{50}}{50}$$

$$= \frac{124 - 8}{50}$$

$$\sigma_u^2 = \frac{58}{25}$$

Variance of the original data will be

$$\sigma_x^2 = h^2 \sigma_u^2 = 25 \times \frac{58}{25} = 58$$

and $\sigma_x = +\sqrt{58}$

$= 7.61$ (approx)

We, of course, get the same variance, and hence, standard deviation as before.

Example 29.9 Find the standard deviation for the following distribution giving wages of 230 persons.

Wages (in Rs)	No. of persons	Wages (in Rs)	No. of persons
70 – 80	12	110 – 120	50
80 – 90	18	120 – 130	45
90 – 100	35	130 – 140	20
100 – 110	42	140 – 150	8

Solution :

Wages (in Rs.)	No. of persons f_i	class mark x_i	$u_i = \frac{x_i - 105}{10}$	u_i^2	$f_i u_i$	$f_i u_i^2$
70 – 80	12	75	– 3	9	– 36	108
80 – 90	18	85	– 2	4	– 36	72
90 – 100	35	95	– 1	1	– 35	35
100 – 110	42	105	0	0	0	0
110 – 120	50	115	+1	1	50	50
120 – 130	45	125	+2	4	90	180
130 – 140	20	135	+3	9	60	180
140 – 150	8	145	+4	16	32	128
Total	230				125	753

$$\sigma^2 = h^2\left[\frac{1}{N}\sum\left[f_i u_i^2\right] - \left(\frac{1}{N}\sum\left[f_i u_i\right]\right)^2\right]$$

$$= 100\left[\frac{753}{230} - \left(\frac{125}{230}\right)^2\right]$$

$$= 100\left(3.27 - 0.29\right) = 298$$

$\therefore$ $\sigma = +\sqrt{298} = 17.3$ (approx)

CHECK YOUR PROGRESS 29.4

1. The data written below gives the daily earnings of 400 workers of a flour mill.

Weekly earning (in Rs.)	No. of Workers
80 – 100	16
100 – 120	20
120 – 140	25
140 – 160	40
160 – 180	80
180 – 200	65
200 – 220	60
220 – 240	35
240 – 260	30
260 – 280	20
280 – 300	9

Calculate the variance and standard deviation using step deviation method.

2. The data on ages of teachers working in a school of a city are given below:

Age (in years)	20 – 25	25 – 30	30 – 35	35 – 40
No. of teachers	25	110	75	120
Age (in years)	40 – 45	45 – 50	50 – 55	55 – 60
No. of teachers	100	90	50	30

Calculate the variance and standard deviation using step deviation method.

3. Calculate the variance and standard deviation using step deviation method of the following data :

Age (in years)	25 – 30	30 – 35	35 – 40
No. of persons	70	51	47
Age (in years)	40 – 50	45 – 50	50 – 55
No. of persons	31	29	22

29.9 PROPERTIES OF VARIANCE AND STANDARD DEVIATION

Property I : The variance is independent of change of origin.

To verify this property let us consider the example given below.

Example : 29.10 The marks of 10 students in a particular examination are as follows:

10 12 15 12 16 20 13 17 15 10

Later, it was decided that 5 bonus marks will be awarded to each student. Compare the variance and standard deviation in the two cases.

Solution : ***Case – I***

x_i	f_i	$f_i x_i$	$x_i - \overline{x}$	$(x_i - \overline{x})^2$	$f_i(x_i - \overline{x})^2$
10	2	20	– 4	16	32
12	2	24	– 2	4	8
13	1	13	– 1	1	1
15	2	30	1	1	2
16	1	16	2	4	4
17	1	17	3	9	9
20	1	20	6	36	36
	10	140			92

Here $\overline{x} = \dfrac{140}{10} = 14$

Variance $= \dfrac{\sum\left[f_i(x_i-\bar{x})^2\right]}{10}$

$= \dfrac{92}{10} = 9.2$

Standard deviation $= +\sqrt{9.2} = 3.03$

Case – II (By adding 5 marks to each x_i)

x_i	f_i	$f_i x_i$	$x_i-\bar{x}$	$(x_i-\bar{x})^2$	$f_i(x_i-\bar{x})^2$
15	2	30	–4	16	32
17	2	34	–2	4	8
18	1	18	–1	1	1
20	2	40	1	1	2
21	1	21	2	4	4
22	1	22	3	9	9
25	1	25	6	36	36
	10	190			92

$$\bar{x} = \frac{190}{10} = 19$$

∴ Variance $= \dfrac{92}{10} = 9.2$

Standard deviation $= +\sqrt{9.2} = 3.03$

Thus, we see that there is no change in variance and standard deviation of the given data if the origin is changed i.e., if a constant is added to each observation.

Property II : The variance is not independent of the change of scale.

Example 29.11 In the above example, if each observation is multiplied by 2, then discuss the change in variance and standard deviation.

Solution : In case-I of the above example , we have variance = 9.2, standard deviation = 3.03. Now, let us calculate the variance and the Standard deviation when each observation is multiplied by 2.

x_i	f_i	$f_i x_i$	$x_i-\bar{x}$	$(x_i-\bar{x})^2$	$f_i(x_i-\bar{x})^2$
20	2	40	–8	64	128
24	2	48	–4	16	32
26	1	26	–2	4	4
30	2	60	2	4	8
32	1	32	4	16	16
34	1	34	6	36	36
40	1	40	12	144	144
	10	280			368

$$\bar{x} = \frac{280}{10} = 28$$

$$\text{Variance} = \frac{368}{10} = 36.8$$

$$\text{Standard deviation} = +\sqrt{36.8} = 6.06$$

Here we observe that, the variance is four times the original one and consequently the standard deviation is doubled.

In a similar way we can verify that if each observation is divided by a constant then the variance of the new observations gets divided by the square of the same constant and consequently the standard deviation of the new observations gets divided by the same constant.

Property III : Prove that the standard deviation is the least possible root mean square deviation.

Proof : Let $\bar{x} - a = d$

By definition, we have

$$s^2 = \frac{1}{N}\sum\left[f_i(x_i - a)^2\right]$$

$$= \frac{1}{N}\sum\left[f_i(x_i - \bar{x} + \bar{x} - a)^2\right]$$

$$= \frac{1}{N}\sum f_i\left[(x_i - \bar{x})^2 + 2(x_i - \bar{x})(\bar{x} - a) + (\bar{x} - a)^2\right]$$

$$= \frac{1}{N}\sum f_i(x_i - \bar{x})^2 + \frac{2}{N}(\bar{x} - a)\sum f_i(x_i - \bar{x}) + \frac{(\bar{x} - a)^2}{N}\sum f_i$$

$$= \sigma^2 + 0 + d^2$$

$\therefore$ The algebraic sum of deviations from the mean is zero

or $$s^2 = \sigma^2 + d^2$$

Clearly s^2 will be least when $d = 0$ i.e., when $a = \bar{x}$.

Hence the root mean square deviation is the least when deviations are measured from the mean i.e., the standard deviation is the least possible root mean square deviation.

Property IV : The standard deviations of two sets containing n_1, and n_2 numbers are σ_1 and σ_2 respectively being measured from their respective means m_1 and m_2. If the two sets are grouped together as one of $(n_1 + n_2)$ numbers, then the standard deviation σ of this set, measured from its mean m is given by

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(m_1 - m_2)^2$$

Example 29.12 The means of two samples of sizes 50 and 100 respectively are 54.1 and 50.3; the standard deviations are 8 and 7. Find the standard deviation of the sample of size 150 by combining the two samples.

Solution : Here we have

$$n_1 = 50, n_2 = 100, m_1 = 54.1, m_2 = 50.3$$

$$\sigma_1 = 8 \text{ and } \sigma_2 = 7$$

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{(n_1 + n_2)} + \frac{n_1 n_2}{(n_1 + n_2)^2}(m_1 - m_2)^2$$

$$= \frac{(50 \times 64) + (100 \times 49)}{150} + \frac{50 \times 100}{(150)^2}(54.1 - 50.3)^2$$

$$= \frac{3200 + 4900}{150} + \frac{2}{9}(3.8)^2$$

$$= 57.21$$

$\therefore \qquad \sigma = 7.56$ (approx)

Example 29.13 Find the mean deviation (M.D) from the mean and the standard deviation (S.D) of the A.P.

$$a, a + d, a + 2d, \ldots\ldots, a + 2n.d$$

and prove that the latter is greater than the former.

Solution : The number of items in the A.P. is (2n + 1)

$\therefore \qquad \bar{x} = a + nd$

Mean deviation about the mean

$$= \frac{1}{(2n+1)} \sum_{r=0}^{2n} \left|(a + rd) - (a + nd)\right|$$

$$= \frac{1}{(2n+1)} .2\left[nd + (n-1)d + \ldots\ldots + d\right]$$

$$= \frac{2}{(2n+1)}\left[1 + 2 + \ldots. + (n-1) + n\right]d$$

$$= \frac{2n(n+1)}{(2n+1)2}.d \qquad = \frac{n(n+1)d}{(2n+1)} \qquad \ldots..(1)$$

Now $\qquad \sigma^2 = \frac{1}{(2n+1)} \sum_{r=0}^{2n} \left[(a + rd) - (a + nd)\right]^2$

$$= \frac{2d^2}{(2n+1)}\left[n^2 + (n-1)^2 + \ldots. + 2^2 + 1^2\right]$$

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$$= \frac{2d^2}{(2n+1)} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)d^2}{3}$$

$$\therefore \qquad \sigma = d\sqrt{\left(\frac{n(n+1)}{3}\right)} \qquad .....(2)$$

We have further, (2) > (1)

if $$d\sqrt{\left(\frac{n(n+1)}{3}\right)} > \frac{n(n+1)}{(2n+1)}d$$

or if $$(2n+1)^2 > 3n(n+1)$$

or if $n^2 + n + 1 > 0$, which is true for $n > 0$

Hence the result.

Example 29.14 Show that for any discrete distribution the standard deviation is not less than the mean deviation from the mean.

Solution : We are required to show that

$$\text{S.D.} \geq \text{M.D. from mean}$$

or $$(\text{S. D})^2 \geq (\text{M.D. from mean})^2$$

i.e. $$\frac{1}{N}\sum\left[f_i(x_i - \overline{x})^2\right] \geq \left[\frac{1}{N}\sum[f_i|(x_i - \overline{x})|]\right]^2$$

or $$\frac{1}{N}\sum\left[f_i d_i^2\right] \geq \left[\frac{1}{N}\sum[f_i|d_i|]\right]^2, \text{ where } d_i = x_i - \overline{x}$$

or $$N\sum\left(f_i d_i^2\right) \geq \left[\sum\{f_i|d_i|\}\right]^2$$

or $$(f_1 + f_2 + ...)\left(f_1 d_1^2 + f_2 d_2^2 + ...\right) \geq [f_1|d_1| + f_2|d_2| + ...]^2$$

or $$f_1 f_2\left(d_1^2 + d_2^2\right) + ... \geq 2f_1 f_2\, d_1 d_2 + ...$$

or $$f_1 f_2 (d_1 - d_2)^2 + ... \geq 0$$

which is true being the sum of perfect squares.

LET US SUM UP

- Range : The difference between the largest and the smallest value of the given data.

- Mean deviation from mean $= \dfrac{\sum_{i=1}^{n} \left(f_i \left| x_i - \overline{x} \right| \right)}{N}$

  where $N = \sum_{i=1}^{n} f_i$, $\qquad \overline{x} = \dfrac{1}{N} \sum_{i=1}^{n} \left(f_i x_i \right)$

- Variance $\left(\sigma^2 \right) = \dfrac{\sum_{i=1}^{n} \left(x_i - \overline{x} \right)^2}{n}$ [for raw data]

- Standard derivation $(\sigma) = +\sqrt{\dfrac{\sum_{i=1}^{n} \left(x_i - \overline{x} \right)^2}{n}}$

- Variance for grouped data

  $$\sigma_g^2 = \frac{\sum_{i=1}^{k} \left[f_i \left(x_i - \overline{x} \right)^2 \right]}{N}, \; x_i \text{ is the mid value of the class.}$$

  Also, $\sigma_x{}^2 = h^2 \sigma_u{}^2$ and $\sigma_u{}^2 = \dfrac{1}{N} \sum_{i=1}^{k} \left[f_i \left(u_i - \overline{u}^2 \right) \right]$

  $$N = \sum_{i=1}^{k} f_i$$

or $\qquad \sigma_u{}^2 = \dfrac{\sum_{i=1}^{k} \left(f_i u_i{}^2 \right) - \dfrac{\left[\sum_{i=1}^{k} \left(f_i u_i{}^2 \right) \right]^2}{N}}{N}$ where $N = \sum_{i=1}^{k} f_i$

- Standard deviation for grouped data $\sigma_g = +\sqrt{\sigma_g^2}$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. Find the mean deviation for the following data of marks obtained (out of 100) by 10 students in a test

   55 45 63 76 67 84 75 48 62 65

2. The data below presents the earnings of 50 labourers of a factory

Earnings (in Rs.)	1200	1300	1400	1500	1600	1800
No. of Labourers	4	7	15	12	7	5

   Calculate mean deviation.

3. The salary per day of 50 employees of a factory is given by the following data.

Salary (in Rs.)	20 – 30	30 – 40	40 – 50	50 – 60
No. of employees	4	6	8	12
Salary (in rupees)	60 – 70	70 – 80	80 – 90	90 – 100
No. of employees	7	6	4	3

   Calculate mean deviation.

4. Find the batting average and mean deviation for the following data of scores of 50 innings of a cricket player:

Run Scored	0 – 20	20 – 40	40 – 60	60 – 80
No. of Innings	6	10	12	18
Run scored	80 – 100	100 – 120		
No. of innings	3	1		

5. The marks of 10 students in test of Mathematics are given below:

   6 10 12 13 15 20 24 28 30 32

   Find the variance and standard deviation of the above data.

6. The following table gives the masses in grams to the nearest gram, of a sample of 10 eggs.

   46 51 48 62 54 56 58 60 71 75

   Calculate the standard deviation of the masses of this sample.

7. The weekly income (in rupees) of 50 workers of a factory are given below:

Income	400	425	450	500	550	600	650
No of workers	5	7	9	12	7	6	4

Find the variance and standard deviation of the above data.

8. Find the variance and standard deviation for the following data:

Class	$0-20$	$20-40$	$40-60$	$60-80$	$80-100$
Frequency	7	8	25	15	45

9. Find the standard deviation of the distribution in which the values of x are 1,2,......, N. The frequency of each being one.

MODULE - VI
Statistics

Notes

ANSWERS

CHECK YOUR PROGRESS 29.1

1. 15 2. 22 3. 9.4 4. 15.44

5. 13.7 6. 136 7. 5.01 8. 14.4

CHECK YOUR PROGRESS 29.2

1. Variance = 311, Standard deviation = 17.63
2. Variance = 72.9, Standard deviation = 8.5
3. Variance = 42.6, Standard deviation = 6.53
4. Standard deviation = 4
5. Variance = 13.14, Standard deviation = 3.62
6. Standard deviation = 17.6

CHECK YOUR PROGRESS 29.3

1. Variance = 734.96, Standard deviation = 27.1
2. Variance = 12.16, Standard deviation = 3.49
3. Variance = 5489 , Standard deviation = 74.09

CHECK YOUR PROGRESS 29.4

1. Variance = 2194, Standard deviation = 46.84
2. Variance = 86.5 , Standard deviation = 9.3
3. Variance = 67.08 , Standard deviation = 8.19

TERMINAL EXERCISE

1. 9.4 2. 124.48
3. 15.44 4. 52, 19.8
5. Variance = 74.8, Standard deviation = 8.6
6. 8.8
7. Variance = 5581.25,Standard deviation = 74.7
8. Variance = 840, Standard deviation = 28.9
9. Standard deviation $= \sqrt{\dfrac{N^2 - 1}{12}}$

30

RANDOM EXPERIMENTS AND EVENTS

In day-to-day life we see that before commencement of a cricket match two captains go for a toss. Tossing of a coin is an activity and getting either a 'Head' or a "Tail' are two possible outcomes. (Assuming that the coin does not stand on the edge). If we throw a die (of course fair die) the possible outcomes of this activity could be any one of its faces having numerals, namely 1,2,3,4,5 and 6..... at the top face.

An activity that yields a result or an outcome is called an experiment. Normally there are variety of outcomes of an experiment and it is a matter of chance as to which one of these occurs when an experiment is performed. In this lesson, we propose to study various experiments and their outcomes.

OBJECTIVES

After studying this lesson, you will be able to :

- explain the meaning of a random experiments and cite examples thereof;
- explain the role of chance in such random experiments;
- define a sample space corresponding to an experiment;
- write a sample space corresponding to a given experiment; and
- differentiate between various types of events such as equally likely, mutually exclusive, exhaustive, independent and dependent events.

EXPECTED BACKGROUND KNOWLEDGE

- Basic concepts of probability

30.1 RANDOM EXPERIMENT

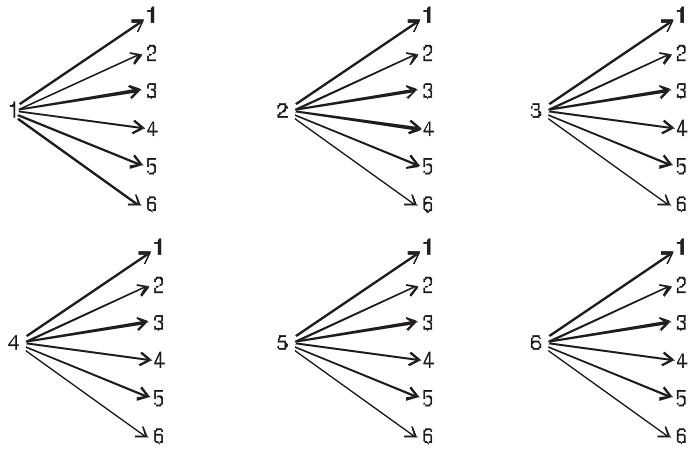
Let us consider the following activities :

(i) Toss a coin and note the outcomes. There are two possible outcomes, either a head (H) or a tail (T).

(ii) In throwing a fair die, there are six possible outcomes, that is, any one of the six faces 1,2,...... 6.... may come on top.

(iii) Toss two coins simultaneously and note down the possible outcomes. There are four possible outcomes, HH,HT,TH,TT.

(iv) Throw two dice and there are 36 possible outcomes which are represented as below :



i.e. outcomes are (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)

$$(2,1),(2,2),....,(2,6)$$
$$\vdots \quad \vdots \quad \vdots$$
$$(6,1),(6,2),....,(6,6)$$

Each of the above mentioned activities fulfil the following two conditions.

(a) The activity can be repeated number of times under identical conditions.

(b) Outcome of an activity is not predictable beforehand, since the chance play a role and each outcome has the same chance of being selection. Thus, due to the chance playing a role, an activity is

(i) repeated under identical conditions, and

(ii) whose outcome is not predictable beforehand is called a random experiment.

Example 30.1 Is drawing a card from well shuffled deck of cards, a random experiment ?

Solution :

(a) The experiment can be repeated, as the deck of cards can be shuffled every time before drawing a card.

(b) Any of the 52 cards can be drawn and hence the outcome is not predictable beforehand.

Hence, this is a random experiment.

Example 30.2 Selecting a chair from 100 chairs without preference is a random experiment. Justify.

Solution :

(a) The experiment can be repeated under identical conditions.

(b) As the selection of the chair is without preference, every chair has equal chances of selection. Hence, the outcome is not predictable beforehand. Thus, it is a random experiment.

Can you think of any other activities which are not random in nature.

Let us consider some activities which are not random experiments.

(i) Birth of Manish : Obviously this activity, that is, the birth of an individual is not repeatable and hence is not a random experiment.

(ii) Multiplying 4 and 8 on a calculator.

Although this activity can be repeated under identical conditions, the outcome is always 32. Hence, the activity is not a random experiment.

30.2 SAMPLE SPACE

We throw a die once, what are possible outcomes ? Clearly, a die can fall with any of its faces at the top. The number on each of the faces is, therefore, a possible outcome. We write the set S of all possible outcomes as

$$S = \{1, 2, 3, 4, 5, 6\}$$

Again, if we toss a coin, the possible outcomes for this experiment are either a head or a tail. We write the set S of all possible outcomes as

$$S = \{H, T\}.$$

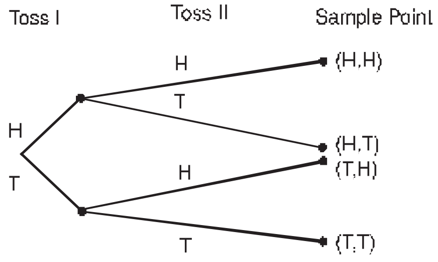
The set S associated with an experiment satisfying the following properties :

(i) each element of S denotes a possible outcome of the experiment.

(ii) any trial results in an outcome that corresponds to one and only one element of the set S is called the sample space of the experiment and the elements are called sample points. Sample space is generally denoted by S.

Example 30.3 Write the sample space in two tosses of a coin.

Solution : Let H denote a head and T denote a tail in the experiment of tossing of a coin.



$$S = \{ (H, H), (H, T), (T, H), (T, T) \}.$$

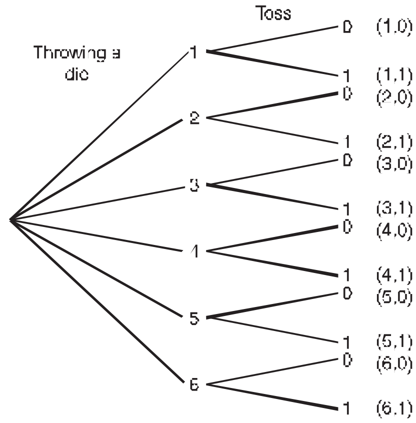
Note : If two coins are tossed simultaneously then the sample space S can be written as

$$S = \{ H H, H T, T H, T T\}.$$

Example 30.4 Consider an experiment of rolling a fair die and then tossing a coin.

Write the sample space.

Solution : In rolling a die possible outcomes are 1, 2, 3, 4, 5 and 6. On tossing a coin the possible outcomes are either a head or a tail. Let H (head) = 0 and T (tail) = 1.

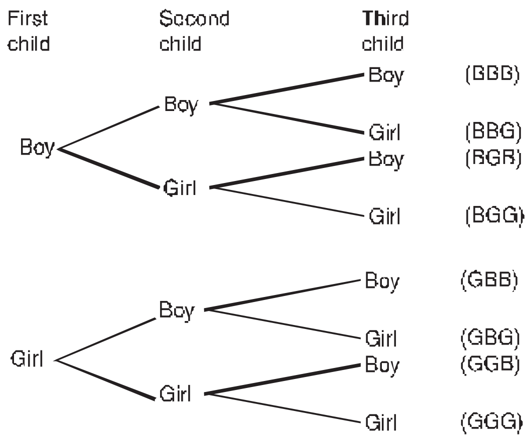


$$S = \{(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1), (4, 0), (4, 1), (5, 0), (5, 1), (6, 0), (6, 1)\}$$

$\therefore\ n(S) = 6 \times 2 = 12$

Example 30.5 Suppose we take all the different families with exactly 3 children. The experiment consists in asking them the sex (or genders) of the first, second and third chid. Write down the sample space.

Solution : Let us write 'B' for boy and 'G' for girl and construct the following tree diagram.



The sample space is

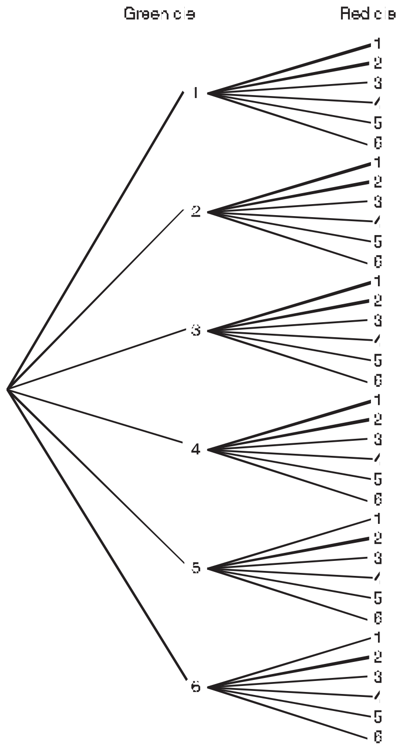
$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$

The advantage of writing the sample space in the above form is that a question such as "Was the second child a girl" ? or " How many families have first child a boy ?" and so forth can be answered immediately.

$$n(S) = 2 \times 2 \times 2 = 8$$

Example 30.6 Consider an experiment in which one die is green and the other is red. When these two dice are rolled, what will be the sample space ?

Solution : This experiment can be displayed in the form of a tree diagram, as shown below :



Let g_i and r_j denote, the number that comes up on the green die and red die respectively. Then an out-come can be represented by an ordered pair (g_i, r_j), where i and j can assume any of the values 1, 2, 3, 4, 5, 6.

Thus, a sample space S for this experiment is the set

$$S = \{(g_i, r_j) : 1 \le i \le 6, 1 \le j \le 6\}.$$

Also, notice that the multiplication principle (principle of counting) shows that the number of elements in S is 36, since there are 6 choices for g and 6 choices for r, and $6 \times 6 = 36$

$$\therefore n(S) = 36$$

Example 30.7 Write the sample space for each of the following experiments :

(i) A coin is tossed three times and the result at each toss is noted.

(ii) From five players A, B, C, D and E , two players are selected for a match.

(iii) Six seeds are sown and the number of seeds germinating is noted.

(iv) A coin is tossed twice. If the second throw results in a head, a die is thrown, otherwise a coin is tossed.

Solution :

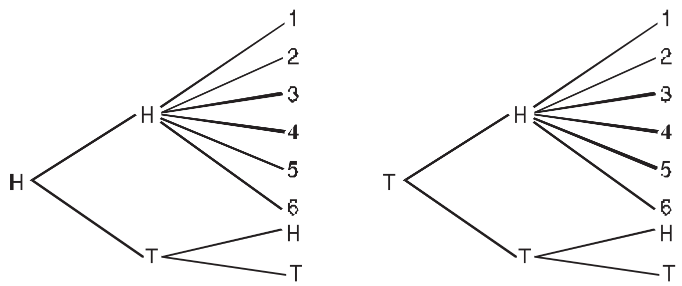
(i) S = { TTT, TTH, THT, HTT, HHT, HTH, THH, HHH}

number of elements in the sample space is $2 \times 2 \times 2 = 8$

(ii) S= { AB, AC, AD, AE, BC, BD, BE, CD, CE, DE}. Here n (S) = 10

(iii) S = { 0, 1, 2, 3, 4, 5, 6}. Here n (S) = 7

(iv) This experiment can be displayed in the form of a tree-diagram as shown below :



Thus S = {HH1, HH2, HH3, HH4, HH5, HH6, HTH, HTT, TH1, TH2,TH3,TH4, TH5, TH6, TTH, TTT}

i.e. there are 16 outcomes of this experiment.

30.3. DEFINITION OF VARIOUS TERMS

Event : Let us consider the example of tossing a coin. In this experiment, we may be interested in 'getting a head'. Then the outcome 'head' is an event.

In an experiment of throwing a die, our interest may be in, 'getting an even number'. Then the outcomes 2, 4 or 6 constitute the event. We have seen that an experiment which, though repeated under identical conditions, does not give unique results but may result in any one of the several possible outcomes, which constitute the sample space.

Some outcomes of the sample space satisfy a specified description, which we call an 'event'.

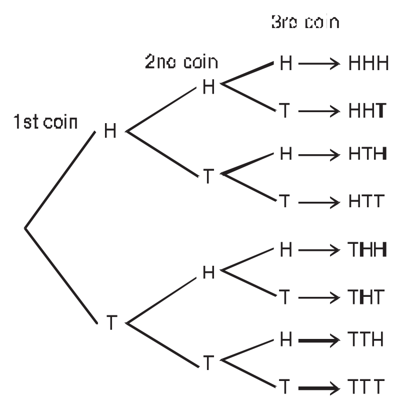
We often use the capital letters A, B, C etc. to represent the events.

Example 30.8 Let E denote the experiment of tossing three coins at a time. List all possible outcomes and the events that

(i) the number of heads exceeds the number of tails.

(ii) getting two heads.

Solution :



The sample space S is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$
$$= \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\} \text{ (say)}$$

If E_1 is the event that the number of heads exceeds the number of tails, and E_2 the event getting two heads. Then

$$E_1 = \{ w_1 , w_2 , w_3 , w_5 \}$$

and $$E_2 = \{ w_2 , w_3 , w_5 \}$$

30.3.1 Equally Likely Events

Outcomes of a trial are said to be equally likely if taking into consideration all the relevant evidences there is no reason to expect one in preference to the other.

Examples :

(i) In tossing an unbiased coin, getting head or tail are equally likely events.

(ii) In throwing a fair die, all the six faces are equally likely to come.

(iii) In drawing a card from a well shuffled deck of 52 cards, all the 52 cards are equally likely to come.

30.3.2 Mutually Exclusive Events

Events are said to be mutually exclusive if the happening of any one of the them preludes the happening of all others, i.e., if no two or more of them can happen simultaneously in the same trial.

Examples :

(i) In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive. If any one of these faces comes at the top, the possibility of others, in the same trial is ruled out.

(ii) When two coins are tossed, the event that both should come up tails and the event that there must be at least one head are mutually exclusive.

Mathematically events are said to be mutually exclusive if their intersection is a null set (i.e., empty)

30.3.3 Exhaustive Events

If we have a collection of events with the property that no matter what the outcome of the experiment, one of the events in the collection must occur, then we say that the events in the collection are exhaustive events.

For example, when a die is rolled, the event of getting an even number and the event of getting an odd number are exhaustive events. Or when two coins are tossed the event that at least one head will come up and the event that at least one tail will come up are exhaustive events.

Mathematically a collection of events is said to be exhaustive if the union of these events is the complete sample space.

30.3.4 Independent and Dependent Events

A set of events is said to be independent if the happening of any one of the events does not affect the happening of others. If, on the other hand, the happening of any one of the events influence the happening of the other, the events are said to be dependent.

Examples :

(i) In tossing an unbiased coin the event of getting a head in the first toss is independent of getting a head in the second, third and subsequent throws.

(ii) If we draw a card from a pack of well shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But, however, if the first card drawn is not replaced then the second card is dependent on the first draw (in the sense that it cannot be the card drawn the first time).

CHECK YOUR PROGRESS 30.1

1. Selecting a student from a school without preference is a random experiment. Justify.
2. Adding two numbers on a calculator is not a random experiment. Justify.
3. Write the sample space of tossing three coins at a time.
4. Write the sample space of tossing a coin and a die.
5. Two dice are thrown simultaneously, and we are interested to get six on top of each of the die. Are the two events mutually exclusive or not ?
6. Two dice are thrown simultaneously. The events A, B, C, D are as below :

   A : Getting an even number on the first die.

   B : Getting an odd number on the first die.

   C : Getting the sum of the number on the dice < 7.

   D : Getting the sum of the number on the dice > 7.

   State whether the following statements are True or False.

   (i) A and B are mutually exclusive.

   (ii) A and B are mutually exclusive and exhaustive.

   (iii) A and C are mutually exclusive.

   (iv) C and D are mutually exclusive and exhaustive.
7. A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. There will be how many sample points, in its sample space?
8. In a single rolling with two dice, write the sample space and its elements.
9. Suppose we take all the different families with exactly 2 children. The experiment consists in asking them the sex of the first and second child.

   Write down the sample space.

LET US SUM UP

- An activity that yields a result or an outcome is called an experiment.
- An activity repeated number of times under identical conditions and outcome of activity is not predictable is called Random Experiment.

- The set of possible outcomes of a random experiment is called sample space and elements of the set are called sample points.
- Some outcomes of the sample space satisfy a specified description, which is called an Event.
- Events are said to be Equally likely, when we have no preference for one rather than the other.
- If happening of an event prevents the happening of another event, then they are called Mutually Exclusive Events.
- The total number of possible outcomes in any trial is known as Exhaustive Events.
- A set of events is said to be Independent events, if the happening of any one of the events does not effect the happening of other events, otherwise they are called dependent events.

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. A tea set has four cups and saucers. If the cups are placed at random on the saucers, write the sample space.
2. If four coins are tossed, write the sample space.
3. If n coins are tossed simultaneously, there will be how many sample points ?

   [Hint : try for n = 1,2, 3,4, ...]
4. In a single throw of two dice, how many sample points are there ?

ANSWERS

CHECK YOUR PROGRESS 30.1

1. Both properties are satisfied   2. Outcome is predictable

3. $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

4. $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T4, T6\}$ 5. No.

6. (i) True (ii) True (ii) False (iv) True   7. 15

8.
$$\begin{aligned}\{ & (1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\ & (2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\ & (3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\ & (4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\\ & (5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\\ & (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$

9. {MM, MF, FM, FF}

TERMINAL EXERCISE

1.
$$\begin{aligned}\{ & C_1S_1, C_1S_2, C_1S_3, C_1S_4, C_2S_1, C_2S_2, C_2S_3, C_2S_4,\\ & C_3S_1, C_3S_2, C_3S_3, C_3S_4, C_4S_1, C_4S_2, C_4S_3, C_4S_4\}\end{aligned}$$

2. $2^4 = 16$, { HHHH, HHHT, HHTH, HTHH, HHTT, HTHT, HTTH, HTTT,
THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

3. 2^n 4. $6^2 = 36$

31

PROBABILITY

In our daily life, we often used phrases such as 'It may rain today', or 'India may win the match' or ' I may be selected for this post.' These phrases involve an element of uncertainty. How can we measure this uncertainty ? A measure of this uncertainty is provided by a branch of Mathematics, called the theory of probability. Probability Theory is designed to measure the degree of uncertainty regarding the happening of a given event. The dictionary meaning of probability is ' likely though not certain to occur. Thus, when a coin is tossed, a head is likely to occur but may not occur. Similarly, when a die is thrown, it may or may not show the number 6.

In this lesson, we shall discuss some basic concepts of probability, addition and multiplication theorem of probability and applications of probability in our day to day life.

OBJECTIVES

After studying this lesson, you will be able to :

- define probability of occurance of an event;
- cite through examples that probability of occurance of an event is a non-negative fraction, not greater than one;
- use permutation and combinations in solving problems in probability;
- state and establish the addition theorems on probability and the conditions under which each holds;
- generalize the addition theorem of probability for mutually exclusive events;
- state the multiplication theorem on probability for any two events;
- state and prove the multiplication theorem on compound probability for independent events;
- solve problems on probability using addition and multiplication theorems;
- define conditional probability of an event; and
- solve problems involving conditional probability.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of random experiments and events.

MODULE - VI
Statistics

Notes

- The meaning of sample space.
- A standard deck of playing cards consists of 52 cards divided into 4 suits of 13 cards each : spades, hearts, diamonds, clubs and cards in each suit are - ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3 and 2. Kings, Queens and Jacks are called face cards and the other cards are called number cards.

31.1 EVENTS AND THEIR PROBABILITY

In the previous lesson, we have learnt whether an activity is a random experiment or not. The study of probability always refers to random experiments. Hence, from now onwards, the word experiment will be used for a random experiment only. In the preceeding lesson, we have defined different types of events such as equally likely, mutually exclusive, exhaustive, independent and dependent events and cited examples of the above mentioned events.

Here we are interested in the chance that a particular event will occur, when an experiment is performed. Let us consider some examples.

What are the chances of getting a ' Head' in tossing an unbiased coin ? There are only two equally likely outcomes, namely head and tail. In our day to day language, we say that the coin has chance 1 in 2 of showing up a head. In technical language, we say that the probability of getting a head is $\frac{1}{2}$.

Similarly, in the experiment of rolling a die, there are six equally likely outcomes 1, 2,3,4,5 or 6. The face with number '1' (say) has chance 1 in 6 of appearing on the top. Thus, we say that the probability of getting 1 is $\frac{1}{6}$.

In the above experiment, suppose we are interested in finding the probability of getting even number on the top, when a die is rolled. Clearly, the numbers possible are 2, 4 and 6 and the chance of getting an even number is 3 in 6. Thus, we say that the probability of getting an even number is $\frac{3}{6}$, i.e., $\frac{1}{2}$.

The above discussion suggests the following definition of probability.

If an experiment with 'n' exhaustive, mutually exclusive and equally likely outcomes, m outcomes are favourable to the happening of an event A, the probability 'p' of happening of A is given by

$$p = \mathrm{P}(\mathrm{A}) = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \frac{m}{n} \qquad \text{....(i)}$$

Since the number of cases favourable to the non-happening of the event A are $n - m$, the probability 'q' that 'A' will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n}$$

$$= 1 - p \quad \text{[Using (i)]}$$

$$\therefore \qquad p + q = 1.$$

Obviously, p as well as q are non-negative and cannot exceed unity.

i.e., $0 \le p \le 1, \quad 0 \le q \le 1$

Thus, the probability of occurrence of an event lies between 0 and 1[including 0 and 1].

Remarks

1. Probability 'p' of the happening of an event is known as the probability of success and the probability 'q' of the non-happening of the event as the probability of failure.
2. Probability of an impossible event is 0 and that of a sure event is 1

   if P (A) = 1, the event A is certainly going to happen and

   if P (A) = 0, the event is certainly not going to happen.
3. The number (m) of favourable outcomes to an event cannot be greater than the total number of outcomes (n).

Let us consider some examples

Example 31.1 A die is rolled once. Find the probability of getting a 5.

Solution : There are six possible ways in which a die can fall, out of these only one is favourable to the event.

$$\therefore \qquad P(5) = \frac{1}{6}.$$

Example 31.2 A coin is tossed once. What is the probability of the coin coming up with head ?

Solution : The coin can come up either 'head' (H) or a tail (T). Thus, the total possible outcomes are two and one is favourable to the event.

$$\text{So,} \qquad P(H) = \frac{1}{2}$$

Example 31. 3 A die is rolled once. What is the probability of getting a prime number ?

Solution : There are six possible outcomes in a single throw of a die. Out of these; 2, 3 and 5 are the favourable cases.

$$\therefore \qquad P(\text{Prime Number}) = \frac{3}{6} = \frac{1}{2}$$

Example 31.4 A die is rolled once. What is the probability of the number '7' coming up ? What is the probability of a number 'less than 7' coming up ?

Solution : There are six possible outcomes in a single throw of a die and there is no face of the die with mark 7.

$$\therefore \qquad P(\text{number } 7) = \frac{0}{6} = 0$$

[**Note :** That the probability of impossible event is zero]

As every face of a die is marked with a number less than 7,

$$\therefore \qquad P(\leq 7) = \frac{6}{6} = 1$$

[**Note :** That the probability of an event that is certain to happen is 1]

Example 31.5 In a simultaneous toss of two coins, find the probability of

(i) getting 2 heads (ii) exactly 1 head

Solution : Here, the possible outcomes are

HH, HT, TH, TT.

i.e., Total number of possible outcomes = 4.

(i) Number of outcomes favourable to the event (2 heads) = 1 (i.e., HH).

$$\therefore \qquad P(\text{ 2 heads }) = \frac{1}{4}.$$

(ii) Now the event consisting of exactly one head has two favourable cases, namely HT and TH .

$$\therefore \qquad P(\text{ exactly one head }) = \frac{2}{4} = \frac{1}{2}.$$

Example 31.6 In a single throw of two dice, what is the probability that the sum is 9?

Solution : The number of possible outcomes is $6 \times 6 = 36$. We write them as given below :

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Now, how do we get a total of 9. We have :

$$3 + 6 = 9$$
$$4 + 5 = 9$$
$$5 + 4 = 9$$
$$6 + 3 = 9$$

In other words, the outcomes (3, 6), (4, 5), (5, 4) and (6, 3) are favourable to the said event, i.e., the number of favourable outcomes is 4.

Hence, $P(\text{a total of 9 }) = \frac{4}{36} = \frac{1}{9}$

Example 31.7 From a bag containing 10 red, 4 blue and 6 black balls, a ball is drawn at random. What is the probability of drawing

(i) a red ball ? (ii) a blue ball ? (iii) not a black ball ?

Solution : There are 20 balls in all. So, the total number of possible outcomes is 20. (Random drawing of balls ensure equally likely outcomes)

(i) Number of red balls = 10

$\therefore \quad P(\text{a red ball}) = \frac{10}{20} = \frac{1}{2}$

(ii) Number of blue balls = 4

$\therefore \quad P(\text{a blue ball}) = \frac{4}{20} = \frac{1}{5}$

(iii) Number of balls which are not black = 10 + 4

= 14

$\therefore \quad P(\text{not a black ball}) = \frac{14}{20} = \frac{7}{10}$

Example 31.8 A card is drawn at random from a well shuffled deck of 52 cards. If A is the event of getting a queen and B is the event of getting a card bearing a number greater than 4 but less than 10, find P(A) and P (B).

Solution : Well shuffled pack of cards ensures equally likely outcomes.

$\therefore$ the total number of possible outcomes is 52.

(i) There are 4 queens in a pack of cards.

$\therefore \quad P(A) = \frac{4}{52} = \frac{1}{13}$

(ii) The cards bearing a number greater than 4 but less than 10 are 5,6, 7,8 and 9.

Each card bearing any of the above number is of 4 suits diamond, spade, club or heart.

Thus, the number of favourable outcomes = 5 × 4 = 20

$\therefore \quad = \frac{20}{52} = \frac{5}{13}$

Example 31.9 What is the chance that a leap year, selected at random, will contain 53 Sundays?

Solution : A leap year consists of 366 days consisting of 52 weeks and 2 extra days. These two extra days can occur in the following possible ways.

(i) Sunday and Monday

(ii) Monday and Tuesday

(iii) Tuesday and Wednesday

(iv) Wednesday and Thursday

(v) Thursday and Friday

(vi) Friday and Saturday

(vii) Saturday and Sunday

Out of the above seven possibilities, two outcomes,
e.g., (i) and (vii), are favourable to the event

$$\therefore \qquad P\,(\,53 \text{ Sundays}\,) = \frac{2}{7}$$

CHECK YOUR PROGRESS 31.1

1. A die is rolled once. Find the probability of getting 3.
2. A coin is tossed once. What is the probability of getting the tail ?
3. What is the probability of the die coming up with a number greater than 3 ?
4. In a simultaneous toss of two coins, find the probability of getting ' at least' one tail.
5. From a bag containing 15 red and 10 blue balls, a ball is drawn 'at random'. What is the probability of drawing (i) a red ball ? (ii) a blue ball ?
6. If two dice are thrown, what is the probability that the sum is (i) 6 ? (ii) 8? (iii) 10? (iv) 12?
7. If two dice are thrown, what is the probability that the sum of the numbers on the two faces is divisible by 3 or by 4 ?
8. If two dice are thrown, what is the probability that the sum of the numbers on the two faces is greater than 10 ?
9. What is the probability of getting a red card from a well shuffled deck of 52 cards ?
10. If a card is selected from a well shuffled deck of 52 cards, what is the probability of drawing

    (i) a spade ? (ii) a king ? (iii) a king of spade ?
11. A pair of dice are thrown. Find the probability of getting

    (i) a sum as a prime number

    (ii) a doublet, i.e., the same number on both dice

    (iii) a multiple of 2 on one die and a multiple of 3 on the other.
12. Three coins are tossed simultaneously. Find the probability of getting

    (i) no head (ii) at least one head (iii) all heads

31.2. CALCULATION OF PROBABILITY USING COMBINATORICS (PERMUTATIONS AND COMBINATIONS)

In the preceding section, we calculated the probability of an event by listing down all the possible outcomes and the outcomes favourable to the event. This is possible when the number of outcomes is small, otherwise it becomes difficult and time consuming process. In general, we do not require the actual listing of the outcomes, but require only the total number of possible outcomes and the number of outcomes favourable to the event. In many cases, these can be found by applying the knowledge of permutations and combinations, which you have already studied.

Let us consider the following examples :

Example 31.10 A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and blue ?

Solution : Total number of balls = 3 + 6 + 7 = 16

Now, out of 16 balls, 2 can be drawn in ${}^{16}C_2$ ways.

$\therefore$ Exhaustive number of cases = ${}^{16}C_2 = \dfrac{16 \times 15}{2} = 120$

Out of 6 white balls, 1 ball can be drawn in ${}^{6}C_1$ ways and out of 7 blue balls, one can be drawn is ${}^{7}C_1$ ways. Since each of the former case is associated with each of the later case, therefore total number of favourable cases are ${}^{6}C_1 \times {}^{7}C_1 = 6 \times 7 = 42$.

$\therefore$ Required probability $= \dfrac{42}{120} = \dfrac{7}{20}$

Remarks

When two or more balls are drawn from a bag containing several balls, there are two ways in which these balls can be drawn.

(i) Without replacement : The ball first drawn is not put back in the bag, when the second ball is drawn. The third ball is also drawn without putting back the balls drawn earlier and so on. Obviously, the case of drawing the balls without replacement is the same as drawing them together.

(ii) With replacement : In this case, the ball drawn is put back in the bag before drawing the next ball. Here the number of balls in the bag remains the same, every time a ball is drawn.

In these types of problems, unless stated otherwise, we consider the problem of without replacement.

Example 31.11 Find the probability of getting both red balls, when from a bag containing 5 red and 4 black balls, two balls are drawn,

(i) with replacement.

(ii) without replacement.

Solution : (i) Total number of balls in the bag in both the draws = 5 + 4 = 9

Hence, by fundamental principle of counting, the total number of possible outcomes = 9 × 9 = 81.

Similarly, the number of favourable cases = 5 × 5 = 25.

Hence, probability (both red balls) $= \dfrac{25}{81}$.

(ii) Total number of possible outcomes is equal to the number of ways of selecting 2 balls out of 9 balls = ${}^{9}C_2$.

Number of favourable cases is equal to the number of ways of selecting

2 balls out of 5 red balls = ${}^{5}C_2$.

Hence, P (both red balls) = $\dfrac{{}^{5}C_2}{{}^{9}C_2} = \dfrac{\dfrac{5\times 4}{1\times 2}}{\dfrac{9\times 8}{1\times 2}} = \dfrac{5}{18}$

Example 31.12 Six cards are drawn at random from a pack of 52 cards. What is the probability that 3 will be red and 3 black?

Solution : Six cards can be drawn from the pack of 52 cards in ${}^{52}C_6$ ways.

i.e., Total number of possible outcomes = ${}^{52}C_6$

3 red cards can be drawn in ${}^{26}C_3$ ways and

3 black cards can be drawn in ${}^{26}C_3$ ways.

$\therefore$ Total number of favourable cases = ${}^{26}C_3 \times {}^{26}C_3$

Hence, the required probability = $\dfrac{{}^{26}C_3 \times {}^{26}C_3}{{}^{52}C_6} = \dfrac{13000}{39151}$

Example 31.13 Four persons are chosen at random from a group of 3 men, 2 women and 4 children. Show that the chance that exactly two of them will be children is $\dfrac{10}{21}$.

Solution : Total number of persons in the group = 3 + 2 + 4 = 9. Four persons are chosen at random. If two of the chosen persons are children, then the remaining two can be chosen from 5 persons (3 men + 2 women).

Number of ways in which 2 children can be selected from 4

children = ${}^{4}C_2 = \dfrac{4\times 3}{1\times 2} = 6$

Number of ways in which remaining of the two persons can be selected

from 5 persons = ${}^{5}C_2 = \dfrac{5\times 4}{1\times 2} = 10$

Total number of ways in which 4 persons can be selected out of

9 persons = ${}^{9}C_4 = \dfrac{9\times 8\times 7\times 6}{1\times 2\times 3\times 4} = 126$

Hence, the required probability = $\dfrac{{}^{4}C_2 \times {}^{5}C_2}{{}^{9}C_4} = \dfrac{6\times 10}{126} = \dfrac{10}{21}$

Example 31.14 Three cards are drawn from a well-shuffled pack of 52 cards. Find the probability that they are a king, a queen and a jack.

Solution : From a pack of 52 cards, 3 cards can be drawn in ${}^{52}C_3$ ways, all being equally likely.

$\therefore$ Exhaustive number of cases = ${}^{52}C_3$

A pack of cards contains 4 kings, 4 queens and 4 jacks .A king, a queen and a Jack can each be drawn in ${}^{4}C_1$ ways and since each way of drawing a king can be associated with each of the ways of drawing a queen and a jack, the total number of favourable cases = ${}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1$

$$\therefore \quad \text{Required probability} = \frac{{}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1}{{}^{52}C_3}$$

$$= \frac{4 \times 4 \times 4}{\frac{52 \times 51 \times 50}{1 \times 2 \times 3}}$$

$$= \frac{16}{5525}$$

Example 31.15 From 25 tickets, marked with the first 25 numerals, one is drawn at random. Find the probability that it is a multiple of 5.

Solution : Numbers (out of the first 25 numerals) which are multiples of 5 are 5, 10, 15, 20 and 25, i.e., 5 in all. Hence, required favourable cases are = 5.

$\therefore$ Required probability = $\frac{5}{25} = \frac{1}{5}$

Example 31.16 If the letters of the word 'REGULATIONS' be arranged at random, what is the probability that there will be exactly 4 letters between R and E?

Solution : The word 'REGULATIONS' consists of 11 letters. The two letters R and E can occupy ${}^{11}P_2$, i.e., $11 \times 10 = 110$ positions.

The number of ways in which there will be exactly 4 letters between R and E are enumerated below :

R is in the first place and E in the 6th place.
R is in the 2nd place and E in the 7th place
------- ------- ------- -------
------- ------- ------- -------
------- ------- ------- -------
R is in the 6th place and E is in the 11th place.

Since R and E can interchange their positions, the required number of favourable cases is $2 \times 6 = 12$.

$\therefore$ The required probability $= \dfrac{12}{110} = \dfrac{6}{55}$

Example 31.17 Out of (2n + 1) tickets consecutively numbered starting with 1, three are drawn at random. Find the chance that the numbers on them are in A.P.

Solution : Since out of (2n + 1) tickets, 3 tickets can be drawn in ${}^{2n+1}C_3$ ways,

Therefore, exhaustive number of cases = ${}^{2n+1}C_3$

$$= \frac{(2n+1)2n(2n-1)}{1 \times 2 \times 3}$$

$$= \frac{n\left(4n^2 - 1\right)}{3}$$

To find the favourable number of cases, we are to enumerate all the cases in which the number on the drawn tickets are in A.P. with common difference (say, d = 1, 2, 3, ..., n – 1, n)

If d = 1, possible cases are as follows :

$$\left.\begin{array}{l} 1,\ 2,\ 3 \\ 2,\ 3,\ 4 \\ : \\ 2n-1,\quad 2n,\ 2n+1 \end{array}\right\}, \text{ i.e., } (2n-1) \text{ cases in all.}$$

If d = 2, the possible cases are as follows :

$$\left.\begin{array}{l} 1,\ 3,\ 5 \\ 2,\ 4,\ 6 \\ : \\ 2n-3,\quad 2n-1,\ 2n+1 \end{array}\right\}, (2n-3) \text{ cases in all.}$$

and so on

If d = n – 1, the possible cases are

$$\left.\begin{array}{l} 1,\ n,\quad 2n-1 \\ 2,\ n+1,\ 2n \\ 3,\ n+2,\quad 2n+1 \end{array}\right\}, \text{ i.e., 3 cases in all.}$$

If d = n, there is only one case, viz., 1, n + 1, 2n + 1

Hence, total number of favourable cases

$= (2n - 1) + (2n - 3) + ... + 5 + 3 + 1$

$= 1 + 3 + 5 + ... + (2n - 1)$

which is a series in A.P. with d = 2 and n terms.

$\therefore$ Number of favourable cases $= \frac{n}{2}[2 + (n-1)2]$

$$= \frac{n}{2}(2n) = n^2$$

Hence, required probability $= \dfrac{n^2}{\frac{n(4n^2-1)}{3}} = \dfrac{3n}{4n^2-1}$

CHECK YOUR PROGRESS 31.2

1. A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn at random are both white?

2. A bag contains 5 red and 8 blue balls. What is the probability that two balls drawn are red and blue ?

3. A bag contains 20 white and 30 black balls. Find the probability of getting 2 white balls, when two balls are drawn at random

   (a) with replacement (b) without replacement

4. Three cards are drawn from a well-shuffled pack of 52 cards. Find the probability that all the three cards are jacks.

5. Two cards are drawn from a well-shuffled pack of 52 cards. Show that the chances of drawing both aces is $\frac{1}{221}$.

6. In a group of 10 outstanding students in a school, there are 6 boys and 4 girls. Three students are to be selected out of these at random for a debate competition. Find the probability that

   (i) one is boy and two are girls.

   (ii) all are boys.

   (iii) all are girls.

7. Out of 21 tickets marked with numbers from 1 to 21, three are drawn at random. Find the probability that the numbers on them are in A.P.

8. Two cards are drawn at random from 8 cards numbered 1 to 8. What is the probability that the sum of the numbers is odd, if the cards are drawn together ?

9. A team of 5 players is to be selected from a group of 6 boys and 8 girls. If the selection is made randomly, find the probability that there are 2 boys and 3 girls in the team.

10. An integer is chosen at random from the first 200 positive integers.Find the probability that the integer is divisible by 6 or 8.

MODULE - VI
Statistics

Notes

31.3 EVENT RELATIONS

Let us consider the example of throwing a fair die. The sample space of this experiment is

$$S = \{ 1, 2, 3, 4, 5, 6 \}$$

If A be the event of getting an even number, then the sample points 2, 4 and 6 are favourable to the event A.

The remaining sample points 1, 3 and 5 are not favourable to the event A. Therefore, these will occur when the event A will not occur.

In an experiment, the outcomes which are not favourable to the event A are called complement of A and defined as follows :

' The outcomes favourable to the complement of an event A consists of all those outcomes which are not favourable to the event A, and are denoted by 'not' A or by $\overline{A}$.

31.3.1 Event 'A or B'

Let us consider the example of throwing a die. A is an event of getting a multiple of 2 and B be another event of getting a multiple of 3.

The outcomes 2, 4 and 6 are favourable to the event A and the outcomes 3 and 6 are favourable to the event B.

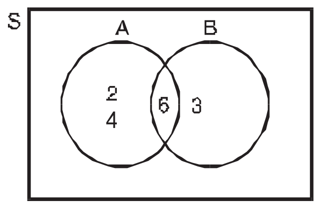


Fig. 31.1

The happening of event A or B is

$$A \cup B = \{ 2, 3, 4, 6 \}$$

Again, if A be the event of getting an even number and B is another event of getting an odd number, then

$A = \{ 2, 4, 6 \}$, $B = \{ 1, 3, 5 \}$

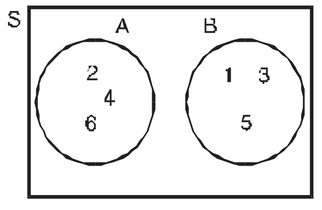


Fig. 31.2

$A \cup B = \{1, 2, 3, 4, 5, 6 \}$

Here, it may be observed that if A and B are two events, then the event ' A or B ' ($A \cup B$) will consist of the outcomes which are either favourable to the event A or to the event B or to both the events.

Thus, the event 'A or B' occurs, if either A or B or both occur.

31.3.2 Event "A and B'

Recall the example of throwing a die in which A is the event of getting a multiple of 2 and B is the event of getting a multiple of 3. The outcomes favourable to A are 2, 4, 6 and the outcomes favourable to B are 3 , 6.

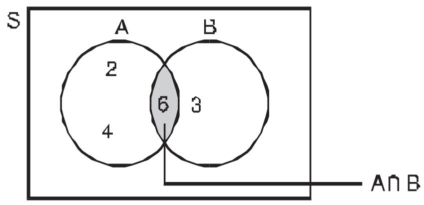


Fig. 31.3

Here, we observe that the outcome 6 is favourable to both the events A and B.

Draw a card from a well shuffled deck of 52 cards. A and B are two events defined as

A : a red card

B : a king

We know that there are 26 red cards and 4 kings in a deck of cards. Out of these 4 kings, two are red.

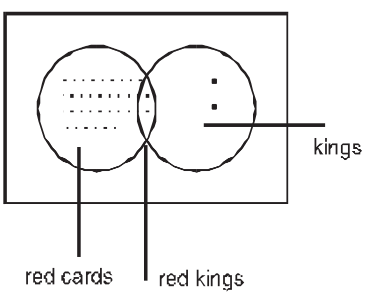


Fig. 31.4

Here, we see that the two red kings are favourable to both the events.

Hence, the event ' A and B' consists of all those outcomes which are favourable to both the events A and B. That is, the event 'A and B' occurs, when both the events A and B occur simultaneously. Symbolically, it is denoted as $A \cap B$.

31.4 ADDITIVE LAW OF PROBABILITY

Let A be the event of getting an odd number and B be the event of getting a prime number in a single throw of a die. What will be the probability that it is either an odd number or a prime number ?

In a single throw of a die, the sample space would be

$$S = \{ 1, 2, 3, 4, 5, 6 \}$$

The outcomes favourable to the events A and B are

$$A = \{ 1, 3, 5 \}$$

$B = \{ 2, 3, 5 \}$

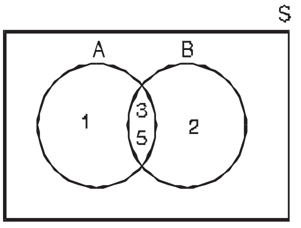


Fig. 31.5

The outcomes favourable to the event 'A or B' are

$$A \cup B = \{1, 2, 3, 5\}.$$

Thus, the probability of getting either an odd number or a prime number will be

$$P(A \text{ or } B) = \frac{4}{6} = \frac{2}{3}$$

To discover an alternate method, we can proceed as follows :

The outcomes favourable to the event A are 1, 3 and 5.

$$\therefore \qquad P(A) = \frac{3}{6}$$

Similarly, $\qquad P(B) = \frac{3}{6}$

The outcomes favourable to the event 'A and B' are 3 and 5.

$$\therefore \qquad P(A \text{ and } B) = \frac{2}{6}$$

Now, $\quad P(A) + P(B) - P(A \text{ and } B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6}$

$$\frac{4}{6} = \frac{2}{3}$$

$$= P(A \text{ or } B)$$

Thus, we state the following law, called additive rule, which provides a technique for finding the probability of the union of two events, when they are not disjoint.

For any two events A and B of a sample space S,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

or $\qquad P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad$ .....(ii)

Example 31.18 A card is drawn from a well-shuffled deck of 52 cards. What is the prob ability that it is either a spade or a king ?

Solution : If a card is drawn at random from a well-shuffled deck of cards, the likelyhood of any of the 52 cards being drawn is the same. Obviously, the sample space consists of 52 sample points.

If A and B denote the events of drawing a 'spade card' and a 'king' respectively, then the event A consists of 13 sample points, whereas the event B consists of 4 sample points. Therefore,

$$P(A) = \frac{13}{52} \quad , \qquad P(B) = \frac{4}{52}$$

The compound event $(A \cap B)$ consists of only one sample point, viz.; king of spade. So,

$$P(A \cap B) = \frac{1}{52}$$

Hence, the probability that the card drawn is either a spade or a king is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$$

$$= \frac{16}{52} = \frac{4}{13}$$

Example 31.19 In an experiment with throwing 2 fair dice, consider the events

A : The sum of numbers on the faces is 8

B : Doubles are thrown.

What is the probability of getting A or B ?

Solution : In a throw of two dice, the sample space consists of $6 \times 6 = 36$ sample points. The favourable outcomes to the event A (the sum of the numbers on the faces is 8) are

$$A = \{ (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) \}$$

The favourable outcomes to the event B (Double means both dice have the same number) are

$$B = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$

$$A \cap B = \{ (4,4) \}.$$

Now $P(A) = \frac{5}{36}$, $P(B) = \frac{6}{36}$, $P(A \cap B) = \frac{1}{36}$

Thus, the probability of A or B is

$$P(A \cup B) = \frac{5}{36} + \frac{6}{36} - \frac{1}{36}$$

$$= \frac{10}{36} = \frac{5}{18}$$

31.5 ADDITIVE LAW OF PROBABILITY FOR MUTUALLY EXCLUSIVE EVENTS

We know that the events A and B are mutually exclusive, if and only if they have no outcomes in common. That is, for mutually exclusive events,

$$P(A \text{ and } B) = 0$$

Substituting this value in the additive law of probability, we get the following law :

$$P(A \text{ or } B) = P(A) + P(B) \qquad .....(iii)$$

Example 31.20 In a single throw of two dice, find the probability of a total of 9 or 11.

Solution : Clearly, the events - a total of 9 and a total of 11 are mutually exclusive.

Now $\qquad P(\text{a total of } 9) = P[(3, 6), (4, 5), (5, 4), (6, 3)] = \dfrac{4}{36}$

$\qquad P(\text{a total of } 11) = P[(5, 6), (6, 5)] = \dfrac{2}{36}$

Thus, $\qquad P(\text{a total of 9 or 11}) = \dfrac{4}{36} + \dfrac{2}{36}$

$$= \frac{1}{6}$$

Example 31.21 The probabilities that a student will receive an A, B , C or D grade are 0.30, 0.35, 0.20 and 0.15 respectively. What is the probability that a student will receive at least a B grade ?

Solution : The event at least a 'B' grade means that the student gets either a B grade or an A grade.

$\therefore \qquad P(\text{at least B grade}) = P(\text{B grade}) + P(\text{A grade})$

$$= 0.35 + 0.30$$

$$= 0.65$$

Example 31.22 Prove that the probability of the non-occurrence of an event A is $1 - P(A)$.

i.e., $\qquad P(\text{not } A) = 1 - P(A) \qquad$ or, $\qquad P(\bar{A}) = 1 - P(A)$.

Solution : We know that the probability of the sample space S in any experiment is 1.

Now, it is clear that if in an experiment an event A occurs, then the event $(\bar{A})$ cannot occur simultaneously, i.e., the two events are mutually exclusive.

Also, the sample points of the two mutually exclusive events together constitute the sample space S. That is,

$$A \cup \bar{A} = S$$

Thus, $\qquad P\left(A \cup \bar{A}\right) = P(S)$

$\Rightarrow P(A) + P(\bar{A}) = 1 \quad (\because$ A and $\bar{A}$ are mutually exclusive and S is sample space)

$\Rightarrow P\left(\bar{A}\right) = 1 - P(A),$

which proves the result.

This is called the law of complementation.

Law of complimentation : $P\left(\bar{A}\right) = 1 - P(A)$

Example 31.23 Find the probability of the event getting at least 1 tail, if four coins are tossed once.

Solution : In tossing of 4 coins once, the sample space has 16 samples points.

$\therefore$ P (at least one tail) = P (1or 2 or 3 or 4 tails)

$= 1 - P(0 \text{ tail})$ (By law of complimentation)

$= 1 - P(H\,H\,H\,H)$

The outcome favourable to the event four heads is 1.

$$\therefore \quad P(H\,H\,H\,H) = \frac{1}{16}$$

Substituting this value in the above equation,

we get

$$P(\text{at least one tail}) = 1 - \frac{1}{16} = \frac{15}{16}$$

In many instances, the probability of an event may be expressed as odds - either odds in favour of an event or odds against an event.

If A is an event :

The odds in favour of $A = \dfrac{P(A)}{P(\bar{A})}$ or $P(A)$ to $P(\bar{A})$,

where P (A) is the probability of the event A and $P(\bar{A})$ is the probability of the event 'not A'.

Similarly, the odds against A are

$$\frac{P(\bar{A})}{P(A)} \text{ or } P(A) \text{ to } P(\bar{A}).$$

Example 31.24 The probability of the event that it will rain is 0.3. Find the odds in favour of rain and odds against rain.

Solution : Let A be the event that it will rain.

$\therefore \quad P(A) = .3$

By law of complementation,

$$P\left(\bar{A}\right) = 1 - .3 = .7 .$$

Now, the odds in favour of rain are $\dfrac{0.3}{0.7}$ or 3 to 7 (or 3 : 7).

The odds against rain are

$$\frac{0.7}{0.3} \text{ or } 7 \text{ to } 3.$$

When either the odds in favour of A or the odds against A are given, we can obtain the probability of that event by using the following formulae

If the odds in favour of A are *a* to *b*, then

$$P(A) = \frac{a}{a+b}.$$

If the odds against A are a to b, then

$$P(A) = \frac{b}{a+b}.$$

This can be proved very easily.

Suppose the odds in favour of A are *a* to *b*. Then, by the definition of odds,

$$\frac{P(A)}{P(\bar{A})} = \frac{a}{b}.$$

From the law of complimentation,

$$P\left(\bar{A}\right) = 1 - P(A)$$

Therefore, $\dfrac{P(A)}{1 - P(A)} = \dfrac{a}{b}$ or $b\,P(A) = a - a\,P(A)$

or $(a + b)\,P(A) = a$ or $P(A) = \dfrac{a}{a+b}$

Similarly, we can prove that

$$P(A) = \frac{b}{a+b}$$

when the odds against A are b to a.

Example 31.25 Determine the probability of A for the given odds

(a) 3 to 1 in favour of A (b) 7 to 5 against A.

Solution : (a) $P(A) = \dfrac{3}{3+1} = \dfrac{3}{4}$ (b) $P(A) = \dfrac{5}{7+5} = \dfrac{5}{12}$

Example 31.26 If two dice are thrown, what is the probability that the sum is

(a) greater than 8 ? (b) neither 7 nor 11 ?

Solution : (a) If S denotes the sum on two dice, then we want $P(S > 8)$. The required event can happen in the following mutually exclusive ways :

(i) S = 9 (ii) S = 10, (iii) S = 11, and (iv) S = 12 .

Hence, by addition probability theorem for mutually exclusive events, we get

$$P(S > 8) = P(S = 9) + P(S = 10) + P(S = 11) + P(S = 12) \qquad ....(1)$$

In a throw of two dice, the sample space contains $6 \times 6 = 36$ points. The number of favourable cases can be enumerated as shown below :

S = 9 : (3, 6), (4, 5), (5, 4), (6, 3) , i.e., 4 sample points.

$\therefore \qquad P(S=9) = \frac{4}{36}.$

S = 10 : $\quad (4,6),\ (5,5),\ (6,4)$, , i.e., 3 sample points.

$\therefore \qquad P(S=10) = \frac{3}{36}.$

S = 11 : $\quad (5,6),\ (6,5)$, i.e., 2 sample points.

$\therefore \qquad P(S=11) = \frac{2}{36}.$

S = 12 : $\quad (6,6)$, i.e., 1 sample point.

$\therefore \qquad P(S=12) = \frac{1}{36}.$

Substituting these values in equation (1), we get

$$P(S > 8) = \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18}.$$

(b) Let A and B denote the events of getting the sum 7 and 11 respectively on a pair of dice.

S = 7 : $(1,6),\ (2,5),\ (3,4),\ (4,3),\ (5,2),\ (6,1)$, i.e., 6 sample points.

$\therefore \qquad P(S=7) = \frac{6}{36}, \text{ or } \quad P(A) = \frac{6}{36}.$

S = 11 : $\quad (5,6),\ (6,5)$, i.e., 2 sample points.

$\therefore P(S=11) = \frac{2}{36}, \text{ or } P(B) = \frac{2}{36}.$

Since A and B are disjoint events, therefore

$$P(\text{either} A \text{or} B) = P(A) + P(B)$$

$$= \frac{6}{36} + \frac{2}{36}$$

$$= \frac{8}{36}$$

Hence, by law of complementation,

P (neither 7 nor 11) = 1 – P (either 7 or 11)

$$= 1 - \frac{8}{36}$$

$$= \frac{28}{36}$$

$$= \frac{7}{9}.$$

Example 31.27 Are the following probability assignments consistent ? Justify your answer.

(a) $P(A) = P(B) = 0.6$, $P(A \text{ and } B) = 0.05$

(b) $P(A) = 0.5$, $P(B) = 0.4$, $P(A \text{ and } B) = 0.1$

(c) $P(A) = 0.2$, $P(B) = 0.7$, $P(A \text{ and } B) = 0.4$

Solution : (a) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$$= 0.6 + 0.6 - 0.05$$

$$= 1.15$$

Since P (A or B) > 1 is not possible, hence the given probabilities are not consistent.

(b) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$$= 0.5 + 0.4 - 0.1$$

$$= 0.8$$

which is less than 1.

As the number of outcomes favourable to event 'A and B' should always be less than or equal to those favourable to the event A,

Therefore, $P(A \text{ and } B) \le P(A)$

and similarly $P(A \text{ and } B) \le P(B)$

In this case, P (A and B) = 0.1, which is less than both P (A) = 0.5 and P (B) = 0.4. Hence, the assigned probabilities are consistent.

(c) In this case, P (A and B) = 0.4, which is more than P (A) = 0.2.

$$[\because P(A \text{ and } B) \le P(A)]$$

Hence, the assigned probabilities are not consistent.

Example 31.28 An urn contains 8 white balls and 2 green balls. A sample of three balls is selected at random. What is the probability that the sample contains at least one green ball ?

Solution : Urn contains 8 white balls and 2 green balls.

$\therefore$ Total number of balls in the urn = 10

Three balls can be drawn in ${}^{10}C_3$ ways = 120 ways.

Let A be the event " at least one green ball is selected".

Let us determine the number of different outcomes in A. These outcomes contain either one green ball or two green balls.

There are ${}^{2}C_1$ ways to select a green ball from 2 green balls and for this remaining two white balls can be selected in ${}^{8}C_2$ ways.

Hence, the number of outcomes favourable to one green ball

$$= {}^{2}C_1 \times {}^{8}C_2$$

$$= 2 \times 28 = 56$$

Similarly, the number of outcomes favourable to two green balls

$$= {}^{2}C_2 \times {}^{8}C_1 = 1 \times 8 = 8$$

Hence, the probability of at least one green ball is

P (at least one green ball)

$$= P\ (\text{one green ball}) + P\ (\text{two green balls})$$

$$= \frac{56}{120} + \frac{8}{120}$$

$$= \frac{64}{120} = \frac{8}{15}$$

Example 31.29 Two balls are drawn at random with replacement from a bag containing 5 blue and 10 red balls. Find the probability that both the balls are either blue or red.

Solution : Let the event A consists of getting both blue balls and the event B is getting both red balls. Evidently A and B are mutually exclusive events.

By fundamental principle of counting, the number of outcomes favourable to $A = 5 \times 5 = 25$.

Similarly, the number of outcomes favourable to B = 10 × 10 = 100.

Total number of possible outcomes = 15 × 15 = 225.

$$\therefore \quad P(A) = \frac{25}{225} = \frac{1}{9} \text{ and } P(B) = \frac{100}{225} = \frac{4}{9}.$$

Since the events A and B are mutually exclusive, therefore

$$P(A \text{ or } B) = P(A) + P(B)$$

$$= \frac{1}{9} + \frac{4}{9}$$

$$= \frac{5}{9}$$

Thus, P (both blue or both red balls) = $\frac{5}{9}$

CHECK YOUR PROGRESS 31.3

1. A card is drawn from a well-shuffled pack of cards. Find the probability that it is a queen or a card of heart.
2. In a single throw of two dice, find the probability of a total of 7 or 12.
3. The odds in favour of winning of Indian cricket team in 2010 world cup are 9 to 7. What is the probability that Indian team wins ?
4. The odds against the team A winning the league match are 5 to 7. What is the probability that the team A wins the league match.
5. Two dice are thrown. Getting two numbers whose sum is divisible by 4 or 5 is considered a success. Find the probability of success.
6. Two cards are drawn at random from a well-shuffled deck of 52 cards with replacement. What is the probability that both the cards are either black or red ?

7. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that the card is an ace or a black card.

8. Two dice are thrown once. Find the probability of getting a multiple of 3 on the first die or a total of 8.

9. (a) In a single throw of two dice, find the probability of a total of 5 or 7.

   (b) A and B are two mutually exclusive events such that P (A) = 0.3 and P (B) = 0.4. Calculate P (A or B).

10. A box contains 12 light bulbs of which 5 are defective. All the bulbs look alike and have equal probability of being chosen. Three bulbs are picked up at random. What is the probability that at least 2 are defective ?

11. Two dice are rolled once. Find the probability

    (a) that the numbers on the two dice are different,

    (b) that the total is at least 3.

12. A couple have three children. What is the probability that among the children, there will be at least one boy or at least one girl ?

13. Find the odds in favour and against each event for the given probability

    (a) P (A) = .7 (b) P (A) $= \frac{4}{5}$

14. Determine the probability of A for the given odds

    (a) 7 to 2 in favour of A (b) 10 to 7 against A.

15. If two dice are thrown, what is the probability that the sum is

    (a) greater than 4 and less than 9 ?

    (b) neither 5 nor 8 ?

16. Which of the following probability assignments are inconsistent ? Give reasons.

    (a) P (A) = 0.5 , P (B) = 0.3 , P (A and B) = 0.4

    (b) P (A) = P (B) = 0.4, P (A and B) = 0.2

    (c) P (A) = 0.85, P (B) = 0.8, P (A and B) = 0.61

17. Two balls are drawn at random from a bag containing 5 white and 10 green balls. Find the probability that the sample contains at least one white ball.

18. Two cards are drawn at random from a well-shuffled deck of 52 cards with replacement. What is the probability that both cards are of the same suit ?

31.6 SOME RESULTS ON PROBABILITY OF EVENTS

Result 1 : Probability of impossible event is zero.

Solution : Impossible event contains no sample points. Therefore, the certain event S and the impossible event ϕ are mutually exclusive.

Hence, $S \cup \phi = S$

$\Rightarrow \quad P\left(S \cup \phi\right) = P\ (S)$

or $\quad P(S) + P(\phi) = P(S)$

$\Rightarrow \quad P(\phi) = 0.$

Result 2 : Probability of the complementary event $\overline{A}$ of A is given by

$$P(\overline{A}) = 1 - P(A)$$

Solution : A and $\overline{A}$ are disjoint events. Also,

$$A \cup \overline{A} = S \quad \Rightarrow \quad P(A \cup \overline{A}) = P(S)$$

Using additive laws (ii) and (iii) , we get

$$P(A) + P(\overline{A}) = 1 \quad \Rightarrow \quad P(\overline{A}) = 1 - P(A).$$

Result 3 : Prove that $0 \le P(A) \le 1$, for any A in S.

Solution : We know that

$$A \subset S \quad \Rightarrow \quad P(A) \le P(S)$$

$\Rightarrow \quad P(A) \le 1.$ [Using additive law]

We know that $P(A) \ge 0.$

Hence, $\quad 0 \le P(A) \le 1.$

Result 4 : If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Solution :

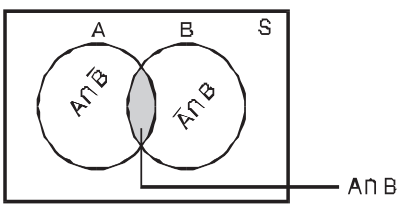


Fig. 31.6

From the above figure, we can write

$$A \cup B = A \cup (\overline{A} \cap B)$$

$\Rightarrow \quad P(A \cup B) = P\left[A \cup (\overline{A} \cap B)\right].$ ....(1)

Since the events A and $(\overline{A} \cap B)$ are disjoint, therefore law (iii) gives

$$P\left[A \cup (\overline{A} \cap B)\right] = P(A) + P(\overline{A} \cap B)$$

Substituting this value in (1) , we get

$$P(A \cup B) = P(A) + P(\overline{A} \cap B)$$

or $\quad P(A \cup B) = P(A) + [P(\overline{A} \cap B) + P(A \cap B)] - P(A \cap B)$ .....(2)

From Fig. 31.6, we see that

$$(\overline{A} \cap B) \cup (A \cap B) = B$$

$$\therefore \quad P[(\overline{A} \cap B) \cup (A \cap B)] = P(B).$$

Further, the events $(\overline{A} \cap B)$ and $(A \cap B)$ are disjoint, so from additive law, we get

$$P(\overline{A} \cap B) + P(A \cap B) = P(B)$$

Substituting this value in (2), we get

$$P(A \cup B) = P(A) + P(B) - P(A \cap B$$

Result 5 : If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

Solution : From additive law, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad .....(1)$$

Since A and B are mutually exclusive events,

Therefore $\quad A \cap B = \phi$

$$\Rightarrow \quad P(A \cap B) = P(\phi) = 0$$

Substituting this value in equation (1), we get

$$P(A \cup B) = P(A) + P(B),$$

which is additive law for mutually exclusive events.

31.7 MULTIPLICATION LAW OF PROBABILITY FOR INDEPENDENT EVENTS

Let us recall the definition of independent events.

Two events A and B are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence (and hence non-occurrence) of the other.

Can you think of some examples of independent events ?

The event of getting 'H' on first coin and the event of getting 'T' on the second coin in a simultaneous toss of two coins are independent events.

What about the event of getting 'H' on the first toss and event of getting 'T' on the second toss in two successive tosses of a coin ? They are also independent events.

Let us consider the event of 'drawing an ace' and the event of 'drawing a king' in two successive draws of a card from a well-shuffled deck of cards without replacement.

Are these independent events ?

No, these are not independent events, because we draw an ace in the first draw with probability $\frac{4}{52}$. Now, we do not replace the card and draw a king from the remaining 51 cards and this affect the probability of getting a king in the second draw, i.e., the probability of getting a king in the second draw without replacement will be $\frac{4}{51}$.

Note : If the cards are drawn with replacement, then the two events become independent.

Is there any rule by which we can say that the events are independent ?

How to find the probability of simultaneous occurrence of two independent events?

If A and B are independent events, then

$$P\ (A \text{ and } B) = P(A)\ .\ P\ (B)$$

or

$$P(A \cap B) = P(A).\ P\ (B)$$

Thus, the probability of simultaneous occurrence of two independent events is the product of their separate probabilities.

Note : The above law can be extended to more than two independent events, i.e.,

$$P(A \cap B \cap C...) = P(A) \cdot P(B) \cdot P(C)...$$

On the other hand, if the probability of the event 'A' and 'B' is equal to the product of the probabilities of the events A and B, then we say that the events A and B are independent.

Example 31.30 A die is tossed twice. Find the probability of a number greater than 4 on each throw.

Solution : Let us denote by A, the event 'a number greater than 4' on first throw. B be the event 'a number greater than 4' in the second throw. Clearly A and B are independent events.

In the first throw, there are two outcomes, namely, 5 and 6 favourable to the event A.

$\therefore$ $$P(A) = \frac{2}{6} = \frac{1}{3}$$

Similarly, $$P(B) = \frac{1}{3}$$

Hence, $P(A \text{ and } B) = P\ (A)\ .\ P\ (B)$

$$= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}.$$

Example 31.31 Arun and Tarun appear for an interview for two vacancies. The probability of Arun's selection is $\frac{1}{3}$ and that of Tarun's selection is $\frac{1}{5}$. Find the probability that

(a) both of them will be selected.

(b) none of them is selected.

(c) at least one of them is selected.

(d) only one of them is selected.

Solution : Probability of Arun's selection $= P(A) = \frac{1}{3}$

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$$\text{Probability of Tarun's selection } = P(T) = \frac{1}{5}$$

(a) P (both of them will be selected) = P (A) P (T)

$$= \frac{1}{3} \times \frac{1}{5}$$

$$= \frac{1}{15}$$

(b) P (none of them is selected)

$$= P(\bar{A})P(\bar{T})$$

$$= \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$$

$$= \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

(c) P (at least one of them is selected)

$$= 1 - P \text{ (None of them is selected)}$$

$$= 1 - P(\bar{A})P(\bar{T})$$

$$= 1 - \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$$

$$= 1 - \left(\frac{2}{3} \times \frac{4}{5}\right)$$

$$= 1 - \frac{8}{15} = \frac{7}{15}$$

(d) P (only one of of them is selected)

$$= P(A)P(\bar{T}) + P(\bar{A})\, P(T)$$

$$= \frac{1}{3} \times \frac{4}{5} + \frac{2}{3} \times \frac{1}{5}$$

$$= \frac{6}{15} = \frac{2}{5}$$

Example 31.32 A problem in statistics is given to three students, whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that problem will be solved ?

Solution : Let p_1, p_2 and p_3 be the probabilities of three persons of solving the problem.

Here, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$ and $p_3 = \frac{1}{4}$.

The problem will be solved, if at least one of them solves the problem.

$\therefore$ P (at least one of them solves the problem)

= 1 − P (None of them solves the problem) .....(1)

Now, the probability that none of them solves the problem will be

$$\text{P (none of them solves the problem)} = (1 - p_1)(1 - p_2)(1 - p_3)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

Putting this value in (1), we get

$$\text{P (at least one of them solves the problem)} = 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

Hence, the probability that the problem will be solved is $\frac{3}{4}$.

Example 31.33 Two balls are drawn at random with replacement from a box containing 15 red and 10 white balls. Calculate the probability that

(a) both balls are red.

(b) first ball is red and the second is white.

(c) one of them is white and the other is red.

Solution :

(a) Let A be the event that first drawn ball is red and B be the event that the second ball drawn is red. Then as the balls drawn are with replacement,

therefore $$P(A) = \frac{15}{25} = \frac{3}{5}, P(B) = \frac{3}{5}$$

As A and B are independent events

therefore P (both red) = P (A and B)

$$= P(A) \times P(B)$$

$$= \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$$

(b) Let A : First ball drawn is red.

B : Second ball drawn is white.

$$\therefore \quad P(\text{A and B}) = P(A) \times P(B)$$

$$= \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}.$$

(c) If WR denotes the event of getting a white ball in the first draw and a red ball in the second draw and the event RW of getting a red ball in the first draw and a white ball in the second draw. Then as 'RW' and WR' are mutually exclusive events, therefore

$\therefore$ P (a white and a red ball)

$$= P(\text{WR or RW})$$
$$= P(\text{WR}) + P(\text{RW})$$
$$= P(\text{W})\,P(\text{R}) + P(\text{R})\,P(\text{W})$$
$$= \frac{2}{5}\cdot\frac{3}{5} + \frac{3}{5}\cdot\frac{2}{5}$$
$$= \frac{6}{25} + \frac{6}{25} = \frac{12}{25}$$

Example 31.34 The odds against Manager X settling the wage dispute with the workers are 8 : 6 and odds in favour of manager Y settling the same dispute are 14 : 16.

(i) What is the chance that neither settles the dispute, if they both try independently of each other ?

(ii) What is the probability that the dispute will be settled ?

Solution : Let A be the event that the manager X will settle the dispute and B be the event that the manager Y will settle the dispute. Then, clearly

(i)
$$P(A) = \frac{6}{14} = \frac{3}{7},\ P\left(\overline{A}\right) = 1 - P(A) = 1 - \frac{3}{7} = \frac{4}{7}$$
$$P(B) = \frac{14}{30} = \frac{7}{15},\ P\left(\overline{B}\right) = 1 - \frac{14}{30} = \frac{16}{30} = \frac{8}{15}$$

The required probability that neither settles the dispute is given by

$$P(\overline{A} \cap \overline{B}) = P\left(\overline{A}\right)P\left(\overline{B}\right)$$
$$= \frac{4}{7}\cdot\frac{8}{15} = \frac{32}{105}$$

(Since A, B are independent, therefore, $\overline{A}, \overline{B}$ also independent)

(ii) The dispute will be settled, if at least one of the managers X and Y settles the dispute. Hence, the required probability is given by

$$P(A \cup B) = P\,[\text{At least one of X and Y settles the dispute}]$$
$$= 1 - P\,[\text{None settles the dispute}]$$
$$= 1 - P\left(\overline{A} \cap \overline{B}\right)$$
$$= 1 - \frac{32}{105} = \frac{73}{105}$$

Example 31.35 A dice is thrown 3 times. Getting a number '5 or 6' is a success. Find the probability of getting

(a) 3 successes (b) exactly 2 successes (c) at most 2 successes (d) at least 2 successes.

Solution : Let S denote the success in a trial and F denote the ' not success' i.e. failure. Therefore,

$$P(S) = \frac{2}{6} = \frac{1}{3}$$

$$P(F) = 1 - \frac{1}{3} = \frac{2}{3}$$

(a) As the trials are independent, by multiplication theorem for independent events,

$$P(SSS) = P(S) \;\; P(S) \;\; P(S)$$

$$= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$$P(SSF) = P(S)\,P(S)\,P(F)$$

$$= \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$$

Since the two successes can occur in ${}^3C_2$ ways

$\therefore$ P (exactly two successes) $= {}^3C_2 \times \frac{2}{27} = \frac{2}{9}$

(c) P (at most two successes) $= 1 - P(3\text{successes})$

$$= 1 - \frac{1}{27} = \frac{26}{27}$$

(d) P (at least two successes) = P (exactly 2 successes) + P (3 successes)

$$= \frac{2}{9} + \frac{1}{27} = \frac{7}{27}$$

Example 31.36 A card is drawn from a pack of 52 cards so that each card is equally likely to be selected. Which of the following events are independent ?

(i) A : the card drawn is a spade
B : the card drawn is an ace

(ii) A : the card drawn is black
B : the card drawn is a king

(iii) A : the card drawn is a king or a queen
B : the card drawn is a queen or a jack

Solution : (i) There are 13 cards of spade in a pack.

$$P(A) = \frac{13}{52} = \frac{1}{4}$$

There are four aces in the pack.

$\therefore$ $P(B) = \frac{4}{52} = \frac{1}{13}$

$A \cap B = \{$ an ace of spade $\}$

$\therefore \qquad P(A \cap B) = \frac{1}{52}$

Now $\qquad P(A)\ P(B) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$

Since $\qquad P(A \cap B) = P(A).\ P(B)$

Hence, the events A and B are independent.

(ii) There are 26 black cards in a pack.

$\therefore \qquad P(A) = \frac{26}{52} = \frac{1}{2}$

There are four kings in the pack.

$\therefore \qquad P(B) = \frac{4}{52} = \frac{1}{13}$

$\qquad A \cap B = \{$ 2 black kings $\}$

$\therefore \qquad P(A \cap B) = \frac{2}{52} = \frac{1}{26}$

Now, $\qquad P(A) \times P(B) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$

Since $\qquad P(A \cap B) = P(A)\ P(B)$

Hence, the events A and B are independent.

(iii) There are 4 kings and 4 queens in a pack of cards.

$\therefore$ Total number of outcomes favourable to the event A is 8.

$\therefore \qquad P(A) = \frac{8}{52} = \frac{2}{13}$

Similarly, $\qquad P(B) = \frac{2}{13}$

$\qquad A \cap B = \{$ 4 queens $\}$

$\therefore \qquad P(A \cap B) = \frac{4}{52} = \frac{1}{13}$

$\therefore \qquad P(A) \times P(B) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169}$

Here, $\qquad P(A \cap B) \neq P(A).P(B)$

Hence, the events A and B are not independent.

Example 31.37 Consider a group of 36 students. Suppose that A and B are two properties that each student either has or does not have. The events are

A : Student has blue eyes

B : Student is a male

Out of 36, there are 12 male and 24 female students and half of them in each has blue eyes. Are these events independent ?

Solution : With regard to the given two properties, i.e., either has or does not have, the 36 students are distributed as follows :

	Blue eyes A	Not blue eyes $(\bar{A})$	Total
Male (B)	6	6	12
Female $(\bar{B})$	12	12	24
Total	18	18	36

If we choose a student at random, the probabilities corresponding to the events A and B are

$$P(A) = \frac{18}{36} = \frac{1}{2}$$

$$P(B) = \frac{12}{36} = \frac{1}{3}$$

$$P(A \cap B) = \frac{6}{36} = \frac{1}{6}$$

Also $\quad P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

Here, $\quad P(A \cap B) = P(A) \cdot P(B)$

Hence, the events A and B are independent.

Example 31.38 Suppose that we toss a coin three times and record the sequence of heads and tails. Let A be the event ' at most one head occurs' and B the event ' both heads and tails occur'. Are these event independent ?

Solution : The sample space in tossing a coin three times will be

$$S = \{HHH, HHT, HTH, HTT, THH, TTH, THT, TTT\}$$

Also, $\quad A \cap B = \{TTH, THT, HTT\}$

$\therefore \quad P(A) = \frac{4}{8} = \frac{1}{2}, \; P(B) = \frac{6}{8} = \frac{3}{4}, \quad P(A \cap B) = \frac{3}{8}$

Moreover, $\quad P(A) \times P(B) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$

Which equals $P(A \cap B)$. Hence, A and B are independent.

CHECK YOUR PROGRESS 31.4

1. A husband and wife appear in an interview for two vacancies in the same department.

The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that

(a) Only one of them will be selected ?

(b) Both of them will be selected ?

(c) None of them will be selected ?

(d) At least one of them will be selected ?

2. Probabilities of solving a specific problem independently by Raju and Soma are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(a) the problem is solved.

(b) exactly one of them solves the problem.

3. A die is rolled twice. Find the probability of a number greater than 3 on each throw.

4. Sita appears in the interview for two posts A and B, selection for which are independent. The probability of her selection for post A is $\frac{1}{5}$ and for post B is $\frac{1}{7}$. Find the probability that she is selected for

(a) both the posts

(b) at least one of the posts.

5. The probabilities of A, B and C solving a problem are $\frac{1}{3}, \frac{2}{7}$ and $\frac{3}{8}$ respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them will solve it.

6. A draws two cards with replacement from a well-shuffled deck of cards and at the same time B throws a pair of dice. What is the probability that

(a) A gets both cards of the same suit and B gets a total of 6 ?

(b) A gets two jacks and B gets a doublet ?

7. Suppose it is 9 to 7 against a person A who is now 35 years of age living till he is 65 and 3:2 against a person B now 45 living till he is 75. Find the chance that at least one of these persons will be alive 30 years hence.

8. A bag contains 13 balls numbered from 1 to 13. Suppose an even number is considered a 'success'. Two balls are drawn with replacement, from the bag. Find the probability of getting

(a) Two successes (b) exactly one success

(c) at least one success (d) no success

9. One card is drawn from a well-shuffled deck of 52 cards so that each card is equally likely to be selected. Which of the following events are independent ?

(a) A : The drawn card is red B : The drawn card is a queen

(b) A : The drawn card is a heart B: The drawn card is a face card

31.8 CONDITIONAL PROBABILITY

Suppose that a fair die is thrown and the score noted. Let A be the event, the score is 'even'. Then

$$A = \{2, 4, 6\}$$

$$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}.$$

Now suppose we are told that the score is greater than 3. With this additional information what will be P (A) ?

Let B be the event, 'the score is greater than 3'. Then B is {4, 5, 6 }. When we say that B has occurred, the event 'the score is less than or equal to 3' is no longer possible. Hence the sample space has changed from 6 to 3 points only. Out of these three points 4, 5 and 6; 4 and 6 are even scores.

Thus, given that B has occurred, P (A) must be $\frac{2}{3}$.

Let us denote the probability of A given that B has already occurred by $P(A \mid B)$.

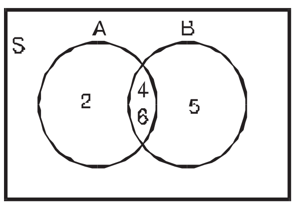

FIg. 31.7

Again, consider the experiment of drawing a single card from a deck of 52 cards. We are interested in the event A consisting of the outcome that a black ace is drawn.

Since we may assume that there are 52 equally likely possible outcomes and there are two black aces in the deck, so we have

$$P(A) = \frac{2}{52}.$$

However, suppose a card is drawn and we are informed that it is a spade. How should this information be used to reappraise the likelihood of the event A ?

Clearly, since the event B "A spade has been drawn " has occurred, the event "not spade" is no longer possible. Hence, the sample space has changed from 52 playing cards to 13 spade cards. The number of black aces that can be drawn has now been reduced to 1.

Therefore, we must compute the probability of event A relative to the new sample space B.

Let us analyze the situation more carefully.

The event A is " a black ace is drawn'. We have computed the probability of the event A knowing that B has occurred. This means that we are computing a probability relative to a new sample space B. That is, B is treated as the universal set. We should consider only that part of A which is included in B.

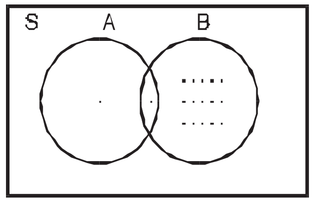

Fig. 31.8

Hence, we consider $A \cap B$ (see figure 31.8).

Thus, the probability of A. given B, is the ratio of the number of entries in $A \cap B$ to the number of entries in B. Since $n(A \cap B) = 1$ and $n(B) = 13$,

then
$$P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{13}$$

Notice that
$$n(A \cap B) = 1 \Rightarrow P(A \cap B) = \frac{1}{52}$$

$$n(B) = 13 \Rightarrow P(B) = \frac{13}{52}$$

$\therefore$
$$P(A|B) = \frac{1}{13} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{P(A \cap B)}{P(B)}.$$

This leads to the definition of conditional probability as given below :

Let A an B be two events defined on a sample space S. Let P(B) > 0, then the conditional probability of A, provided B has already occurred, is denoted by P(A|B) and mathematically written as :

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Similarly,
$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

The symbol P(A | B) is usually read as "the probability of A given B".

Example 31.39 Consider all families "with two children (not twins). Assume that all the elements of the sample space {BB, BG, GB,GG} are equally likely. (Here, for instance, BG denotes the birth sequence "boy girls"). Let A be the event {BB} and B be the event that 'at least one boy'. Calculate P (A | B).

Solution : Here, $A = \{ BB \}$

$B = \{ BB, BG, GB \}$

$A \cap B = \{BB\}$

$\therefore \quad P(A \cap B) = \frac{1}{4}$

$$P(B) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Hence, $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

Example 31.40 Assume that a certain school contains equal number of female and male students. 5 % of the male population is football players. Find the probability that a randomly selected student is a football player male.

Solution : Let M = Male

F = Football player

We wish to calculate $P(M \cap F)$. From the given data,

$$P(M) = \frac{1}{2} \quad (\because \text{ School contains equal number of male and female students})$$

$$P(F|M) = 0.05$$

But from definition of conditional probability, we have

$$P(F \mid M) = \frac{P(M \cap F)}{P(M)}$$

$\Rightarrow$ $$P(M \cap F) = P(M) \times P(F \mid M)$$

$$= \frac{1}{2} \times 0.05 = 0.025$$

Example 31.41 If A and B are two events, such that $P(A) = 0.8$, $P(B) = 0.6$, $P(A \cap B) = 0.5$, find the value of

(i) $P(A \cup B)$ (ii) $P(B \mid A)$ (iii) $P(A \mid B)$.

Solution :(i) $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.8 + 0.6 - 0.5 = 0.9$$

(ii) $$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.5}{0.8} = \frac{5}{8}$$

(iii) $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{0.5}{0.6} = \frac{5}{6}$$

Example 31.42 Find the chance of drawing 2 white balls in succession from a bag containing 5 red and 7 white balls, the balls drawn not being replaced.

Solution : Let A be the event that ball drawn is white in the first draw. B be the event that ball drawn is white in the second draw.

$\therefore$ $$P(A \cap B) = P(A)P(B \mid A)$$

Here, $$P(A) = \frac{7}{12}, P(B \mid A) = \frac{6}{11}$$

$\therefore$ $$P(A \cap B) = \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$$

Example 31.43 A coin is tossed until a head appears or until it has been tossed three times. Given that head does not occur on the first toss, what is the probability that coin is tossed three times ?

Solution : Here, it is given that head does not occur on the first toss. That is, we may get the head on the second toss or on the third toss or even no head.

Let B be the event, " no heads on first toss".

Then $$B = \{TH, TTH, TTT\}$$

These events are mutually exclusive.

$$P(B) = P(TH) + P(TTH) + P(TTT) \qquad .....(1)$$

Now $P(TH) = \frac{1}{4}$ ($\because$ This event has the sample space of four outcomes)

and $P(TTH) = P(TTT) = \frac{1}{8}$ ($\because$ This event has the sample space of eight outcomes)

Putting these values in (1), we get

$$P(B) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Let A be the event "coin is tossed three times".

Then $$A = \{TTH, TTT\}$$

$\therefore$ We have to find $P(A \mid B)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Here, $$A \cap B = A$$

$\therefore$ $$P(A \mid B) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

CHECK YOUR PROGRESS 31.5

1. A sequence of two cards is drawn at random (without replacement) from a well-shuffled deck of 52 cards. What is the probability that the first card is red and the second card is black ?
2. Consider a three child family for which the sample space is

   { BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG }

   Let A be the event " the family has exactly 2 boys " and B be the event " the first child is a boy". What is the probability that the family has 2 boys, given that first child is a boy ?
3. Two cards are drawn at random without replacement from a deck of 52 cards. What is the probability that the first card is a diamond and the second card is red ?
4. If A and B are events with P (A) = 0.4, P (B) = 0.2, $P(A \cap B) = 0.1$, find the probability of A given B. Also find P (B|A).
5. From a box containing 4 white balls, 3 yellow balls and 1 green ball, two balls are drawn one at a time without replacement. Find the probability that one white and one yellow ball is drawn.

31.9 THEOREMS ON MULTIPLICATION LAW OF PROBABILITY AND CONDITIONAL PROBABILITY.

Theorem 1 : For two events A and B,

$$P(A \cap B) = P(A).P(B \mid A),$$

and

$$P(A \cap B) = P(B).P(A \mid B),$$

where P (B|A) represents the conditional probability of occurrence of B, when the event A has already occurred and P (A|B) is the conditional probability of happening of A, given that B has already happened.

Proof : Let n (S) denote the total number of equally likely cases, n (A) denote the cases favourable to the event A, n (B) denote the cases favourable to B and $n(A \cap B)$ denote the cases favourable to both A and B.

$$\therefore \qquad P(A) = \frac{n(A)}{n(S)},$$

$$P(B) = \frac{n(B)}{n(S)}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} \qquad .....(1)$$

For the conditional event A|B , the favourable outcomes must be one of the sample points of B, i.e., for the event A|B, the sample space is B and out of the n (B) sample points, $n(A \cap B)$ pertain to the occurrence of the event A, Hence,

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

Rewriting (1), we get $P(A \cap B) = \frac{n(B)}{n(S)} \cdot \frac{n(A \cap B)}{n(B)} = P(B).P(A|B)$

Similarly, we can prove

$$P(A \cap B) = P(A).P(B \mid A)$$

Note : If A and B are independent events, then

$$P(A \mid B) = P(A) \text{ and } P(B \mid A) = P(B)$$

$$P(A \cap B) = P(A).P(B)$$

Theorem 2 : Two events A and B of the sample space S are independent, if and only if

$$P(A \cap B) = P(A).P(B)$$

Proof : If A and B are independent events,

then $\quad P(A \mid B) = P(A)$

We know that $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$\Rightarrow \quad P(A \cap B) = P(A)P(B)$

Hence, if A and B are independent events, then the probability of 'A and B' is equal to the product of the probability of A and probability of B.

Conversely, if $P(A \cap B) = P(A).P(B)$, then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ gives}$$

$$P(A \mid B) = \frac{P(A)P(B)}{P(B)} = P(A)$$

That is, A and B are independent events.

LET US SUM UP

- **Events Relation :** The complement of an event A consists of all those outcomes which are not favourable to the event A, and is denoted by 'not A' or by $\overline{A}$.
- **Event 'A or B' :** The event 'A or B' occurs if either A or B or both occur.
- **Event 'A and B' :** The event 'A and B' consists of all those outcomes which are favourable to both the events A and B.
- **Addition Law of Probability :** For any two events A and B of a sample space S

  $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- **Additive Law of Probability for Mutually Exclusive Events :** If A and B are two mutually exclusive events, then

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B).$$

- **Odds in Favour of an Event :** If the odds for A are a to b, then

$$P(A) = \frac{a}{a+b}$$

If odds against A are a to b, then

$$P(A) = \frac{b}{a+b}$$

- Two events are mutually exclusive, if occurrence of one precludes the possibility of simultaneous occurrence of the other.
- Two events A and B are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence (and hence non-occurrence) of the other.
- If A and B are independent events, then

$$P(A \text{ and } B) = P(A).P(B)$$

or $$P(A \cap B) = P(A).P(B)$$

- For two events A and B,

$$P(A \cap B) = P(A)P(B|A) \quad , \quad P(A) > 0$$

or $$P(A \cap B) = P(B)P(A|B) \quad , \quad P(B) > 0$$

where P (B|A) represents the conditional probability of occurrence of B, when the event A has already happened and P (A|B) represents the conditional probability of happening of A, given that B has already happened.

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. In a simultaneous toss of four coins, what is the probability of getting
   (a) exactly three heads ? (b) at least three heads ?
   (c) atmost three heads ?
2. Two dice are thrown once. Find the probability of getting an odd number on the first die or a sum of seven.
3. An integer is chosen at random from first two hundred integers. What is the probability that the integer chosen is divisible by 6 or 8 ?
4. A bag contains 13 balls numbered from 1 to 13. A ball is drawn at random. What is the

probability that the number obtained it is divisible by either 2 or 3 ?

5. Find the probability of getting 2 or 3 heads, when a coin is tossed four times.

6. Are the following probability assignments consistent ? Justify your answer.

   (a) P (A) = 0.6, P (B) = 0.5 , P (A and B) = 0.4

   (b) P (A) = 0.2, P (B) = 0.3 , P (A and B) = 0.4

   (c) P (A) = P (B) = 0.7, P (A and B) = 0.2

7. A box contains 25 tickets numbered 1 to 25. Two tickets are drawn at random. What is the probability that the product of the numbers is even ?

8. A drawer contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or is a bolt ?

9. A lady buys a dozen eggs, of which two turn out to be bad. She chose four eggs to scramble for breakfast. Find the chances that she chooses

   (a) all good eggs (b) three good and one bad eggs

   (c) two good and two bad eggs (d) at least one bad egg.

10. Two cards are drawn at random without replacement from a well-shuffled deck of 52 cards. Find the probability that the cards are both red or both kings.

11. Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, 1 girl and 3 boys. One child is selected at random from each group. Show the chances that three selected children consist of 1 girl and 2 boys is $\frac{13}{32}$.

12. A die is thrown twice. Find the probability of a prime number on each throw.

13. Kamal and Monika appears for an interview for two vacancies. The probability of Kamal's selection is $\frac{1}{3}$ and that of Monika's rejection is $\frac{4}{5}$. Find the probability that only one of them will be selected.

14. A bag contains 7 white, 5 black and 8 red balls. Four balls are drawn without replacement. Find the probability that all the balls are black.

15. For two events A and B, it is given that

$$P(A) = 0.4,\ P(B) = p \text{ and } P(A \cup B) = 0.6$$

   (a) Find p so that A and B are independent events.

   (b) For what value of p, of A and B are mutually exclusive ?

16. The odds against A speaking the truth are 3 : 2 and the odds against B speaking the truth are 5 : 3. In what percentage of cases are they likely to contradict each other on a identical issue ?

17. Let A and B be the events such that

$$P\left(\overline{A}\right) = \frac{1}{2},\ P\left(\overline{B}\right) = \frac{2}{3},\ P(A \cap B) = \frac{1}{4}.$$

   Compute P (A|B) and P (B|A).

ANSWERS

CHECK YOUR PROGRESS 31.1

1. $\frac{1}{6}$ 2. $\frac{1}{2}$ 3. $\frac{1}{2}$ 4. $\frac{3}{4}$

5. (i) $\frac{3}{5}$ (ii) $\frac{2}{5}$

6. (i) $\frac{5}{36}$ (ii) $\frac{5}{36}$ (iii) $\frac{1}{12}$ (iv) $\frac{1}{36}$

7. $\frac{5}{9}$ 8. $\frac{1}{12}$ 9. $\frac{1}{2}$

10. (i) $\frac{1}{4}$ (ii) $\frac{1}{13}$ (iii) $\frac{1}{52}$

11. (i) $\frac{5}{12}$ (ii) $\frac{1}{6}$ (iii) $\frac{11}{36}$

12. (i) $\frac{1}{8}$ (ii) $\frac{7}{8}$ (iii) $\frac{1}{8}$

CHECK YOUR PROGRESS 31.2

1. $\frac{1}{8}$ 2. $\frac{20}{39}$ 3.(a) $\frac{4}{25}$ (b) $\frac{38}{245}$

4. $\frac{1}{5525}$ 6. (i) $\frac{3}{10}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{30}$

7. $\frac{10}{133}$ 8. $\frac{4}{7}$ 9. $\frac{60}{143}$ 10. $\frac{1}{4}$

CHECK YOUR PROGRESS 31.3

1. $\frac{4}{13}$ 2. $\frac{7}{36}$ 3. $\frac{9}{16}$ 4. $\frac{7}{12}$

5. $\frac{4}{9}$ 6. $\frac{1}{2}$ 7. $\frac{7}{13}$ 8. $\frac{5}{12}$

9. (a) $\frac{5}{18}$ (b) 0.7 10. $\frac{4}{11}$

11. (a) $\frac{5}{6}$ (b) $\frac{35}{36}$ 12. $\frac{3}{4}$

13. (a) The odds for A are 7 to 3 . The odds against A are 3 to 7

13. (b) The odds for A are 4 to 1 and The odds against A are 1 to 4

14. (a) $\frac{7}{9}$ (b) $\frac{7}{17}$ 15. (a) $\frac{5}{9}$ (b) $\frac{3}{4}$

16. (a), (c) 17. $\frac{4}{7}$ 18. $\frac{1}{4}$

CHECK YOUR PROGRESS 31.4

1. (a) $\frac{2}{7}$ (b) $\frac{1}{35}$ (c) $\frac{24}{35}$ (d) $\frac{11}{35}$

2. (a) $\frac{2}{3}$ (b) $\frac{1}{2}$ 3 . $\frac{1}{4}$

4. (a) $\frac{1}{35}$ (b) $\frac{11}{35}$ 5. $\frac{1}{2}$

6. (a) $\frac{5}{144}$ (b) $\frac{1}{1014}$ 7. $\frac{53}{80}$

8. (a) $\frac{36}{169}$ (b) $\frac{84}{169}$ (c) $\frac{120}{169}$ (d) $\frac{49}{169}$

9. (a) Independent (b) Independent

CHECK YOUR PROGRESS 31.5

1. $\frac{13}{51}$ 2. $\frac{1}{2}$ 3. $\frac{25}{204}$

4. $\frac{1}{2}, \frac{1}{4}$ 5. $\frac{3}{7}$

TERMINAL EXERCISE

1. (a) $\frac{1}{4}$ (b) $\frac{5}{16}$ (c) $\frac{15}{16}$ 2. $\frac{7}{12}$

3. $\frac{1}{4}$ 4. $\frac{8}{13}$ 5. $\frac{5}{8}$

6. Only (a) is consistent

7. $\frac{456}{625}$ 8. $\frac{5}{8}$ 9. (a) $\frac{14}{33}$ (b) $\frac{16}{33}$

(c) $\frac{1}{11}$ (d) $\frac{19}{33}$

10. $\frac{55}{221}$ 12. $\frac{1}{4}$ 13. $\frac{2}{15}$ 14. $\frac{1}{969}$

15. $\frac{1}{3}$ 16. $\frac{19}{40}$ 17. $\frac{3}{4}, \frac{1}{2}$ respectively.

Random Variables and Probability Distributions

31.10 Definition

Let S be the sample space associated with a random experiment A function $X: S \to R$ is called a random variable.

Note : If X is a random variable then $X^{-1}(P(R)) = P(S)$

Here P stands for the probability function and P(S) stands for the power set of S .

Example 1

Let S be the sample space of the experiment of rolling a fair die.

Then $X : S \to R$ given by $X(n) = 0$, if n is even.

$= 1$, If n $X : S \to$ is odd.

is a random variable.

Here $S = \{1, 2, 3, 4, 5, 6\}$ and $X(1) = X(3) = X(5) = 1; X(2) = X(4) = X(6) = 0$.

31.10.1 Definition

Suppose $X: S \to R$ is a random variable. If the range of X is either finite or countably infinite, then X is called a discrete random variable.

A random variable which can take all real values in an interval (a, b) is called a conitnuous random variable.

31.10.2 Definition

The mean (μ) and variance (σ^2) of a discrete random variable X are

$$\mu = \sum x_n \, p(X = x_n), \; \sigma^2 = \sum (x_n - \mu)^2 \; p(X = x_n)$$

$$\sigma^2 = \sum \left[x_n^{\,2} \, p(X = x_n) \right] - \mu^2$$

The standard deviation σ is the square root of the variance.

Example 2

A cubical die is thrown. Find the mean and variance of X, giving the number on the face that shows up.

Sol : Let S be the sample space and X be the random variable associated with S, where p(X) is given by the following table

$X = x_i$	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Mean of X $= \mu = \sum_{i=1}^{6} X_i \, P(X = x_i)$

$$= 1.\frac{1}{6} + 2.\frac{1}{6} + 3.\frac{1}{6} + 4.\frac{1}{6} + 5.\frac{1}{6} + 6.\frac{1}{6}$$

$$= \frac{1}{6}\left(\frac{6 \times 7}{2}\right) = \frac{7}{2}$$

Variance of X $= \sigma^2 = \sum_{i=1}^{6} x_i^{\,2} \, P(X = x_i) - \mu^2$

$$= 1^2.\frac{1}{6} + 2^2.\frac{1}{6} + 3^2.\frac{1}{6} + 4^2.\frac{1}{6} + 5^2.\frac{1}{6} + 6^2.\frac{1}{6} - \left(\frac{7}{2}\right)^2$$

$$= \frac{1}{6}\left(\frac{6 \times 7 \times 13}{6}\right) - \frac{49}{4} = \frac{35}{12}$$

31.11 Theoretical discrete distributions Binomial and possion distributions

Suppose that an experiment has only two possible outcomes. For instance, when a coin is tossed the possible out comes are head and tail. Each performance of an experiment with two possible outcomes are termed as success and failure. If p is taken as the probability of success and q is the probability of failure, it follows that p + q = 1. Many problems can be solved by determining the probability of x successes when an experiment consists of n independent Bernoulli trials.

We shall discuss two theoretical frequency distributions in this section. One is Binomial distribution and the other is poission distribution.

31.11.1 Binomial distribution

Definition

A discrete random variable X is said to follow a binomial distribution (or simply a binomial variable with parameters n and p) where $0 < p < 1$ if

$$P(X = x) = n_{C_x} p^x q^{n-x}, x \in \{0, 1, 2, ..., n\}$$

If X is a binomial variate with parameters n, p; then it is also described by writing $X \sim B(n, p)$

Theorem

If $X \sim B(n, p)$, then the mean μ and the variance σ^2 of X are equal to np and npq respectively.

n is number of trials, p be the probability of sucess and q be the probability of failure.

31.12 Poisson distribution

In binomial distribution, the number of trials n is precisely known and is finite. But there are some situations where this may not be possible. Also if the event is rare and casual, the successful events are quite a few in the sample space. If we know the number of times of times an event occurred but not how many times it did not occur, then the bonomial distribution is inapplicable. In such cases we compute the binomial probabilities approximately by using Poisson distribution

Objectives

From this chapter we can get to know following points

- the number of telephone calls received at a telephone exchange in a given time interval.
- the number of printing errors in a page of a book.
- the number of road/rail accidents reported in a city in a given period of time (say a day)
- The number of defective blades in a packet of 100.

31.12.1 Definition

If X is a discrete random variable that can assume value 0, 1, 2, 3, ... such that for some fixed $\lambda > 0$.

$p(X = x) = \dfrac{e^{-\lambda}\lambda^x}{\lfloor x}$, $x = 0, 1, 2.......$ then X is said to follow a Possion distribution with parameter λ then its mean μ, variance $\sigma^2 = \lambda$.

Example 3

8 coins are tossed simultaneously. If the number of heads turned up is denoted by the variable X. Then find the mean and variance of X.

Sol : Here $n = 8$

prab. of geting in a head on a coin $p = \dfrac{1}{2}$

$$p = \frac{1}{2} \Rightarrow q = 1 - p = \frac{1}{2},\ n = 8$$

$$\text{Mean } (\mu)\ np = 8\left(\frac{1}{2}\right) = 4$$

$$\text{Variance } (\sigma^2) = npq = 8\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 2$$

Example 4

Find the maximum number of times a fair coin must be tossed so that the probability of geting atleast one head is atleast 0.8.

Sol : Let n be the number of times a fair coin tossed X denotes the number of heads getting X follows binomial distribution with parameters n and

$$p = \frac{1}{2} \Rightarrow q = 1 - p = \frac{1}{2},\ p(X \geq 1) \geq 0.8$$

$$\Rightarrow 1 - p(X = 0) \geq 0.8$$

$$\Rightarrow p(X = 0) \leq 0.2$$

$$\Rightarrow n_{C_0}\left(\frac{1}{2}\right)^n \leq 0.2$$

$$\Rightarrow \frac{1}{2^n} \leq \frac{1}{5}$$

$$\Rightarrow \frac{1}{2^n} \geq 5$$

$\therefore$ The Maximum value of n is 3.

Example 5

Find the parameters of the binomial variate whose mean and variance are $\frac{15}{2}, \frac{15}{4}$ respectively.

Sol : n, p are parameters of the binomial distribution $= n\,p = \frac{15}{2}$

$$\text{its mean } (\mu) = npq = \frac{15}{4}$$

$$\frac{npq}{np} = \frac{\left(\frac{15}{4}\right)}{\left(\frac{15}{2}\right)} = \frac{1}{2}$$

$$\Rightarrow q = \frac{1}{2}, \qquad \because\ p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\therefore \quad np = \frac{15}{2}$$

$$\Rightarrow \ n\left(\frac{1}{2}\right) = \frac{15}{2} \Rightarrow n = 15$$

$$\therefore \ n = 15,\ p = \frac{1}{2}.$$

Example 6

If X is a binomial variate with $9\,p(X = 4) = p(X = 2)$ and $n = 6$ then, find the parameter p.

Sol : Given $p + q = 1$

$$\because \qquad 9\,(X = 4) = p(X = 2)$$

$$\Rightarrow 9({}^6C_4)\, q^2 \cdot p^4 = {}^6C_2\, q^4\, p^2$$

$$\Rightarrow \ 9 = \frac{q^4 p^2}{p^4 q^2} = \left(\frac{q}{p}\right)^2$$

$$\Rightarrow \ \frac{q}{p} = 3 \Rightarrow 1 - p = 3p$$

$$\therefore \quad p = \frac{1}{4}$$

$$\Rightarrow \ q = \frac{3}{4}$$

$\therefore$ Parameter $p = \frac{1}{4}$

Example 7

If X is a poisson variate such that $p(X = 0) = p(X = 1) = k$, Then show that $k = e^{-1}$.

Sol : Let $\lambda = 0$ be the parameter of a poisson variate X

Given $p(X=0) = p(X=1) = k$

$$\frac{e^{-\lambda}.\lambda^0}{0!} = \frac{e^{-\lambda}.\lambda^1}{1!} = k$$

$$\Rightarrow e^{-\lambda} = e^{-\lambda}.\lambda \Rightarrow \lambda = 1$$

$$\therefore \quad k = \frac{e^{-1}\lambda^0}{0!} = e^{-1}$$

$$\therefore \quad k = e^{-1}.$$

Example 8.

If X follows a poisson distribution and $p(X=1) = 3\,p(X=2)$, Then find the variance of X.

Sol : Let $\lambda > 0$ be the parameter of a poission variate X

Given $p(X=1) = 3(p(x=2))$

$$\frac{e^{-\lambda}.\lambda^1}{1!} = 3.\frac{e^{-\lambda}.\lambda^2}{2!}$$

$$\Rightarrow \lambda = \frac{2}{3}$$

$$\therefore \quad \text{Variance } \lambda = \frac{2}{3}$$

Check Your Progress

1. 8 coins are tossed simultaneously. Find the probability of getting atleast 6 heads.
2. The mean and variance of a binomial distribution are 4 and 3 respectively Fix the distribution and find $p(X \geq 1)$.
3. If $X - B(n, p), \mu = 20, \sigma^2 = 10$, then find n and p.

4. For a poisson variate X, $p(X = 2) = p(X = 3)$. Find the variance of X.

5. A poisson variable satisfies $p(X = 1) = p(X = 2)$ Find $p(X = 5)$.

Let Us Sum Up

1. If p is the probability of a success, q be the probability of a failure such the $p + q = 1$ and n is the number of Bernoulli trials, then the probability distribution of a discrete random variable X, called a binomial variate is given by

   $$p(X = k) = n_{C_k}\, p^k\, q^{n-k},\ k = 0, 1, 2, ..., n$$

   This is called the binomial distribution.

   Here n and p are called the parameters of the distribution.

   In this case X is expressed as $X \sim B(n, p)$.

2. If X is a binomial variate with parameters n and p i.e., $X \sim B(n, p)$, then the mean of the distribution $\mu = np$ and the variance of the distribution $\sigma^2 = npq$ The standard deviation of this distribution is given by $\sqrt{npq}$.

3. The probability distribution of a discrete random variable X (called the Poisson variable) given by $p(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k\,!}$, k = 0, 1, 2, ...and $\lambda = 0$,is called the Poisson distribution.

   Here λ is called the parameter of X .

4. If x is a Poisson variate with parameter λ then its mean μ = variance $\sigma^2 = \lambda$.

Supportive Websites

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

Answers

Check Your Progress

(1) $\dfrac{37}{256}$

(2) $1-\left(\dfrac{3}{4}\right)^{16}$

(3) $n = 40,\ p = \dfrac{1}{2}$

(4) 3

(5) $\dfrac{e^{-2}\,2^{5}}{5\,!}$

MODEL QUESTION PAPER

Intermediate (TOSS) **MATHEMATICS** **311 (E) (Set – II)**

Time: 3 hrs Max. Marks: 100

Instructions:-

1) All questions are Compulsory.
2) This question paper consists of two parts viz., A and B
3) All questions from Part A are to be attempted
4) Part-B has two options. Candidates are required to attempt questions from one option only.

(Part – A)

SECTION – 1

Each question carries 1 mark. (13 x 1 = 13 marks)

1) If Z_1 = x + iy, and Z_2 = a + ib then what is $Z_1 + Z_2$?
2) What are the roots of $x^2 + 5x + 6 = 0$?
3) Show the region represented by the in-equations $x \leq 0$ and $y \leq 0$
4) If the …………… two straight lines one perpendicular, then what is the product of their slopes?
5) Find the distance between the points (-1,2) and (0,-6) ?
6) Convert 270^0 into Radians.
7) What is the domain of the function?

   f = { (0,1), (2,3), (4,5), (6,7), …….., (100, 101)}
8) Evaluate $\mathrm{Lt}_{x\to 0}[(2x+1)^3 - 5]$
9) What is the derivative of x^n w.r.t x ?
10) What is the slope of the ……………………………………………………………………………… = x. Write the formulae of slope of the tangent of the curve y = f (x) at (x_1, y_1)
11) What is $\int e^x\,[f\,(x) + f^1(x)]d(x)? \ldots..]\,dx$
12) Write the Integrating factor of $\frac{dy}{dx}$ + f (x) y = Q (x) …………………………..
13) What is the principle of standard deviation for grouped data?

Section – 2

(Each question carries 2 marks)

14) If A = $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}_{2x2}$ and B = $\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}_{2x2}$ then prove that AB $\neq$ BA

15) What is the expanded form of $\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$

16) Find the inverse of the matrix $\begin{bmatrix} Cos\ x & -Sin\ x \\ Sin\ x & Cos\ x \end{bmatrix}$

17) Find the equation of the parabola whose focus is (2,3), directrix is 3x + 4y = 1.

18) What is the sum of the series 3 + 5 + 7 + ……………… + (2n + 1) …….

19) Draw the Venn diagram for A ⋃ B, when A and B are disjoint sets.

20) What is the principle value of $\cos^{-1}(\frac{\sqrt{3}}{2})$

21) If y = $\cos^{-1}(4x^3 - 3^x)$ find $\frac{dy}{dx}$.

22) What is the Area bounded by the curve x = y, Y-axis and the line y = 0, y = 3.

23) A die is tossed twice. Find the probability of a number greater than 4 in each throw.

Section - 3

(Each question carries 5 marks)

24) There are 5 Mathematics, 4 Physics and 5 Chemistry books. In how many ways can you arrange 4 Mathematics, 3 Physics and 4 Chemistry books, if the books on the same subjects are arranged together?

25) If the 4th term and the 9th term of a G.P. are 8 and 256 respectively, then write the G.P.

26) Find the Maximum and Minimum value of the function f (x) = Sin x (1 + Cos x) in (0, π)

27) The data below presents the earning of 50 labours of a factory. Calculate the Mean deviation.

Earning in Rs:	1200	1300	1400	1500	1600	1800
No. of Labors:	4	7	15	12	7	3

Section – 4

(Each question carries 8 marks)

28) Using Principle of Mathematical induction, prove the following

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots+\frac{1}{(2n-1).(2n+1)}= \frac{\mathrm{n}}{(2n+1)}$$

29) Find the equation of the circle which passes through the points (0,2), (2,0) and (0,0)

30) In $\Delta\, ABC, if\ < A = 60^0$ prove that $\frac{b}{c+a}+\frac{c}{a+b}= 1$

31) If y = $(\,Tan\ x)^{Cot\ x}$ + $(\,Cot\ x)^{\,x}$ then find $\frac{dy}{dx}$.

Part – B

Optional – I (VECTORS and 3 – D Geometry)

32) If $\overline{r_1}$ = $\bar{i}$ - $\bar{j}$ + $\bar{k}$ and $\overline{r_2}$ = $\overline{2i}$ - $\overline{4j}$ - $\overline{3k}$, then the magnitude of $\overline{r_1}$ + $\overline{r_2}$ 1 Mark

33) Write the position vector of the centroid of the triangle, whose vertices are A ($\vec{a}$), B ($\vec{b}$), C ($\vec{c}$) 4 Marks.

34) If the foot of the perpendicular drawn from the origin to the plane is (4, -2, -5), then find the equation of the plane. (5 Marks)

35) Find the centre and radius of the circle $x^2 + y^2 + z^2 – 6x – 4y + 12z – 36 = 0$, $x + 2y – 2z = 1$

(Optional – II)

32) Define Share 1 Mark

33) Write any one use of Index numbers 1 Mark

34) Discuss the characteristics of VAT 5 Marks

35) For a new product a manufacturer spends Rupees 1,00,000 on the infrastructure and the variable cost is estimated as Rs. 150 per unit of the product the sale price per unit fixed at Rs. 200.

Calculate (1) Cost function

(2) Revenue function

(3) Profit function and

(4) the breakeven point

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