Group A

Mathematics

1. (a) Solve the following equation in real x:

$$8^x + 2^x = 130.$$

(b) Consider the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \infty.$$

Find from the first principle, the set of real numbers x for which the series converges.

(c) Consider the following system of linear equations:

$$\begin{aligned} x - 2y + 3z &= 0 \\ 2x + y - 4z &= 0 \\ x - y + z &= 0. \end{aligned}$$

Find the solution space and its dimension.

[$8 + 10 + 6 = 24$]

Probability & Statistics

2. (a) Let (X, Y) have the joint probability density function

$$f(x, y) = 2e^{-(x+y)},\ 0 < x < y < \infty.$$

Find the conditional distribution of Y given $X = x$. Find $E(Y|X = x)$.

(b) Let X_1 and X_2 be independent random variables with $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$. Find the correlation coefficient between X_1 and X_1X_2.

(c) Determine the equivalence class(es) and the periods of the different states of a Markov chain whose transition probability matrix is given by

$$P = \begin{bmatrix} 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0.3 & 0 & 0 & 0 & 0.2 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Identify the equivalence classes by name(s).

$[(4+3)+5+(5+5+2)=24]$

3. (a) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) Poisson(θ), $\theta > 0$. We wish to estimate $\psi(\theta) = e^{-\theta}$. Consider the estimator of $\psi(\theta)$ given by $\psi(\hat{\theta}) = e^{-\bar{X}_n}$, where $\hat{\theta} = \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Let $b_n(\theta)$ be the bias of this estimator.

(i) Show that

$$b_n(\theta) = \exp[-n\theta(1 - \exp(-1/n))] - \exp(-\theta).$$

(ii) Show that $\lim_{n\to\infty} b_n(\theta) = 0$ for all $\theta > 0$.

(b) Consider the probability density functions

$$f_0(x) = \begin{cases} 2x, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_1(x) = \begin{cases} 3x^2, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Based on n i.i.d. observations $X_1, \ldots, X_n$ from a distribution with pdf f, we wish to test $H_0 : f(x) = f_0(x)$ versus

$H_1 : f(x) = f_1(x)$. Show using the central limit theorem that large sample approximation to the most powerful level-α $(0 < \alpha < 1)$ test is given by

$$\psi(X_1, \ldots, X_n) = \begin{cases} 1, & \text{if } \frac{1}{n}\sum_{i=1}^{n} \log X_i > \frac{z_\alpha}{2\sqrt{n}} - \frac{1}{2} \\ 0, & \text{otherwise,} \end{cases}$$

where z_α is the $(1-\alpha)$th quantile of standard normal distribution.

[(8 + 6) + 10 = 24]

Group B

Operations Research

4. (a) Consider the following linear programming problem

$$\begin{aligned} \text{Maximize} \quad & z = 3y_1 + 6y_2 + 2y_3 + 2y_4 \\ \text{subject to} \quad & y_1 + 3y_2 + y_4 \leq 8 \\ & 2y_1 + y_2 + y_4 \leq 6 \\ & y_2 + y_3 \leq 3 \\ & y_1 + y_2 + y_3 \leq 6 \\ & y_i \geq 0, i = 1, 2, 3, 4. \end{aligned}$$

Show that $(y_1, y_2, y_3, y_4) = (2, 2, 1, 0)$ is an optimal solution to the problem.

(b) A baking company sells bakery product in kg. wt. It makes a profit of Rs. 5 per kg. on each kg. sold on the day it is baked. It disposes all products not sold on the date it is made at a loss of Rs. 1.2 per kg. Demand per day is known to follow a uniform distribution in the range 2000 - 3000 kg. Determine the optimum daily amount to be baked.

(c) Ms. Alisha Choubey runs a one-person, unisex hair salon, on a first-come, first-served basis. Having obtained a master's degree in operations research prior to embarking upon

her career, she wants to analyze the crowd situation carefully before making a decision. She observes the following from her data keeping records:
a) customers seem to arrive according to a Poisson process with a mean arrival rate of 5 per hour, and
b) customers' service time is exponentially distributed with an average of 10 minutes.

Answer the following questions.

(i) What is the average number of customers waiting when there is at least one person waiting?

(ii) What is the percentage of customers that can go directly into service?

(iii) What is the probability that a customer, upon arrival, will not be able to find a seat when the waiting room in the salon has only 4 seats at present?

(iv) What is the probability that an arriving customer has to wait more than 45 minutes?

$[8 + 8 + (2 + 1 + 2 + 3) = 24]$

5. (a) Let S be a nonempty open convex set in R^n and $f : S \to R$ be a differentiable function on S. Consider the set $F = \{(x, y) : x \in S, y \in R, y \geq f(x)\}$. Prove that F is a convex set if and only if

$$[\nabla f(x_2) - \nabla f(x_1)]^t(x_2 - x_1) \geq 0$$

for $x_1, x_2 \in S$.

(b) Write down the basic difference between lot size system and order level system.

(c) A matrix game is called a symmetric game if the corresponding matrix is skew-symmetric. For such a game, the value is 0 and the optimal strategies are same for both the players. You may use this fact to solve the following problem.

Consider a two-person zero-sum game with the associated matrix A given by

$$A = \begin{bmatrix} -3 & 3 & 0 & 2 \\ -4 & -1 & 2 & -2 \\ 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & -1 \end{bmatrix}.$$

where we assume that the player who chooses the row is the maximizer. Find the value of this matrix game and identify the optimal strategies of the players.

[12 + 2 + (4 + 6) = 24]

Reliability

6. (a) Let T be a continuous random variable denoting the lifetime of a component with reliability function

$$R(t) = e^{-\theta t}\left(1 + \theta t + \frac{\theta^2 t^2}{2}\right), \quad t > 0.$$

   (i) Find whether the component lifetime distribution is IFR.

   (ii) Consider a series system consisting of n such independent and identical components. Find the hazard rate of the system lifetime.

   (b) The hazard rate of an item is given by

$$\lambda(t) = \begin{cases} a, & \text{if } 0 < t \leq t_0 \\ a + b(t - t_0), & \text{if } t > t_0, \end{cases}$$

   where a, b and t_0 are positive constants.

   Derive the relibility function of the item. Hence or otherwise find the mean time to failure of the item.

[(6 + 8) + (4 + 6) = 24]

7. (a) Suppose n identical items are put on a life test at time zero. The test is stopped at a pre-specified time T_0. Suppose the lifetime of each unit has hazard rate $\lambda(t) = \alpha\beta t^{\beta-1}, t > 0, \alpha > 0, \beta > 0$. Find the expected number of failures and expected duration of the test.

(b) Suppose 12 items are put on a life test at time zero. The test is stopped at a prespecified time 100 hours. The failure times of the items, which failed by 100 hours, are as follows

$$22, 27, 35, 46, 58, 72, 75, 86, 95.$$

Assume that the lifetime follows exponential distribution with mean θ. Find the expression of reliability estimate at 100 hours.

(c) Consider a system subject to shocks arriving according to a Poisson process N_t with rate parameter λ. The system fails at the arrival of the kth shock for a fixed $k > 1$. Derive the failure time distribution of the system.

$[(5 + 7) + 7 + 5 = 24]$

Statistical Quality Control

8. (a) For an important quality characteristic of a manufacturing process, the upper and lower specification limits are given as U and L respectively, and the target is $T = (U + L)/2$. The quality characteristic is assumed to follow a normal distribution. The process capability ratio C_p of this process is known to be 1.5. The quality control engineer wants to introduce an Individual-Moving Range (I-MR) chart for online monitoring of this quality characteristic. Derive the upper and lower control limits of the I-MR chart in terms of U, L and T.

(b) Mean of a normally distributed process is monitored using an $\bar{X}$ control chart. Samples are collected every hour with a sub-group of size 4. The process mean and standard deviation are 50 mm and 0.02 mm respectively. Suppose the process mean has shifted to 50.03 mm. On an average, how many samples are required to detect this process shift?

(c) The average number of defects per unit product is observed as 1. A quality control engineer wants to introduce a control chart for monitoring defects using a sample of size 5. The number of defects in a sample will be plotted in an appropriately designed control chart. Suggest the control chart and determine its control limits.

(d) A quality characteristic in a manufacturing process is monitored using $\bar{X}$-R charts. The process is found to be operating at a mean equal to $(U + L)/2$, where U and L are upper and lower specification limits respectively. Both $\bar{X}$ and R charts show that the process is in statistical control. The supervisor noted that although the width of the control limits is less than the width of the specification limits provided by the customer, quite a large number of products do not satisfy the specification limits. How can this happen? Explain your answer.

$[8 + 8 + 4 + 4 = 24]$

9. (a) The process capability indices of four similar machines are found as follows.
Machine 1 : $C_p = 1.4, C_{pl} = 0.85$.
Machine 2 : $C_p = 0.90, C_{pu} = 0$.
Machine 3 : $C_p = 1.65, C_{pl} = 1.0$.
Machine 4 : $C_p = 0.75, C_{pl} = 0.75$.
where, C_p is the potential process capability index, C_{pl} and C_{pu} are the performance capability indices with respect to

the lower specification limit (LSL) and the upper specification limit (USL) respectively. Assume that the quality characteristic follows normal distribution. For each machine, sketch (without derivation of actual mean and shift) an approximate distribution of the quality characteristic in relation to LSL and USL of the characteristic. Also, give your recommendations for minimizing the nonconformance items produced by each machine in terms of desired changes in process mean and variance.

(b) The quality characteristic of a process is assumed to follow normal distribution with process capability indices C_p and C_{pk}. Find the relationship between process nonconformance with C_p and C_{pk}.

(c) A double sampling plan for lot sentencing is defined as $n_1 = 50, c_1 = 1, n_2 = 50, c_2 = 3$.

(i) Describe the operating procedure of this double sampling plan in terms of sample selection from the lot and acceptance/rejection decision of the lot.

(ii) Suppose the fraction defective in an incoming lot is p. Then, what will be the probability of acceptance of a lot based on the combined sample (write down the expressions only)?

$[(8 + 4) + 4 + (4 + 4) = 24]$