Part-I

Mathematical and Logical Reasoning

Answer <u>*all*</u> *questions. Each question carries* 5 *marks.*

1. Let $\vec{r}_1$ and $\vec{r}_2$ be vectors joining the points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) to a variable point P (x, y, z). Show that curl $(\vec{r}_1 \times \vec{r}_2) = 2(\vec{r}_2 - \vec{r}_1)$. [5]

2. The work done by a force in moving a particle of mass M from a point (x, y) to a neighboring point $(x + dx, y + dy)$ is given as $dW = 2xydx + x^2dy$. Find the work done for a complete cycle around a unit circle. [5]

3. Considering the distribution function of x as $f(x) = xe^{-\frac{x}{\lambda}}$ (λ being a positive constant) over the interval $0 < x < \infty$, evaluate the mean value of x. [5]

4. Let A be a 2×2 matrix with real elements. What is the maximum number of negative elements that A^2 could have? Justify your answer. [5]

5. Find the solutions of $\cos z = 0$, $z \in \mathbb{C}$ and evaluate $\displaystyle\int_\gamma \frac{\sin z}{\cos z}\, dz$, where γ is a circle of radius π centered at π. [5]

6. Let $y(x)$ be a continuous solution of the initial value problem

$$y' + 2y = f(x), \quad y(0) = 0,$$

where $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1. \end{cases}$

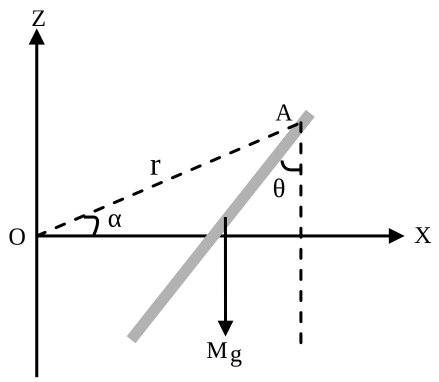
Determine the value of $y(x)$ at $x = \frac{3}{2}$. [5]

Part-II

Physics

Answer any five questions. Each question carries 14 marks.

1. (a) A uniform rod of length L and mass M moves in vertical xz-plane in presence of gravity (g is the acceleration due to gravity). One of the end points of the rod (the point A) moves along the trajectory $z = x \tan\alpha$, where α is a constant angle made by the point A



with x-axis (see the above figure). Considering the generalized coordinates q_i as $q_1 = r$ and $q_2 = \theta$,

   i. write down the Lagrangian in terms of r and θ,

   ii. derive the equations of motion.

(b) With respect to an inertial observer, one rocket and an iron beam (aligned along x-axis) are moving towards each other along x-axis with uniform velocity. The length of the beam is $100\,\text{m}$ as measured by the inertial observer. The distance between them is decreasing at a rate of $7c/6$ and the velocity of the rocket is $2c/3$ with respect to the inertial observer (c being the velocity of light in vacuum). Find the length of the rod as measured from the frame of the rocket.

[(5+2)+7]

2. (a) i. Show that if $\{q, p\}$ are canonical variables, then $\{Q, P\}$ are also canonical under the following transformations

$$Q = \ln(1 + \sqrt{q}\,\cos p), \quad P = 2(1 + \sqrt{q}\,\cos p)\sqrt{q}\,\sin p.$$

ii. Show that the generating function of this transformation is

$$F_3 = -(e^Q - 1)^2 \tan p.$$

(b) Consider a pendulum hanging from a rigid ceiling and another one hanging from the bob of the first one, both via massless inextensible strings. Both the bobs have same mass and the strings have same length. Using the theory of small oscillations,

i. write down the kinetic energy matrix for leading order contribution,

ii. write down the potential energy matrix for the same,

iii. find the modes of frequency of the system.

[(4+3)+(3+2+2)]

3. (a) A circular loop of radius R carries a uniform line charge density λ.

i. Evaluate the electric field at a distance z directly above the center of the loop.

ii. Find the value of z in terms of R for which the electric field would be maximum.

(b) Consider a long solenoid of radius R carrying a current I and having n turns per unit length.

i. Find the self-inductance per unit length.

ii. What is the energy stored in a section of length ℓ of the solenoid?

(c) Let a vector potential be given by

$$\vec{A}(\vec{r}) = \frac{1}{2}\vec{F} \times \vec{r} + \frac{3}{r^3}\vec{r},$$

where $\vec{F}$ is a constant vector having equal components along $\hat{x}, \hat{y}$, and $\hat{z}$. Evaluate the corresponding magnetic field.

[(2+5)+(3+1)+3]

4. (a) The energy spectrum of a particle consists of four states with energies $0, \epsilon, 2\epsilon$, and 3ϵ. Let $Z_B(T)$, $Z_F(T)$, and $Z_C(T)$ denote respectively the Bosonic, Fermionic, and Classical partition functions for four non-interacting particles at temperature T, which can be written as polynomials in $x = \exp[-\epsilon/(k_B T)]$. What are the degrees of these three polynomials?

   (b) Consider a system of two Ising spins s_1 and s_2 taking values ± 1 with interaction energy given by $E = -J s_1 s_2$ when it is in thermal equilibrium at temperature T. Find the average energy of the system.

   (c) For a thermodynamic system, the entropy S is related to the internal energy U and volume V by

$$S = \eta\, U^{3/4} V^{1/4},$$

   where η is a constant. Calculate the Gibbs potential for the system.

[3+5+6]

5. (a) A particle of mass m is confined in a one-dimensional box described by $0 \le x \le a$. The wave function at $t = 0$ is given by

$$\psi(x, t=0) = \sqrt{\frac{8}{5a}} \left[1 + \cos\left(\frac{3\pi x}{a}\right)\right] \sin\left(\frac{3\pi x}{a}\right).$$

      i. Find the wave function and average energy of the system at a later time $t = T$.

      ii. Suppose a position measurement is performed on the system at $t = T$. What is the probability of finding the particle in the region $0 \le x \le \frac{a}{2}$?

   (b) Let ψ_0 and ψ_1 be respectively the normalized ground and first excited state energy eigenfunctions of a linear harmonic oscillator. Consider $A\psi_0 + B\psi_1$ (A and B are real) to be the wave function of the oscillator at some instant of time.

      i. Show that $\langle x \rangle$ (average value of x) is in general non-zero.

      ii. What values of A and B maximize $\langle x \rangle$?

[(6+4)+(2+2)]

6. (a) Using the uncertainty principle, show that the lowest energy of a harmonic oscillator is $\hbar\omega/2$. Assume the motion is confined to the region $-a/2 \leq x \leq a/2$.

   (b) Consider the one-dimensional motion of an electron confined to a potential well $V(x) = \frac{1}{2}kx^2$ and also subjected to a perturbing electric field $\vec{F} = F\hat{x}$. Find the shift in the ground state energy of this system due to the electric field.

   (c) The spin functions for a free electron in a basis where $\hat{S}_z$ is diagonal can be written as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalues of $\hat{S}_z$ being $+1/2$ and $-1/2$, respectively. Using this basis, find a normalized eigenfunction of $\hat{S}_y$ with eigenvalue $-1/2$ (set $\hbar = 1$).

[4+5+5]

7. (a) Consider a line of $2N$ ions of alternating charges $\pm q$ with a repulsive potential A/R^n (A being a constant) between nearest neighbors in addition to the Coulomb potential. Neglect the surface effects.

      i. Find the equilibrium separation R_0 for such a system and evaluate the equilibrium energy $U(R_0)$.

      ii. Let the crystal be compressed so that $R_0 \to R_0(1-\delta)$. Calculate the work done (up to order δ^2) in compressing a unit length of the crystal.

   (b) Consider a two-dimensional square lattice of lattice constant a. The kinetic energy of a free electron at a corner of the first Brillouin zone is larger than that of an electron at the midpoint of a side face of the zone by a factor b. Calculate b.

[(4+5)+5]

8. (a) The Universe is filled with a background of photons with average energy 10^{-3} eV and number density $300\,\text{cm}^{-3}$. High energy γ-rays collide with these photons resulting in electron-positron pair production with cross-section $(8\pi)/9r_e^2$. (Mass of electron $= 0.51$ MeV, classical radius of electron $r_e = 2.8 \times 10^{-13}$ cm)

   i. What is the minimum energy required for pair production in this process?

   ii. What is the average distance the photons will travel before being converted to e^+e^- pairs?

   (b) Show that the Schrödinger equation

$$i\frac{\partial}{\partial t}\begin{pmatrix}\Theta\\ \chi\end{pmatrix} = H\begin{pmatrix}\Theta\\ \chi\end{pmatrix}$$

   reduces to the Klein-Gordon equation for ϕ of mass m, where

$$H = \frac{-\nabla^2}{2m}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix} + m\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

$$\Theta = \frac{1}{2}\left(\phi + \frac{i}{m}\frac{\partial\phi}{\partial t}\right) \text{ and } \chi = \frac{1}{2}\left(\phi - \frac{i}{m}\frac{\partial\phi}{\partial t}\right).$$

   (c) Consider the Lagrangian

$$L = \frac{1}{2}\left(\partial^\mu\phi_1\partial_\mu\phi_1 + \partial^\mu\phi_2\partial_\mu\phi_2\right) - \frac{m^2}{2}\left({\phi_1}^2 + {\phi_2}^2\right) - \frac{\lambda}{4}\left({\phi_1}^2 + {\phi_2}^2\right)^2$$

   where m, λ are constants and Minkowski metric $(1,-1,-1,-1)$ is used.

   i. Show that the Lagrangian is invariant under the transformation $\phi_1 \to \phi_1' = \phi_1\cos\theta - \phi_2\sin\theta$, $\phi_2 \to \phi_2' = \phi_1\sin\theta + \phi_2\cos\theta$.

   ii. Find the conserved current corresponding to this invariance.

[(3+2)+3+(2+4)]

Part-III

Mathematics

Answer any five questions. Each question carries 14 marks.

1. (a) If
$$M(x,y)dx + N(x,y)dy = 0$$
has a general solution $F(x,y) = c$, then show that there exist infinitely many integrating factors.

   (b) Consider the following second order linear differential equation
$$(1-x^2)y'' - 2xy' + 2y = 0.$$
   i. Find any two linearly independent solutions of the above differential equation in terms of power series of x, and
   ii. show that they are convergent for $|x| < 1$.

[6+(6+2)]

2. (a) Consider the first order dynamical system
$$\frac{dx}{dt} = rx - \frac{x}{1+x^2},$$
where r is a real parameter. Find the values of r at which bifurcation occurs and classify them as saddle-node, transcritical, or pitchfork bifurcation.

   (b) Solve the following initial-boundary value problem
$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{\partial^2 y}{\partial t^2}, \\ \text{such that} \quad y(0,t) &= y(\pi,t) = 0, \quad 0 \le t \le \infty \\ y(x,0) &= \sin 2x, \quad 0 \le x \le \pi \\ \frac{\partial y(x,0)}{\partial t} &= 0, \quad 0 \le x \le \pi. \end{aligned}$$

[6+8]

3. (a) Let S_n be the set of $n \times n$ matrices defined by

$$S_n := \left\{ \begin{pmatrix} a & a & \dots & a \\ a & a & \dots & a \\ \vdots & \vdots & \dots & \vdots \\ a & a & \dots & a \end{pmatrix} \middle| \, a \in \mathbb{R} - \{0\} \right\},$$

where $2 \leq n \in \mathbb{N}$. Determine if S_n forms a group under usual matrix multiplication.

(b) How many irreducible monic factors does the polynomial $x^9 - x$ have over $\mathbb{Z}_3[x]$? Justify your answer.

[7+7]

4. (a) A square matrix A is said to be *diagonalizable*, if there exists a non-singular matrix S and a diagonal matrix D such that $A = SDS^{-1}$. Give an example, with proper justification, of a real matrix A which is diagonalizable, but not by any *real* nonsingular matrix S.

(b) A real $n \times n$ matrix A satisfies

$$x^T A x \geq 0,$$

for all $x \in \mathbb{R}^n$. Show that the real part of all eigenvalues of A are non-negative.

[7+7]

5. (a) A two-dimensional steady velocity field of a particle is described by $u = 3x^2 - 2y^2, v = -6xy$.

i. Derive the streamline pattern, and
ii. sketch a few streamlines in the upper-half plane.

(b) Given that a two-dimensional dipole source at the origin produces steady incompressible flow with the stream function

$$\psi = \frac{y}{(x^2 + y^2)}$$

i. determine the direction of motion of a fluid particle at the point $x = 6, \ y = 9$, and
ii. sketch the streamline.

[(3+3)+(4+4)]

6. (a) If the Lagrangian function of a dynamical system has the form

$$L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) = \sum_{k=1}^{n} \left[q_k \dot{q}_k - \sqrt{(1 - \dot{q}_k^2)} \right],$$

then

i. find the Lagrange's equations of motion, and

ii. show that the generalized accelerations are zero.

(b) i. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$,

ii. and hence show that $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{12}$.

[(3+4)+(5+2)]

7. (a) Is it possible to find a real function $v(x, y)$ so that $x^3 + y^3 + iv$ is holomorphic (x and y are real variables)? Justify your answer.

(b) i. Find all the singular points and corresponding residues of $1/(e^z - 1)$.

ii. Find $\displaystyle\int_\gamma \frac{dz}{e^z - 1}$, where γ is a circle of radius 9 centered at 0.

[6+(4+4)]

8. (a) Consider $\mathbb{Q}$ as a metric space with the usual distance function $d(x, y) = |x - y|$, and define $S = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$. Show that S is closed and bounded in $\mathbb{Q}$, but S is not compact.

(b) Discuss the continuity and differentiability of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} x^2 \sin \frac{1}{x} + y^2 \sin \frac{1}{y}, & xy \neq 0; \\ x^2 \sin \frac{1}{x}, & x \neq 0, y = 0; \\ y^2 \sin \frac{1}{y}, & y \neq 0, x = 0; \\ 0, & x = 0 = y. \end{cases}$$

[7+7]